2025 MATHEMATICA BOHEMICA 16 pp
Online first

QUASIMODULES OVER BOUNDED LATTICES

Ivan Chajda, Helmut Länger

Received November 1, 2024. Published online October 9, 2025.
Communicated by Sándor Radeleczki

Abstract. We define a quasimodule $\mathbf{Q}$ over a bounded lattice $\mathbf{L}$ in an analogous way as a module over a semiring is defined. The essential difference is that $\mathbf{L}$ need not be distributive. Also for quasimodules there can be introduced the concepts of inner product, orthogonal elements, orthogonal subsets, bases and closed subquasimodules. We show that the set of all closed subquasimodules forms a complete lattice having orthogonality as an antitone involution. We describe important properties of closed subquasimodules. We prove that every canonical quasimodule has a nontrivial subquasimodule having a basis and we show that orthogonality can be introduced via the inner product. We call a subquasimodule $\mathbf{P}$ of a quasimodule $\mathbf{Q}$ splitting if the sum of P and its orthogonal companion is the whole set Q and the intersection of P and its orthogonal companion is trivial. We show that every splitting subquasimodule is closed and that its orthogonal companion is splitting, too. Our results are illuminated by several examples.

Keywords: quasimodule; canonical quasimodule; lattice; 0-distributive lattice; orthogonal set; basis; closed subquasimodule; splitting subquasimodule

MSC 2020: 13C13, 06B05, 06D99

In our previous paper [4] we investigated semimodules over commutative semirings. Since every bounded distributive lattice is a commutative semiring, the natural question arises if this study can be extended to semimodule-like structures over lattices that need not be distributive. We call such structures "quasimodules" and their treaty is the aim of the present paper.

We start with the definition of our main concept.

Definition 1. By a *quasimodule over a bounded lattice* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ we mean a quadruple $\mathbf{Q} = (Q, +, \cdot, \vec{0})$ satisfying the following conditions:

Support of the research of the first author by the Czech Science Foundation (GAČR), project 25-20013L, and by IGA, project PřF 2025 008, and support of the research of the second author by the Austrian Science Fund (FWF), project 10.55776/PIN5424624, is gratefully acknowledged.

DOI: 10.21136/MB.2025.0143-24

(i) $(Q, +, \vec{0})$ is a commutative monoid,
(ii) $\cdot\colon L \times Q \to Q$,
(iii) $a(b\vec{x}) = (a \wedge b)\vec{x}$,
(iv) $0\vec{x} = \vec{0}$ and $1\vec{x} = \vec{x}$,

where the elements of Q, i.e., the vectors, are denoted by $\vec{x}$, whereas the elements of L, i.e., the scalars, are denoted by Roman letters.

If $\mathbf{L}$ is distributive, then it is a commutative semiring and semimodules over commutative semirings were treated in [4]. If both $\mathbf{L}$ and $\mathbf{Q}$ are distributive lattices, then we obtain an algebra similar to a quantale, see [5] for details. Hence, in the sequel we will not assume distributivity of $\mathbf{L}$. Recall that a *quantale* is an ordered triple $(S, \cdot, \bigvee)$ such that $(S, \bigvee)$ is a $\bigvee$-semilattice, $(S, \cdot)$ is a semigroup, and multiplication distributes over arbitrary joins in both coordinates, that is,

$$a\Big(\bigvee M\Big) = \bigvee(aM) \quad \text{and} \quad \Big(\bigvee M\Big)a = \bigvee(Ma)$$

for all $a \in S$ and all $M \subseteq S$.

Definition 2. Let $\mathbf{Q} = (Q, +, \cdot, \vec{0})$ be a quasimodule over $(L, \vee, \wedge, 0, 1)$ and $P \subseteq Q$. Then $(P, +, \cdot, \vec{0})$ is called a *subquasimodule* of $\mathbf{Q}$ if $\vec{x} + \vec{y}$, $a\vec{x}$, $\vec{0} \in P$ for all $\vec{x}, \vec{y} \in P$ and all $a \in L$. Let $L(\mathbf{Q})$ denote the set of all subquasimodules of $\mathbf{Q}$ and put $\mathbf{L}(\mathbf{Q}) := (L(\mathbf{Q}), \subseteq)$.

Recall the following concepts.

Let $\mathbf{L} = (L, \vee, \wedge, 0)$ be a lattice with 0, M_i a set for all $i \in I$ and $k \in I$.

(i) $\mathbf{L}$ is called 0-*distributive* if $x, y, z \in L$ and $x \wedge z = y \wedge z = 0$ together imply $(x \vee y) \wedge z = 0$.
(ii) p_k denotes the kth projection from $\prod_{i \in I} M_i$ to M_k.

Note that the concept of a 0-distributive lattice was mentioned in [6]. That a 0-distributive lattice need not be distributive is shown in the following example.

E x a m p l e 3. The nonmodular lattice $\mathbf{N}_5 = (N_5, \vee, \wedge)$ visualized in Figure 1 is 0-distributive.

One method how to construct quasimodules in a natural way is described in the following lemma whose proof is straightforward.

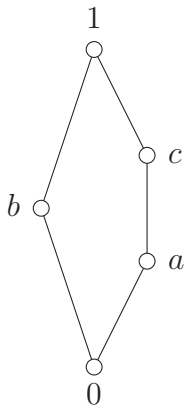

Figure 1. Nonmodular 0-distributive lattice $\mathbf{N}_5$.

Lemma 4. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice, I a nonempty index set, for every $i \in I$ let L_i be a nonempty ideal of $\mathbf{L}$, put $Q := \prod_{i \in I} L_i$ and $\vec{0} := (0, \dots, 0)$ and define $+\colon Q \times Q \to Q$ and $\cdot\colon L \times Q \to Q$ by*

$$(a_i; i \in I) + (b_i; i \in I) := (a_i \vee b_i; i \in I),$$
$$c(a_i; i \in I) := (c \wedge a_i; i \in I)$$

for all $(a_1; i \in I)$, $(b_i; i \in I) \in Q$ and all $c \in L$. Then $\prod_{i \in I} \mathbf{L}_i := (Q, +, \cdot, \vec{0})$ is a quasimodule over $\mathbf{L}$.

Definition 5. We call the quasimodule $\prod_{i \in I} \mathbf{L}_i$ over $\mathbf{L}$ constructed in Lemma 4 the *canonical quasimodule* $\prod_{i \in I} \mathbf{L}_i$ over $\mathbf{L}$.

It is worth noticing that such a canonical quasimodule is not a quantale provided $\mathbf{L}$ is not distributive.

It is evident that $\mathbf{L}(\mathbf{Q})$ forms a complete lattice since $L(\mathbf{Q})$ is closed with respect to arbitrary intersections, see e.g. [1].

In the following, let $\bigvee_{j \in J} P_j$ denote the supremum of the subquasimodules P_j, $j \in J$, of $\mathbf{Q}$ in $\mathbf{L}(\mathbf{Q})$.

Definition 6. Let $\mathbf{Q} = (Q, +, \cdot, \vec{0})$ be a quasimodule and $A \subseteq Q$. Then $\langle A \rangle$ denotes the smallest subquasimodule of $\mathbf{Q}$ including A. This subquasimodule is called the *subquasimodule* of $\mathbf{Q}$ *generated* by A. The set A is called a *generating set* of $\mathbf{Q}$ if $\langle A \rangle = Q$. A minimal generating set of $\mathbf{Q}$ is called a *basis* of $\mathbf{Q}$.

It is clear that finite (sub-)quasimodules always have a basis. For certain canonical quasimodules the next theorem describes a possibly infinite subquasimodule having a basis.

Theorem 7. *Let* $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ *be a bounded lattice and* $q_i \in L$ *for all* $i \in I$. *Further, let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} [0, q_i]$ *be a canonical quasimodule over* $\mathbf{L}$ *and let* P *denote the set of all elements of* Q *having only finitely many nonzero components. For any* $i \in I$ *let* $\vec{b}_i$ *denote the element of* P *satisfying*

$$p_j(\vec{b}_i) = \begin{cases} q_i & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

and put $B := \{\vec{b}_i \mid i \in I\}$. *Then* $\mathbf{P} := (P, +, \cdot, \vec{0})$ *is a subquasimodule of* $\mathbf{Q}$ *and* B *a basis of* $\mathbf{P}$.

P r o o f. That $\mathbf{P}$ is a subquasimodule of $\mathbf{Q}$ including B is easy to see. Let $\vec{a} \in P$. Then there exists a finite subset F of I satisfying $p_i(\vec{a}) = 0$ for all $i \in I \setminus F$. Now $\vec{a} = \sum_{i \in F} p_i(\vec{a})\vec{b}_i$ showing B to be a generating set of $\mathbf{P}$. Similarly, one can see that for every $k \in I$ we have

$$\langle \{\vec{b}_i \mid i \in I \setminus \{k\}\} \rangle = \{\vec{x} \in P \mid p_k(\vec{x}) = 0\}$$

and hence B is a minimal generating set, i.e., a basis of $\mathbf{P}$. □

For canonical quasimodules we can introduce the concept of inner product and orthogonality similarly as it was done for subspaces of a vector space in [3].

Definition 8. Let $(Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ be a canonical quasimodule over a complete lattice and $A \subseteq Q$. Define the *inner product* of $(a_i; i \in I)$, $(b_i; i \in I) \in Q$ by

$$(a_i; i \in I)(b_i; i \in I) := \bigvee_{i \in I} (a_i \wedge b_i)$$

and a binary relation $\perp$ on Q, the so-called orthogonality, by

$$\vec{x} \perp \vec{y} \text{ if and only if } \vec{x}\vec{y} = 0$$

$(\vec{x}, \vec{y} \in Q)$. Moreover, put

$$A^{\perp} := \{\vec{x} \in Q \mid \vec{x} \perp \vec{y} \text{ for all } \vec{y} \in A\},$$

especially,

$$\vec{y}^{\perp} := \{\vec{x} \in Q \mid \vec{x} \perp \vec{y}\}$$

$(\vec{y} \in Q)$. The set A is called *orthogonal* if its elements are pairwise orthogonal, and it is called an *orthogonal basis* if it is both orthogonal and a basis.

R e m a r k 9. Obviously, $(a_i; i \in I) \perp (b_i; i \in I)$ if and only if $a_i \wedge b_i = 0$ for all $i \in I$. Hence, orthogonality of two elements of Q can be defined also in the case when $\mathbf{L}$ is not complete.

Lemma 10. *Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice and $q_i \in L$ for all $i \in I$. Further, let $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} [0, q_i]$ be a canonical quasimodule over $\mathbf{L}$ and let $\vec{x}, \vec{y} \in Q$. If $\vec{x}\vec{z} = \vec{y}\vec{z}$ for all $\vec{z} \in Q$, then $\vec{x} = \vec{y}$.*

P r o o f. If $\vec{x}\vec{z} = \vec{y}\vec{z}$ for all $\vec{z} \in Q$ and if for all $i \in I$ the element $\vec{b}_i$ of Q is defined as in Theorem 7, then $p_i(\vec{x}) = \vec{x}\vec{b}_i = \vec{y}\vec{b}_i = p_i(\vec{y})$ for all $i \in I$, that is, $\vec{x} = \vec{y}$. □

Now we prove that for a canonical quasimodule $\mathbf{Q} = (Q, +, \cdot, \vec{0})$ over a 0-distributive bounded lattice and for a subset A of Q we have that $(A^\perp, +, \cdot, \vec{0})$ is a subquasimodule of $\mathbf{Q}$.

Proposition 11. *Let $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ be a canonical quasimodule over $(L, \vee, \wedge, 0, 1)$, assume every $\mathbf{L}_i$ to be 0-distributive and let $A \subseteq Q$. Then $(A^\perp, +, \cdot, \vec{0})$ is a subquasimodule of $\mathbf{Q}$.*

P r o o f. Let $\vec{x}, \vec{y} \in A^\perp$, say $\vec{x} = (a_i; i \in I)$ and $\vec{y} = (b_i; i \in I)$, and $\vec{z} \in A$, say $\vec{z} = (c_i; i \in I)$. Then $a_i \wedge c_i = b_i \wedge c_i = 0$ for all $i \in I$. Since every $\mathbf{L}_i$ is 0-distributive, we conclude $(a_i \vee b_i) \wedge c_i = 0$ for all $i \in I$ and hence $\vec{x} + \vec{y} = (a_i \vee b_i; i \in I) \perp \vec{z}$. Since $\vec{z}$ was an arbitrary element of A, we obtain $\vec{x} + \vec{y} \in A^\perp$. Now let $a \in L$. Since $a_i \wedge c_i = 0$ for all $i \in I$, we have $(a \wedge a_i) \wedge c_i = 0$ for all $i \in I$ showing $a\vec{x} = (a \wedge a_i; i \in I) \perp \vec{z}$. Because $\vec{z}$ was an arbitrary element of A, we conclude $a\vec{x} \in A^\perp$. Finally, $0 \wedge c_i = 0$ for all $i \in I$ and hence $\vec{0} \in A^\perp$. □

That the condition that all $\mathbf{L}_i$ should be 0-distributive cannot be omitted from Proposition 11 is demonstrated by the following example.

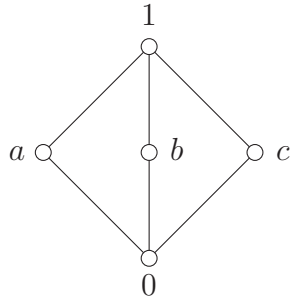

Figure 2. Modular nondistributive lattice $\mathbf{M_3}$.

E x a m p l e 12. The modular nondistributive lattice $\mathbf{M}_3 = (M_3, \vee, \wedge)$ depicted in Figure 2 is not 0-distributive since $a \wedge c = b \wedge c = 0$, but $(a \vee b) \wedge c = 1 \wedge c =$

$c \neq 0$. Consider the canonical quasimodule $\mathbf{Q} := \mathbf{M}_3 \times [0,a]$ over $\mathbf{M}_3$ and put $P := [0,a] \times [0,a]$. Then $P \in L(\mathbf{Q})$. Of course, $[0,a]$ is 0-distributive. Now $P^\perp \notin L(\mathbf{Q})$ since $(b,0),(c,0) \in P^\perp$, but $(b,0)+(c,0) = (b \vee c,0) = (1,0) \notin P^\perp$ because $(1,0) \not\perp (a,0) \in P$. Observe that $B := \{(0,a),(a,0),(b,0)\}$ is an orthogonal basis of $\mathbf{Q}$. First of all, B is a generating set since $(1,0) = (a,0)+(b,0)$, $(x,0) = x(1,0)$ for all $x \in M_3$ and $(x,a) = (x,0)+(0,a)$ for all $x \in M_3$. Moreover, it is easy to see that

$$\begin{aligned}
\langle\{(0,a),(a,0)\}\rangle &= [0,a] \times [0,a] \subsetneqq M_3 \times [0,a],\\
\langle\{(0,a),(b,0)\}\rangle &= [0,b] \times [0,a] \subsetneqq M_3 \times [0,a],\\
\langle\{(a,0),(b,0)\}\rangle &= M_3 \times \{0\} \subsetneqq M_3 \times [0,a].
\end{aligned}$$

In the following we are interested in subquasimodules closed with respect to the operator ${}^{\perp\perp}$. For this purpose we define:

Definition 13. Let $\mathbf{Q} = (Q,+,\cdot,\vec{0})$ be a canonical quasimodule. A subset A of Q is called *closed* if $A^{\perp\perp} = A$. Let $L_C(\mathbf{Q})$ denote the set of all closed subquasimodules of $\mathbf{Q}$ and put $\mathbf{L}_C(\mathbf{Q}) := (L_C(\mathbf{Q}),\subseteq)$.

It is easy to verify assertions in the following remark, see e.g. [1].

R e m a r k 14. Let $(Q,+,\cdot,\vec{0})$ be a canonical quasimodule. Then $({}^\perp,{}^\perp)$ is the Galois correspondence between $(2^Q,\subseteq)$ and $(2^Q,\subseteq)$ induced by $\perp$. From this we obtain for all $A,B \subseteq Q$:

(i) $A \subseteq A^{\perp\perp}$,
(ii) $A \subseteq B$ implies $B^\perp \subseteq A^\perp$,
(iii) $A^{\perp\perp\perp} = A^\perp$,
(iv) $A \subseteq B^\perp$ if and only if $B \subseteq A^\perp$.

In the following we list some elementary properties of orthogonality.

Lemma 15. *Let $(Q,+,\cdot,\vec{0})$ be a canonical quasimodule and $A, A_j \subseteq Q$ for all $j \in J$ with $A \neq \emptyset$. Then the following hold:*

(i) $\bigcap\limits_{j\in J} A_j^\perp = \Big(\bigcup\limits_{j\in J} A_j\Big)^\perp$,
(ii) $\Big(\bigcap\limits_{j\in J} A_j\Big)^{\perp\perp} \subseteq \bigcap\limits_{j\in J} A_j^{\perp\perp}$,
(iii) $Q^\perp = \{\vec{0}\}$,
(iv) $A \cap A^\perp = \{\vec{0}\}$,
(v) $\{\vec{0}\}^\perp = Q$.

P r o o f. (i) For all $\vec{a} \in Q$ the following are equivalent:

$$\begin{aligned}
&\vec{a} \in \bigcap_{j\in J} A_j^{\perp},\\
&\vec{a} \in A_j^{\perp} \quad \text{for all } j \in J,\\
&\vec{a} \perp \vec{x} \quad \text{for all } j \in J \text{ and all } \vec{x} \in A_j,\\
&\vec{a} \perp \vec{x} \quad \text{for all } \vec{x} \in \bigcup_{j\in J} A_j,\\
&\vec{a} \in \Big(\bigcup_{j\in J} A_j\Big)^{\perp}.
\end{aligned}$$

(ii) If $k \in J$, then we have $\bigcap\limits_{j\in J} A_j \subseteq A_k$ and hence $\Big(\bigcap\limits_{j\in J} A_j\Big)^{\perp\perp} \subseteq A_k^{\perp\perp}$ according to (ii) of Remark 14, which implies $\Big(\bigcap\limits_{j\in J} A_j\Big)^{\perp\perp} \subseteq \bigcap\limits_{j\in J} A_j^{\perp\perp}$.

(iii) follows from the fact that $\vec{x} \in Q$ and $\vec{x} \perp \vec{x}$ together imply $\vec{x} = \vec{0}$.

(iv) follows from the same fact as (iii).

(v) is straightforward. □

If, moreover, all sublattices $\mathbf{L}_i$ of the lattice $\mathbf{L}$ are considered to be 0-distributive, then we can prove the following.

Theorem 16. *Let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod\limits_{i\in I} \mathbf{L}_i$ *be a canonical quasimodule with* 0*-distributive* $\mathbf{L}_i$*, assume* $A \subseteq Q$ *and* $B \subseteq C \subseteq Q$*, assume* C *to be orthogonal and let* $P_j \in L_C(Q)$ *for all* $j \in J$*. Then the following hold:*

(i) $L_C(\mathbf{Q}) = \{D^{\perp} \mid D \subseteq Q\}$,

(ii) $A^{\perp\perp}$ *is the smallest closed subquasimodule of* $\mathbf{Q}$ *including* A,

(iii) $\overset{c}{\bigvee\limits_{j\in J}} P_j = \Big(\bigcup\limits_{j\in J} P_j\Big)^{\perp\perp}$, *where* $\overset{c}{\bigvee\limits_{j\in J}} P_j$ *denotes the supremum of the closed subquasimodules* $P_j, j \in J$*, of* $\mathbf{Q}$ *in* $\mathbf{L}_C(\mathbf{Q})$,

(iv) $^{\perp}$ *is an antitone involution on* $\mathbf{L}_C(\mathbf{Q})$,

(v) $\mathbf{L}_C(\mathbf{Q})$ *is a complete lattice,*

(vi) $A^{\perp} = \langle A\rangle^{\perp}$,

(vii) $\langle C \setminus B\rangle \subseteq \langle B\rangle^{\perp}$.

P r o o f. (i) follows from Proposition 11 and from (iii) of Remark 14.

(ii) follows from (i) and (ii) of Remark 14 and from (i).

(iii) According to (ii), $\Big(\bigcup\limits_{j\in J} P_j\Big)^{\perp\perp}$ is the smallest closed subquasimodule of $\mathbf{Q}$ including $\bigcup\limits_{j\in J} P_j$ and hence, also the smallest closed subquasimodule of $\mathbf{Q}$ including all $P_j, j \in J$.

(iv) follows from (ii) and (iii) of Remark 14 and from (i).

(v) follows from (i) of Lemma 15 and from (i).

(vi) Because of $A \subseteq \langle A \rangle$ and (ii) of Remark 14 we have $\langle A \rangle^\perp \subseteq A^\perp$. On the other hand, $A \subseteq A^{\perp\perp} \in L(\mathbf{Q})$ according to (i) of Remark 14 and Proposition 11 and hence, $\langle A \rangle \subseteq A^{\perp\perp}$, which implies $A^\perp \subseteq \langle A \rangle^\perp$ by (ii) and (iii) of Remark 14.

(vii) According to the orthogonality of C, (vi) and Proposition 11 we have

$$C \setminus B \subseteq B^\perp = \langle B \rangle^\perp \in L(\mathbf{Q})$$

and hence $\langle C \setminus B \rangle \subseteq \langle B \rangle^\perp$. □

Lemma 17. *Let $\mathbf{Q} = \prod_{i \in I} \mathbf{L}_i$ with $\mathbf{L}_i = (L_i, \vee, \wedge, 0)$ be a canonical quasimodule over $(L, \vee, \wedge, 0)$ and $M_i \subseteq L_i$ for all $i \in I$. Then the following hold:*

(i) *If $P \in L(\mathbf{Q})$, then $p_i(P) \in L(\mathbf{L}_i)$ for all $i \in I$,*

(ii) *$\prod_{i \in I} M_i \in L(\mathbf{Q})$ if and only if $M_i \in L(\mathbf{L}_i)$ for all $i \in I$.*

Proof. (i) Let $j \in I$, $a, b \in p_j(P)$ and $c \in L$. Then there exist $\vec{x}, \vec{y} \in P$ with $p_j(\vec{x}) = a$ and $p_j(\vec{y}) = b$. Now we have $a \vee b = p_j(\vec{x} + \vec{y}) \in p_j(P)$ and $c \wedge a = p_j(c\vec{x}) \in p_j(P)$. Moreover, $0 = p_j(\vec{0}) \in p_j(P)$.

(ii) If $\prod_{i \in I} M_i \in L(\mathbf{Q})$, then $M_i \in L(\mathbf{L}_i)$ for all $i \in I$ according to (i). Now assume $M_i \in L(\mathbf{L}_i)$ for all $i \in I$ and let $\vec{x}, \vec{y} \in \prod_{i \in I} M_i$ and $c \in L$. Then $p_i(\vec{x})$, $p_i(\vec{y}) \in M_i$ for all $i \in I$. From this we obtain $p_i(\vec{x}) \vee p_i(\vec{y})$, $c \wedge p_i(\vec{x}) \in M_i$ for all $i \in I$ and hence, $\vec{x} + \vec{y} = (p_i(\vec{x}) \vee p_i(\vec{y}); i \in I) \in \prod_{i \in I} M_i$ and $c\vec{x} = (c \wedge p_i(\vec{x}); i \in I) \in \prod_{i \in I} M_i$. Finally, $0 \in M_i$ for all $i \in I$ and hence $\vec{0} = (0; i \in I) \in \prod_{i \in I} M_i$. □

We show that the set of all elements being orthogonal to a fixed subset of a canonical quasimodule is a product of its projections.

Lemma 18. *Let $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ be a canonical quasimodule and $P \subseteq Q$. Then $P^\perp = \prod_{i \in I} (p_i(P))^\perp$.*

Proof. For any $\vec{a} \in Q$ the following are equivalent:

$$\begin{aligned}
&\vec{a} \in P^\perp,\\
&\vec{a} \perp \vec{x} \text{ for all } \vec{x} \in P,\\
&p_i(\vec{a}) \perp p_i(\vec{x}) \text{ for all } \vec{x} \in P \text{ and all } i \in I,\\
&p_i(\vec{a}) \perp y \text{ for all } i \in I \text{ and all } y \in p_i(P),
\end{aligned}$$

$$p_i(\vec{a}) \in (p_i(P))^{\perp} \text{ for all } i \in I,$$
$$\vec{a} \in \prod_{i \in I} (p_i(P))^{\perp}.$$

□

If all sublattices $\mathbf{L}_i$ of the lattice $\mathbf{L}$ are 0-distributive, then the projections of a closed subquasimodule of a canonical quasimodule over $\mathbf{L}$ are again closed subquasimodules of the canonical quasimodules $\mathbf{L}_i$, see the following result.

Theorem 19. *Let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ *be a canonical quasimodule with* 0*-distributive* $\mathbf{L}_i = (L_i, \vee, \wedge, 0)$ *and* $P \subseteq Q$*. Then* $(P, +, \cdot, \vec{0})$ *is a closed subquasimodule of* $\mathbf{Q}$ *if and only if for every* $i \in I$ *there exists a closed subquasimodule* $(P_i, \vee, \wedge, 0)$ *of* $(L_i, \vee, \wedge, 0)$ *with* $\prod_{i \in I} P_i = P$*.*

P r o o f. First assume $(P, +, \cdot, \vec{0})$ to be a closed subquasimodule of $\mathbf{Q}$. According to Lemma 18 we have

$$P^{\perp} = \prod_{i \in I} (p_i(P))^{\perp},$$
$$P^{\perp\perp} = \prod_{i \in I} (p_i(P^{\perp}))^{\perp} = \prod_{i \in I} (p_i(P))^{\perp\perp}.$$

Because of

$$P^{\perp\perp} = P \subseteq \prod_{i \in I} p_i(P) \subseteq \prod_{i \in I} (p_i(P))^{\perp\perp}$$

according to (i) of Remark 14 we conclude

$$P = \prod_{i \in I} p_i(P) = \prod_{i \in I} (p_i(P))^{\perp\perp},$$

which together with $p_i(P) \subseteq (p_i(P))^{\perp\perp}$ for all $i \in I$ (again according to (i) of Remark 14) implies $p_i(P) = (p_i(P))^{\perp\perp}$ for all $i \in I$, i.e., $p_i(P)$ is a closed subset of L_i for all $i \in I$. Because of Lemma 17 we conclude that $(p_i(P), \vee, \wedge, 0)$ is a closed subquasimodule of $(L_i, \vee, \wedge, 0)$ for all $i \in I$. If, conversely, for every $i \in I$ there exists a closed subquasimodule $(P_i, \vee, \wedge, 0)$ of $(L_i, \vee, \wedge, 0)$ with $\prod_{i \in I} P_i = P$, then

$$P^{\perp} = \prod_{i \in I} (p_i(P))^{\perp} = \prod_{i \in I} P_i^{\perp},$$
$$P^{\perp\perp} = \prod_{i \in I} (p_i(P^{\perp}))^{\perp} = \prod_{i \in I} P_i^{\perp\perp} = \prod_{i \in I} P_i = P,$$

i.e., P is a closed subset of Q and, moreover, $(P, +, \cdot, \vec{0})$ is a closed subquasimodule of $\mathbf{Q}$ because of Lemma 17. □

For the lattice of closed subquasimodules, the following result plays an essential role.

Corollary 20. *Let* $\mathbf{Q} = \prod_{i\in I} \mathbf{L}_i$ *be a canonical quasimodule with* 0*-distributive* $\mathbf{L}_i$. *Then* $\mathbf{L}_C(\mathbf{Q}) \cong \prod_{i\in I} \mathbf{L}_C(\mathbf{L}_i)$.

P r o o f. From Theorem 19 we see that the mapping $(P_i; i \in I) \mapsto \prod_{i\in I} P_i$ is a bijection from $\prod_{i\in I} L_C(\mathbf{L}_i)$ to $L_C(\mathbf{Q})$. Since for $(P_i; i \in I)$, $(Q_i; i \in I) \in \prod_{i\in I} L_C(\mathbf{L}_i)$ we have $\prod_{i\in I} P_i \subseteq \prod_{i\in I} Q_i$ if and only if $P_i \subseteq Q_i$ for all $i \in I$, we conclude that this bijection is an isomorphism from $\prod_{i\in I} \mathbf{L}_C(\mathbf{L}_i)$ to $\mathbf{L}_C(\mathbf{Q})$. □

We illustrate the concepts and results investigated above in the following example.

E x a m p l e 21. The lattice $\mathbf{N}_5$ from Example 3 can be considered as a canonical quasimodule over $\mathbf{N}_5$. Since $A^\perp = \bigcap_{\vec{x}\in A} \vec{x}^\perp$ for all $A \subseteq N_5$ and

$$\begin{aligned} 0^\perp &= \{0, a, b, c, 1\},\\ a^\perp &= \{0, b\},\\ b^\perp &= \{0, a, c\},\\ c^\perp &= \{0, b\},\\ 1^\perp &= \{0\}, \end{aligned}$$

we have $L_C(\mathbf{N}_5) = \{\{0\}, \{0, b\}, \{0, a, c\}, \{0, a, b, c, 1\}\}$. From this and Corollary 20 we conclude that for any positive integer n, $\mathbf{L}_C(\mathbf{N}_5^n)$ is the 2^{2n}-element Boolean algebra. Further, the canonical quasimodule $\mathbf{Q} := \mathbf{N}_5 \times [0, a]$ over $\mathbf{N}_5$ has the following subquasimodules:

$$\begin{aligned} P_1 &= \{(0,0)\},\\ P_2 &= \{(0,0), (0,a)\},\\ P_3 &= \{(0,0), (a,0)\},\\ P_4 &= \{(0,0), (a,a)\},\\ P_5 &= \{(0,0), (b,0)\},\\ P_6 &= \{(0,0), (0,a), (a,a)\},\\ P_7 &= \{(0,0), (a,0), (a,a)\},\\ P_8 &= \{(0,0), (a,0), (c,0)\},\\ P_9 &= \{(0,0), (a,a), (c,a)\},\\ P_{10} &= \{(0,0), (0,a), (a,0), (a,a)\}, \end{aligned}$$

$$
\begin{aligned}
P_{11} &= \{(0,0),(0,a),(a,a),(c,a)\},\\
P_{12} &= \{(0,0),(0,a),(b,0),(b,a)\},\\
P_{13} &= \{(0,0),(0,a),(a,0),(a,a),(c,a)\},\\
P_{14} &= \{(0,0),(a,0),(a,a),(c,0),(c,a)\},\\
P_{15} &= \{(0,0),(a,0),(b,0),(c,0),(1,0)\},\\
P_{16} &= \{(0,0),(a,a),(b,0),(c,a),(1,a)\},\\
P_{17} &= \{(0,0),(0,a),(a,0),(a,a),(c,0),(c,a)\},\\
P_{18} &= \{(0,0),(0,a),(a,a),(b,0),(b,a),(c,a),(1,a)\},\\
P_{19} &= \{(0,0),(a,0),(a,a),(b,0),(c,0),(c,a),(1,0),(1,a)\},\\
P_{20} &= \{(0,0),(0,a),(a,0),(a,a),(b,0),(b,a),(c,0),(c,a),(1,0),(1,a)\}.
\end{aligned}
$$

Moreover, $\{(0,a),(1,0)\}$ and $\{(0,a),(b,0),(c,0)\}$ are orthogonal bases of $\mathbf{Q}$ since

$$
\begin{aligned}
\langle\{(0,a),(1,0)\}\rangle &= P_{20},\\
\langle\{(0,a)\}\rangle &= P_2,\\
\langle\{(1,0)\}\rangle &= P_{15}
\end{aligned}
$$

for the first one and

$$
\begin{aligned}
\langle\{(0,a),(b,0),(c,0)\}\rangle &= P_{20},\\
\langle\{(0,a),(b,0)\}\rangle &= P_{12},\\
\langle\{(0,a),(c,0)\}\rangle &= P_8,\\
\langle\{(b,0),(c,0)\}\rangle &= P_{15}
\end{aligned}
$$

for the second one. Hence, a quasimodule can have more than one basis and these bases may be of different cardinality contrary to the case of vector spaces, see e.g. [2].

According to the description of $L_C(\mathbf{N}_5)$ above, Theorem 19 and Corollary 20, we have $L_C(\mathbf{Q}) = \{P_1, P_2, P_5, P_8, P_{12}, P_{15}, P_{17},\ P_{20}\}$ and $\mathbf{L}_C(\mathbf{Q})$ is the eight-element Boolean algebra. The lattices $\mathbf{L}(\mathbf{Q})$ and $\mathbf{L}_C(\mathbf{Q})$ are visualized in Figure 3 and 4, respectively.

The mappings $^{\perp}$ and $^{\perp\perp}$ on $L(\mathbf{Q})$ are described by the following tables:

P	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
$P^{\perp}$	P_{20}	P_{15}	P_{12}	P_5	P_{17}	P_5	P_5	P_{12}	P_5	P_5
$P^{\perp\perp}$	P_1	P_2	P_8	P_{17}	P_5	P_{17}	P_{17}	P_8	P_{17}	P_{17}

P	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}
$P^{\perp}$	P_5	P_8	P_5	P_5	P_2	P_1	P_5	P_1	P_1	P_1
$P^{\perp\perp}$	P_{17}	P_{12}	P_{17}	P_{17}	P_{15}	P_{20}	P_{17}	P_{20}	P_{20}	P_{20}

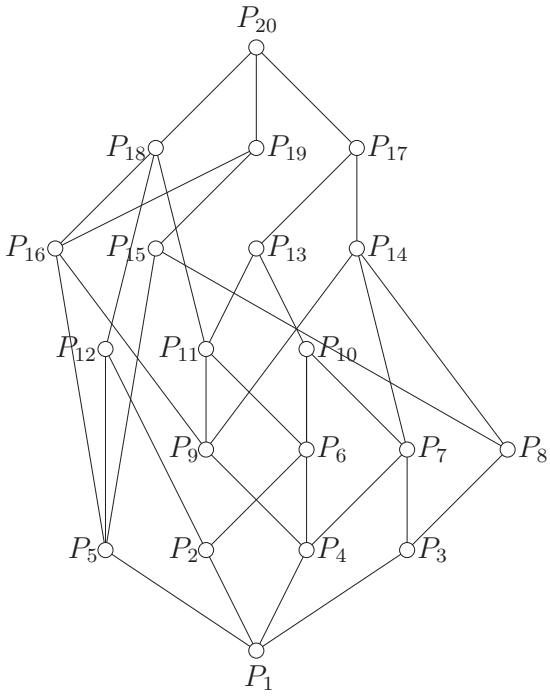

Figure 3. Lattice $\mathbf{L}(\mathbf{N}_5 \times [0, a])$.

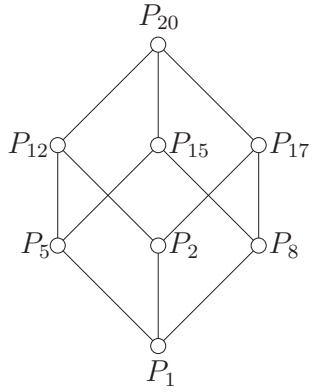

Figure 4. Lattice $\mathbf{L}_C(\mathbf{N}_5 \times [0, a])$.

Similarly, the restrictions of these mappings to $L_C(\mathbf{Q})$ are described in the table below:

P	P_1	P_2	P_5	P_8	P_{12}	P_{15}	P_{17}	P_{20}
$P^{\perp}$	P_{20}	P_{15}	P_{17}	P_{12}	P_8	P_2	P_5	P_1
$P^{\perp\perp}$	P_1	P_2	P_5	P_8	P_{12}	P_{15}	P_{17}	P_{20}

The next result solves the problem when the mapping $^{\perp\perp}$ is a homomorphism from $\mathbf{L}(\mathbf{Q})$ onto $\mathbf{L}_C(\mathbf{Q})$.

Theorem 22. *Let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i\in I} \mathbf{L}_i$ *be a canonical quasimodule with* 0*-distributive* $\mathbf{L}_i$*. Then the following hold:*

(i) *If* $(P \cap R)^{\perp\perp} = P^{\perp\perp} \cap R^{\perp\perp}$ *for all* $P, R \in L(\mathbf{Q})$*, then* $^{\perp\perp}$ *is a homomorphism from* $(L(\mathbf{Q}), \vee, \cap, ^{\perp}, \{0\}, Q)$ *onto* $(L_C(\mathbf{Q}), \overset{c}{\vee}, \cap, ^{\perp}, \{0\}, Q)$*.*

(ii) *If* $\left(\bigcap_{j\in J} P_j\right)^{\perp\perp} = \bigcap_{j\in J} P_j^{\perp\perp}$ *for every family* P_j*,* $j \in J$*, of subquasimodules of* $\mathbf{Q}$*, then* $^{\perp\perp}$ *is a complete homomorphism from* $(L(\mathbf{Q}), \vee, \cap, ^{\perp}, \{0\}, Q)$ *onto* $(L_C(\mathbf{Q}), \overset{c}{\vee}, \cap, ^{\perp}, \{0\}, Q)$*, that is,* $\left(\bigvee_{j\in J} P_j\right)^{\perp\perp} = \overset{c}{\bigvee}_{j\in J} P_j^{\perp\perp}$ *and* $\left(\bigcap_{j\in J} P_j\right)^{\perp\perp} = \bigcap_{j\in J} P_j^{\perp\perp}$ *for every family* $P_j, j \in J$*, of subquasimodules of* $\mathbf{Q}$*.*

P r o o f. Let $P, R, P_j \in L(\mathbf{Q})$ for every $j \in J$.

(i) We have

$$\begin{aligned}
(P \vee R)^{\perp\perp} &= (P \cup R)^{\perp\perp} = (P^{\perp} \cap R^{\perp})^{\perp} = P^{\perp\perp} \overset{c}{\vee} R^{\perp\perp},\\
(P \cap R)^{\perp\perp} &= P^{\perp\perp} \cap R^{\perp\perp},\\
(P^{\perp})^{\perp\perp} &= (P^{\perp\perp})^{\perp},\\
\{0\}^{\perp\perp} &= \{0\},\\
Q^{\perp\perp} &= Q.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
\left(\bigvee_{j\in J} P_j\right)^{\perp\perp} &= \left(\bigcup_{j\in J} P_j\right)^{\perp\perp} = \left(\bigcap_{j\in J} P_j^{\perp}\right)^{\perp} = \overset{c}{\bigvee_{j\in J}} P_j^{\perp\perp},\\
\left(\bigcap_{j\in J} P_j\right)^{\perp\perp} &= \bigcap_{j\in J} P_j^{\perp\perp},\\
(P^{\perp})^{\perp\perp} &= (P^{\perp\perp})^{\perp},\\
\{0\}^{\perp\perp} &= \{0\},\\
Q^{\perp\perp} &= Q.
\end{aligned}$$

□

We are interested in the case when for a subquasimodule $(P, +, \cdot, \vec{0})$ of a quasimodule $\mathbf{Q}$, $P^{\perp}$ is a complement of P in the lattice $\mathbf{L}(\mathbf{Q})$. For this, we introduce the following concept.

Definition 23. Let $\mathbf{Q} = (Q, +, \cdot, \vec{0})$ be a canonical quasimodule. A subset A of Q is called *splitting* if $A + A^\perp = Q$. Here $A + A^\perp$ means $\{\vec{x} + \vec{y} \mid \vec{x} \in A, \vec{y} \in A^\perp\}$. Let $L_S(\mathbf{Q})$ denote the set of all splitting subquasimodules of $\mathbf{Q}$ and put $\mathbf{L}_S(\mathbf{Q}) := (L_S(\mathbf{Q}), \subseteq)$.

Note that according to (iv) of Lemma 15 we have $A \cap A^\perp = \{0\}$ for every nonempty subset A of Q and hence, a subquasimodule $(P, +, \cdot, \vec{0})$ of $\mathbf{Q}$ is splitting if and only if both $P + P^\perp = Q$ and $P \cap P^\perp = \{0\}$.

E x a m p l e 24. Consider the canonical quasimodule $\mathbf{Q} := \mathbf{N}_5 \times \{0, a\}$ from Example 21. Then $L_S(\mathbf{Q}) = L_C(\mathbf{Q})$.

However, the situation from the previous example, where splitting subquasimodules coincide with the closed ones, does not hold in general. We can prove only that every splitting subquasimodule is closed.

Theorem 25. *Let* $\mathbf{Q} = \prod_{i \in I} \mathbf{L}_i$ *be a canonical quasimodule. Then* $L_S(\mathbf{Q}) \subseteq L_C(\mathbf{Q})$.

P r o o f. Assume $L_S(\mathbf{Q}) \not\subseteq L_C(\mathbf{Q})$. Then there exists some $P \in L_S(\mathbf{Q}) \setminus L_C(\mathbf{Q})$. Since P is not closed, there exists some $\vec{x} \in P^{\perp\perp} \setminus P$. Because P is splitting there exist $\vec{y} \in P$ and $\vec{z} \in P^\perp$ with $\vec{y} + \vec{z} = \vec{x}$. Let $x_i, i \in I$, denote the components of $\vec{x}$. We use analogous notations for the components of $\vec{y}$ and $\vec{z}$. Since $\vec{y} + \vec{z} = \vec{x}$, we have $y_i \vee z_i = x_i$ for all $i \in I$. Because $\vec{x} \in P^{\perp\perp}$ and $\vec{z} \in P^\perp$, we have $\vec{x} \perp \vec{z}$, that is, $x_i \wedge z_i = 0$ for all $i \in I$. Together we obtain $z_i = (y_i \vee z_i) \wedge z_i = x_i \wedge z_i = 0$ for all $i \in I$, that is, $\vec{z} = \vec{0}$, which implies $\vec{x} = \vec{y} + \vec{z} = \vec{y} \in P$, a contradiction. This shows $L_S(\mathbf{Q}) \subseteq L_C(\mathbf{Q})$. □

That the converse inclusion does not hold in general is demonstrated by the following example.

E x a m p l e 26. Let $\mathbf{L}$ denote the nonmodular 0-distributive lattice depicted in Figure 5.

Consider the canonical quasimodule $\mathbf{Q} := \mathbf{L} \times \mathbf{L}$ over $\mathbf{L}$. Then $P := [0, b] \times [0, c]$ is a closed subquasimodule of $\mathbf{Q}$ since $[0, b]$ and $[0, c]$ are closed subquasimodules of the canonical quasimodule $\mathbf{L}$ over $\mathbf{L}$, but P is not splitting since $(1, 1) \notin P + P^\perp = ([0, b] \times [0, c]) + ([0, c] \times [0, b])$.

In the following proposition we show that for every splitting subquasimodule $(P, +, \cdot, \vec{0})$ of a canonical quasimodule $\mathbf{Q}$ with 0-distributive components, its orthogonal companion $(P^\perp, +, \cdot, \vec{0})$ is again splitting.

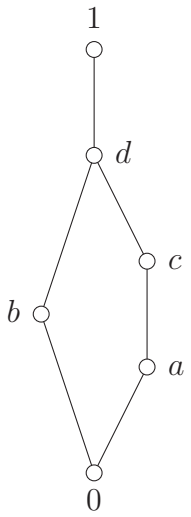

Figure 5. Nonmodular 0-distributive lattice $\mathbf{L}$.

Proposition 27. *Let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ *be a canonical quasimodule with* 0*-distributive* $\mathbf{L}_i$*. Then the following hold:*

(i) *For every splitting subquasimodule* $(P, +, \cdot, \vec{0})$ *of* $\mathbf{Q}$*, the subquasimodule* $(P^\perp, +, \cdot, \vec{0})$ *of* $\mathbf{Q}$ *is again splitting.*

(ii) *The mapping* $^\perp$ *is an antitone mapping from* $\mathbf{L}_S(\mathbf{Q})$ *to* $\mathbf{L}_C(\mathbf{Q})$*.*

P r o o f. Let $P \in L_S(\mathbf{Q})$.

(i) We have $P + P^\perp = Q$. According to (i) of Remark 14 we conclude $P^\perp + P^{\perp\perp} = Q$ showing $P^\perp \in P_S(\mathbf{Q})$.

(ii) According to (i) of Theorem 16 we have $P^\perp \in L_C(\mathbf{Q})$. The rest of the proof follows from (ii) of Remark 14. □

The final proposition shows that for a canonical quasimodule $\mathbf{Q} = \prod_{i \in I} \mathbf{L}_i$ with $\mathbf{L}_i = (L_i, \vee, \wedge, 0)$ for all $i \in I$, a direct product of certain subsets of L_i is a splitting subquasimodule of $\mathbf{Q}$ if and only if its factors are splitting subquasimodules of $\mathbf{L}_i$.

Proposition 28. *Let* $\mathbf{Q} = (Q, +, \cdot, \vec{0}) = \prod_{i \in I} \mathbf{L}_i$ *with* $\mathbf{L}_i = (L_i, \vee, \wedge, 0)$ *be a canonical quasimodule and* $M_i \subseteq L_i$ *for all* $i \in I$*. Then* $\prod_{i \in I} M_i \in L_S(\mathbf{Q})$ *if and only if* $M_i \in L_S(\mathbf{L}_i)$ *for all* $i \in I$*.*

P r o o f. Because of Lemma 17 we have $\prod_{i \in I} M_i \in L(\mathbf{Q})$ if and only if $M_i \in L(\mathbf{L}_i)$ for all $i \in I$. Now assume $\prod_{i \in I} M_i \in L(\mathbf{Q})$. Then, according to Lemma 18, the

following are equivalent:

$$\prod_{i\in I} M_i \in L_S(\mathbf{Q}),$$
$$\prod_{i\in I} M_i + \Big(\prod_{i\in I} M_i\Big)^{\perp} = Q,$$
$$\prod_{i\in I} M_i + \prod_{i\in I} M_i^{\perp} = Q,$$
$$\prod_{i\in I} (M_i \vee M_i^{\perp}) = Q,$$
$$M_i \vee M_i^{\perp} = L_i \quad \text{for all } i \in I,$$
$$M_i \in L_s(\mathbf{L}_i) \quad \text{for all } i \in I.$$

□

A c k n o w l e d g e m e n t. The authors are grateful to the anonymous referee whose valuable remarks helped to increase the quality of the paper.

References

[1] *G. Birkhoff*: Lattice Theory. American Mathematical Society Colloquium Publications 25. AMS, Providence, 1979. zbl MR doi
[2] *I. Chajda, H. Länger*: The lattice of subspaces of a vector space over a finite field. Soft Comput. *23* (2019), 3261–3267. zbl doi
[3] *I. Chajda, H. Länger*: Orthogonality and complementation in the lattice of subspaces of a finite vector space. Math. Bohem. *147* (2022), 141–153. zbl MR doi
[4] *I. Chajda, H. Länger*: Semimodules over commutative semirings and modules over unitary commutative rings. Linear Multilinear Algebra *70* (2022), 1329–1344. zbl MR doi
[5] *D. Kruml, J. Paseka*: Algebraic and categorical aspects of quantales. Handbook of Algebra. Volume 5. Elsevier, Amsterdam, 2008, pp. 323–362. zbl MR doi
[6] *G. N. Rao, R. V. A. Raju*: 0-distributive almost lattices. Bull. Int. Math. Virtual Inst. *10* (2020), 239–248. zbl MR

Authors' addresses: *Ivan Chajda*, Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: `ivan.chajda@upol.cz`; *Helmut Länger* (corresponding author), TU Wien, Fakultät für Mathematik und Geoinformation, Institut für Diskrete Mathematik und Geometrie, Wiedner Hauptstraße 8-10, 1040 Wien, Austria, and Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: `helmut.laenger@tuwien.ac.at`.