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STANDARD AND DISTRIBUTIVE IDEALS IN TRELLISES

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Abstract. The concept of standard ideals in lattices, introduced by G. Grätzer and E. T. Schmidt, plays a significant role in the development of lattice theory. It serves as a generalization of normal subgroups in groups and ideals in rings to lattices. Similarly, the notion of distributive ideals, introduced by O. Ore, is another important tool in lattice theory. Building on these ideas, we extend the concepts of standard ideals and distributive ideals to the broader context of trellises. A trellis is a pseudo-ordered set where every pair of elements has both the least upper bound and the greatest lower bound. Within this framework, many of the results of Grätzer and Schmidt are effectively generalized to trellises.

Keywords: trellis; weakly-associative lattice; pseudo-ordered set; standard ideal; distributive ideal; congruence relation

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1. Introduction

Let L be a lattice. An ideal S of L is called *standard* [4] if S is a standard element of the ideal lattice $I(L)$, that is, if

$$I \wedge (S \vee J) = (I \wedge S) \vee (I \wedge J) \quad \text{for all } I, J \in I(L). \tag{1.1}$$

An equivalent condition for the ideal S to be a standard ideal of L [4, Theorem 2] is that the binary relation Θ defined on L, where for $a, b \in L$, $a \equiv b\ (\Theta)$ if and only if $a \vee b = (a \wedge b) \vee s$ for some $s \in S$, is a congruence relation on L. An ideal D of L is called *distributive* [3, Definition III.3.1], if D is a distributive element of $I(L)$, that is, if

$$D \vee (I \wedge J) = (D \vee I) \wedge (D \vee J) \quad \text{for all } I, J \in I(L). \tag{1.2}$$

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Again, an equivalent characterization of the ideal D as a distributive ideal of L (see Theorem III.3.4 of [3]), is that the binary relation Θ on L, defined by declaring $x \equiv y\ (\Theta)$ for $x, y \in L$ whenever there exists some $a \in D$ such that $x \vee a = y \vee a$, is a congruence relation on L. These two concepts, together with the concept of a *neutral ideal*, form an integral part of the development of lattice theory. Standard ideals play the same role for lattices as normal subgroups do for groups and ideals do for rings. In this work, we generalize the concepts of standard ideals and distributive ideals to certain generalizations of lattices, namely to trellises.

The concepts of a pseudo-ordered set (psoset for short) and a trellis (also known as a weakly-associative lattice) were first introduced independently by Fried [2] and Skala [6]. A pseudo-order defined on a nonempty set P constitutes a reflexive and an anti-symmetric relation $\trianglelefteq$ on it, making $(P, \trianglelefteq)$ a pseudo-ordered set. If a pseudo-order is transitive, then it is a partial order. The notions of upper (lower) bound, least upper bound, and greatest lower bound in psosets are defined analogously to those in posets. A trellis T is essentially a psoset in which each pair of elements possesses both a least upper bound and a greatest lower bound. A lattice is a trellis whose pseudo-order is a partial order. An equivalent definition of a trellis (see [7]) is that it is a nonempty set T, together with two binary operations $\wedge$ and $\vee$ on it, satisfying the following three conditions:

(i) Commutativity: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in T$.

(ii) Absorption: $a \wedge (b \vee a) = a \vee (b \wedge a) = a$ for all $a, b \in T$.

(iii) Part-preservation: $a \vee ((a \wedge b) \vee (a \wedge c)) = a \wedge ((a \vee b) \wedge (a \vee c)) = a$ for all $a, b, c \in T$.

The pseudo-order on $(T, \wedge, \vee)$ is defined by: for $a, b \in T$, $a \trianglelefteq b$ if and only if $a \wedge b = a$ and/or $a \vee b = b$. A trellis T is called normal [5] if $a \wedge b \trianglelefteq a \vee b$ for all $a, b \in T$.

Let T be a trellis with pseudo-order $\trianglelefteq$. A nonempty subset I of T is called an ideal of T if it satisfies the following conditions: whenever $a, b \in I$, then $a \vee b \in I$; and whenever $x \in T$ and $a \in I$ with $x \trianglelefteq a$, then $x \in I$. The ideal generated by a nonempty subset H of T is the intersection of all ideals of T containing H, and it is denoted by $(H]$. If $H = \{a\}$, then we write $(a]$ in place of $(H]$. An ideal I of T is called principal if $I = (a]$ for some $a \in T$.

The notions of a congruence relation Θ on a trellis T and the quotient trellis T/Θ are defined standardly. For an element $a \in T$, the congruence class of T containing a is denoted by $[a]\Theta$. If T/Θ has a least element $[a]\Theta$ for some $a \in T$, then $I = [a]\Theta$ is an ideal of T, called the ideal kernel of Θ. For a nonempty subset H of T, the congruence relation generated by H, denoted by $\Theta[H]$, is the smallest congruence

relation on T under which all pairs of elements of H are congruent. For an ideal I of T, the quotient trellis $T/\Theta[I]$ is denoted by T/I.

A trellis T is said to be modular [7] if for every $x, y, z \in T$ with $x \trianglelefteq z$,

$$x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{1.3}$$

An element d of T is called distributive [5] if d is $\vee$-associative, that is,

$$d \vee (x \vee y) = (d \vee x) \vee y \quad \text{for all } x, y \in T, \tag{1.4}$$

and

$$d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y) \quad \text{for all } x, y \in T. \tag{1.5}$$

Finally, T is called sectionally complemented [1] if T has the least element 0, and for every $a, b \in T$ with $b \trianglelefteq a$, there is $c \in T$ such that $b \wedge c = 0$ and $b \vee c = a$.

For terminologies and notations not mentioned here, the reader may refer to [3], [4], [6], [7].

2. Standard ideals in trellises

Definition 2.1. An ideal S of a trellis T is called a *'standard ideal'* if S is normal, and the binary relation Θ defined on T by for $a, b \in T$, $a \equiv b\ (\Theta) \Leftrightarrow \exists\, s \in S\colon\ a \vee b = (a \wedge b) \vee s$, is a congruence relation on T.

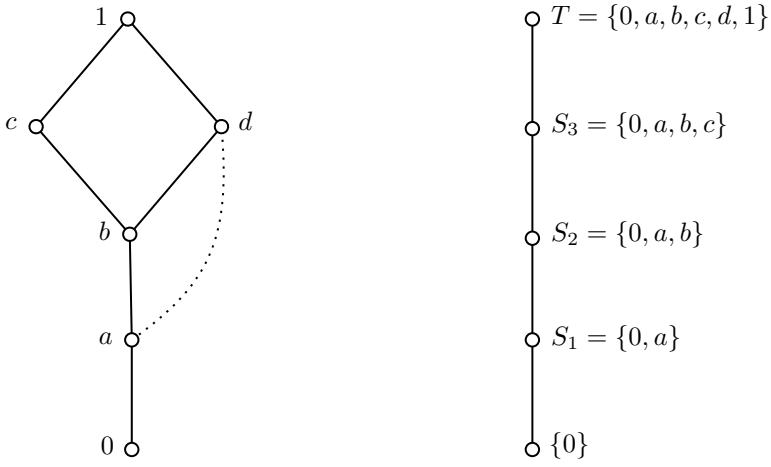
R e m a r k 2.1. If T is a lattice, the definition in Definition 2.1 coincides with the previously established definition of standard ideals in lattices.

Theorem 2.1. *Let S be a standard ideal of a trellis T. Define a relation Θ on T by for $a, b \in T$, $a \equiv b\ (\Theta) \Leftrightarrow \exists\, s \in S\colon\ a \vee b = (a \wedge b) \vee s$. Then $\Theta = \Theta[S]$, and S is the ideal kernel of $\Theta[S]$.*

P r o o f. Since S is a standard ideal of the trellis T, the relation Θ is a congruence relation on T. Let $a, b \in S$. As S is normal, $a \wedge b \trianglelefteq a \vee b$. Put $s = a \vee b$. Then $s \in S$ and $a \vee b = (a \wedge b) \vee s$. Thus $a \equiv b\ (\Theta)$. Consequently, $a \equiv b\ (\Theta)$ for all $a, b \in S$, and hence $\Theta[S] \leqslant \Theta$. Next, suppose $a \equiv b\ (\Theta)$ for some $a, b \in T$. Then $a \vee b = (a \wedge b) \vee s$ for some $s \in S$. As $s, (a \wedge b) \wedge s \in S$, we have $(a \wedge b) \wedge s \equiv s\ (\Theta[S])$. Hence, $a \wedge b \equiv (a \wedge b) \vee s = a \vee b\ (\Theta[S])$. Therefore, $a \equiv b\ (\Theta[S])$. Thus $\Theta \leqslant \Theta[S]$, and hence $\Theta = \Theta[S]$.

Let $a \equiv b\ (\Theta[S])$ with $a \in T$ and $b \in S$. Then $a \vee b = (a \wedge b) \vee s$ for some $s \in S$. Since $a \wedge b \in S$, we get $a \vee b \in S$, and hence $a \in S$. Thus, S is the ideal kernel of $\Theta[S]$. □

R e m a r k 2.2. If every ideal of a trellis T is a standard ideal, then it follows from the above theorem that every ideal of T is the kernel of a congruence relation, that is, T is a Hashimoto trellis (see [1]), and hence $I(T)$ is a distributive lattice [1, Theorem 3.6]. However, the converse does not hold. For example, consider the trellis T in Figure 1 (the dashed line indicates the removal of transitivity between its end points). Clearly, the ideal lattice $I(T)$ is a distributive lattice, but not all ideals of T are standard. In fact, the ideals S_1 and S_2 are not standard.



A trellis T The ideal lattice $I(T)$

Figure 1. A trellis T and the corresponding ideal lattice $I(T)$.

Lemma 2.1. *Let S be a standard ideal of a trellis T. Then for any ideal I of T, every element of $I \vee S$ can be expressed as $x \vee s$ for some $x \in I$ and $s \in S$.*

P r o o f. Using [1, Lemma 3.4], we have $I \vee S = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = I \cup S$ and for $n \geqslant 1$,

$$A_n = \{a \in T\colon\ a \trianglelefteq p(a_1, \dots, a_k) \text{ for some } k \geqslant 1, \text{ where } p \text{ is a trellis polynomial and } a_j \in A_{n-1} \text{ for } 1 \leqslant j \leqslant k\}. \tag{2.1}$$

We now prove, by induction on n, that every element of A_n can be written as $x \vee s$ for some $x \in I$ and $s \in S$. The assertion holds trivially for $n = 0$. Assume that the assertion holds for some $n \geqslant 0$. Let $a \in A_{n+1}$. Then $a \trianglelefteq p(a_1, \dots, a_k)$ for some $k \geqslant 1$, where p is a trellis polynomial and $a_j \in A_n$ for $1 \leqslant j \leqslant k$. By the induction hypothesis, for each j, there are elements $x_j \in I$ and $s_j \in S$ such that $a_j = x_j \vee s_j$ for each j. Note that $a_j \equiv x_j\ (\Theta[S])$ for each j, and hence $p(a_1, \dots, a_k) \equiv p(x_1, \dots, x_k)$

$(\Theta[S])$. Thus, $a = a \wedge p(a_1, \ldots, a_k) \equiv a \wedge p(x_1, \ldots, x_k)$ $(\Theta[S])$. The definition of $\Theta[S]$ implies now that $a = (a \wedge p(x_1, \ldots, x_k)) \vee s$ for some $s \in S$. Note that $a \wedge p(x_1, \ldots, x_k) \in I$. This completes the proof of the lemma by induction. □

R e m a r k 2.3. The converse of Lemma 2.1 holds in the case of lattices [4, Theorem 2], but not in the case of trellises. For example, consider the ideal S_1 of the trellis T in Figure 1. Clearly, for any ideal I of T, every element of $I \vee S_1$ can be written in the form $x \vee s$ for some $x \in I$ and $s \in S_1$. However, S_1 is not a standard ideal of T.

Theorem 2.2. *Let S be a standard ideal of a trellis T. Then S is a standard element of the ideal lattice $I(T)$.*

P r o o f. Using the fundamental characterization theorem of standard elements in lattices [4, Theorem 1], it is sufficient to show that the binary relation Θ_S on $I(T)$ defined by for $I, J \in I(T)$, $I \equiv J$ $(\Theta_S) \Leftrightarrow \exists\, S_1 \in I(T)$, $S_1 \subseteq S$: $I \vee J = (I \wedge J) \vee S_1$, is a congruence relation on $I(T)$.

Since S is a standard ideal, the binary relation Θ defined on T by for $a, b \in T$, $a \equiv b$ $(\Theta) \Leftrightarrow \exists\, s \in S$: $a \vee b = (a \wedge b) \vee s$, is a congruence relation on T. Now, define a binary relation $\overline{\Theta}$ on $I(T)$ by: for $I, J \in I(T)$, $I \equiv J$ $(\overline{\Theta})$ if and only if for each $a \in I$ there is $b \in J$ such that $a \equiv b$ (Θ), and for each $b \in J$, there is $a \in I$ such that $a \equiv b$ (Θ). Then $\overline{\Theta}$ is a congruence relation on $I(T)$, by [1, Lemma 3.5]. We complete the proof by showing that $\Theta_S = \overline{\Theta}$.

Let $I, J \in I(T)$, $J \equiv I$ $(\overline{\Theta})$ and $I \subseteq J$. Let $y \in J$ be arbitrary. Since $J \equiv I$ $(\overline{\Theta})$, there is $x \in I$ such that $x \equiv y$ (Θ), and hence $x \vee y = (x \wedge y) \vee s_{xy}$ for some $s_{xy} \in S$. Note that $x \vee y \in J$, and so $s_{xy} \in J$. This holds for every $y \in J$. Let S_1 be the ideal of T generated by all such s_{xy}. Then $S_1 \subseteq S$ and $J = I \vee S_1$. All $s_{xy} \in J$ imply $S_1 \subseteq J$, and so $I \vee S_1 \subseteq J$. For the reverse inclusion, consider any $y \in J$. Then there is $x \in I$ and $s_{xy} \in S_1$ such that $x \vee y = (x \wedge y) \vee s_{xy}$. Note that $x \wedge y \in I$, and hence $x \vee y = (x \wedge y) \vee s_{xy} \in I \vee S_1$. Thus, $y \in I \vee S_1$. Therefore, $I \equiv J$ (Θ_S).

Next, let $I, J \in I(T)$, $J \equiv I$ (Θ_S) and $I \subseteq J$. Then $J = I \vee S_1$ for some $S_1 \in I(T)$ with $S_1 \subseteq S$. Let $b \in J$. Then $b \in I \vee S$, and hence $b = a \vee s$ for some $a \in I$ and $s \in S$ by Lemma 2.1, as S is a standard ideal of T. Then clearly $a \equiv b$ (Θ). Hence, for each $b \in J$, there is $a \in I$ such that $a \equiv b$ (Θ). Also, for each $a \in I$, there is $b \in J$ such that $a \equiv b$ (Θ), as $I \subseteq J$. Thus, $J \equiv I$ $(\overline{\Theta})$.

Hence $\Theta_S = \overline{\Theta}$. This completes the proof. □

R e m a r k 2.4. The converse of Theorem 2.2 does not hold for trellises. Consider the trellis T in Figure 1. The ideals S_1 and S_2 are standard elements of $I(T)$, but they are not standard ideals of T.

Lemma 2.2. *Let S be a standard ideal of a trellis T, and I be an arbitrary ideal of T. Then $S \cap I$ is a standard ideal of the trellis I.*

P r o o f. Define a binary relation Φ on I by: for $a, b \in I$, $a \equiv b\ (\Phi)$ if and only if $a \vee b = (a \wedge b) \vee s$ for some $s \in S \cap I$. We need to prove that Φ is a congruence relation on I.

Since S is a standard ideal of T, the binary relation Θ defined on T by for $a, b \in T$, $a \equiv b\ (\Theta) \Leftrightarrow \exists\, s \in S\colon\ a \vee b = (a \wedge b) \vee s$, is a congruence relation on T.

Clearly, Φ is the restriction of Θ on I, and hence Φ is a congruence relation on I. Therefore, $S \cap I$ is a standard ideal of the trellis I. □

The following is an extension of the first isomorphism theorem for standard ideals in lattices [4, Theorem 13], to trellises.

Theorem 2.3. *Let S be a standard ideal of a trellis T, and I be an arbitrary ideal of T. Then $S \cap I$ is a standard ideal of the trellis I, and*

$$(I \vee S)/S \cong I/(I \cap S). \tag{2.2}$$

P r o o f. The isomorphism statement can be proved by using the first general isomorphism theorem of Rédei [4]. We have to prove that every congruence class, modulo $\Theta[S]$, of the trellis $I \vee S$ may be represented by an element of I.

Let $x \in I \vee S$. Since S is a standard ideal, using Lemma 2.1, we have $x = a \vee s$ for some $a \in I$ and $s \in S$. Then clearly $x \equiv a\ (\Theta[S])$, and hence $[x]\Theta[S] = [a]\Theta[S]$, where $a \in I$. This completes the proof. □

Theorem 2.4. *Let S be a standard ideal of a trellis T, and let I be an arbitrary ideal of T. Then we have the following isomorphism of two intervals in $I(T)$:*

$$[I \cap S, I] \cong [S, I \vee S]. \tag{2.3}$$

An isomorphism is given by the correspondence

$$X \overset{\Phi}{\to} X \vee S \quad (X \in [I \cap S, I]). \tag{2.4}$$

The inverse correspondence is

$$Y \overset{\Psi}{\to} Y \cap I \quad (Y \in [S, I \vee S]). \tag{2.5}$$

P r o o f. Since S is a standard ideal of T, by Theorem 2.2 we get that S is a standard element (and thus a distributive element) of the ideal lattice $I(T)$. Hence, it follows that the map Φ is a homomorphism.

To prove Φ is one-to-one, let $X_1, X_2 \in [I \cap S, I]$ be such that $S \vee X_1 = S \vee X_2$. Note that $S \cap X_1 = S \cap (X_1 \cap I) = (S \cap I) \cap X_1 = S \cap I$, and similarly, $S \cap X_2 = S \cap I$. Since S is a standard element of $I(T)$, S can have at most one relative complement in any interval containing it [4, Theorem 1], and hence it follows that $X_1 = X_2$.

Next we prove that Ψ is the inverse of Φ. Let $Y \in [S, I \vee S]$. Then since S is a distributive element of $I(T)$, we get $\Phi(\Psi(Y)) = (Y \cap I) \vee S = (Y \vee S) \cap (I \vee S) = Y \cap (I \vee S) = Y$. Also, for any $X \in [I \cap S, I]$, as S is a standard element of $I(T)$, we have $\Psi(\Phi(X)) = (X \vee S) \cap I = (X \cap I) \vee (S \cap I) = X \vee (I \cap S) = X$.

This completes the proof of the theorem. □

The following is a generalization of [4, Theorem 11], to trellises.

Theorem 2.5. *Let T be a sectionally complemented normal trellis. Then, an ideal kernel of a congruence relation on T is a standard ideal of T. Furthermore, every standard ideal is the kernel of precisely one congruence relation.*

P r o o f. Let an ideal S of T be the kernel of a congruence relation Θ on T. Then clearly $\Theta[S] \leqslant \Theta$. To show S is a standard ideal of T, define a relation Φ on T by: for $a, b \in T$, $a \equiv b\ (\Phi)$ if and only if $a \vee b = (a \wedge b) \vee s$ for some $s \in S$. We show that $\Phi = \Theta$.

Let $a, b \in T$ and $a \equiv b\ (\Theta)$. The trellis T is normal and so $a \wedge b \trianglelefteq a \vee b$. Since T is sectionally complemented, there is $s \in T$ such that $(a \wedge b) \vee s = a \vee b$ and $(a \wedge b) \wedge s = 0$. As $a \vee b \equiv a \wedge b\ (\Theta)$, we get $(a \vee b) \wedge s \equiv (a \wedge b) \wedge s\ (\Theta)$. But $(a \vee b) \wedge s = [(a \wedge b) \vee s] \wedge s = s$. Therefore $s \equiv 0\ (\Theta)$, and so $s \in [0]\Theta = S$. Hence, we have $a \vee b = (a \wedge b) \vee s$, where $s \in S$. Thus, $a \equiv b\ (\Phi)$.

Next, let $a, b \in T$ and $a \equiv b\ (\Phi)$. Then $a \vee b = (a \wedge b) \vee s$ for some $s \in S$. Clearly, $(a \wedge b) \wedge s \equiv s\ (\Theta[S])$. This gives $(a \wedge b) \wedge s \equiv s\ (\Theta)$, as $\Theta[S] \leqslant \Theta$. Hence $a \wedge b \equiv s \vee (a \wedge b) = a \vee b\ (\Theta)$. Thus $a \equiv b\ (\Theta)$.

Therefore $\Theta = \Phi$ and hence Φ is a congruence relation, which implies that S is a standard ideal of T. Hence, we get $\Phi = \Theta[S]$, and so $\Theta = \Theta[S]$. This shows that if S is the ideal kernel of a congruence relation Θ on T, then $\Theta = \Theta[S]$. Hence, every ideal kernel of a congruence relation on T is the kernel of at most one congruence relation on T. Now, since every standard ideal of T is the kernel of at least one congruence relation on T by Theorem 2.1, it follows that every standard ideal of T is the kernel of precisely one congruence relation on T. □

Below we have a generalization of [4, Lemma 8], to trellises.

Theorem 2.6. *Let S be a standard ideal of a trellis T, and I be an arbitrary ideal such that both $I \vee S$ and $I \wedge S$ are principal. Then I itself is principal.*

P r o o f. Let $I \vee S = (a]$ and $I \wedge S = (b]$. As $a \in I \vee S$, and S is a standard ideal, it follows that $a = x \vee s$ for some $x \in I$ and $s \in S$, by Lemma 2.1. We prove that $I = (x \vee b]$.

We first note the following. Let $w \in I$ with $w \trianglerighteq x \vee b$. Then

$$(a] = S \vee I \supseteq S \vee (w] \supseteq S \vee (x \vee b] \supseteq S \vee (x] \supseteq (s \vee x] = (a]. \tag{2.6}$$

Thus, $S \vee (w] = S \vee (x \vee b]$. Furthermore,

$$(b] = S \cap I \supseteq S \cap (w] \supseteq S \cap (x \vee b] \supseteq S \cap (b] = (b], \tag{2.7}$$

as $(b] = S \cap I \subseteq S$. Thus, $S \wedge (w] = S \wedge (x \vee b]$. Now, as S is a standard element of the lattice $I(T)$, it cannot have more than one relative complement in any interval containing it [4, Theorem 1]. Thus $(w] = (x \vee b]$.

We now show that $I = (x \vee b]$. As $x, b \in I$, we have $x \vee b \in I$, and thus $(x \vee b] \subseteq I$. To prove the reverse inclusion, consider any $p \in I$. Put $w = p \vee (x \vee b)$. Then $w \in I$ and $w \trianglerighteq x \vee b$. It follows from the above that $(w] = (x \vee b]$. Now as $p \trianglelefteq w$, $p \in (w] = (x \vee b]$. Since $p \in I$ is arbitrary, we have $I \subseteq (x \vee b]$. □

Corollary 2.1. *If the join and meet of two standard ideals in a trellis are principal, then each of these standard ideals must be principal.*

Theorem 2.7. *Let S_1 and S_2 be any two standard ideals of a trellis T. Then $\Theta[S_1 \cap S_2] = \Theta[S_1] \cap \Theta[S_2]$.*

P r o o f. Clearly, $\Theta[S_1 \cap S_2] \leqslant \Theta[S_1]$ and $\Theta[S_1 \cap S_2] \leqslant \Theta[S_2]$. Thus $\Theta[S_1 \cap S_2] \leqslant \Theta[S_1] \cap \Theta[S_2]$.

Let $a \equiv b\ (\Theta[S_1] \cap \Theta[S_2])$. Then $a \equiv b\ (\Theta[S_1])$ and $a \equiv b\ (\Theta[S_2])$. The first congruence implies that $a \vee b = (a \wedge b) \vee s_1$ for some $s_1 \in S_1$, as S_1 is a standard ideal. Using this with the congruence $a \vee b \equiv a \wedge b\ (\Theta[S_2])$, we have $s_1 = (a \vee b) \wedge s_1 \equiv (a \wedge b) \wedge s_1\ (\Theta[S_2])$. As S_2 is a standard ideal, there is $s_2 \in S_2$ such that $s_1 = ((a \wedge b) \wedge s_1) \vee s_2$. Note that $s_2 \trianglelefteq s_1$ and hence $s_2 \in S_1 \cap S_2$. Then clearly $((a \wedge b) \wedge s_1) \wedge s_2 \equiv s_2\ (\Theta[S_1 \cap S_2])$. Thus $(a \wedge b) \wedge s_1 \equiv ((a \wedge b) \wedge s_1) \vee s_2 = s_1\ (\Theta[S_1 \cap S_2])$. Consequently, $a \wedge b \equiv s_1 \vee (a \wedge b) = a \vee b\ (\Theta[S_1 \cap S_2])$. Thus $a \equiv b\ (\Theta[S_1 \cap S_2])$. Therefore $\Theta[S_1] \cap \Theta[S_2] \leqslant \Theta[S_1 \cap S_2]$. □

Theorem 2.8. *Let I and J be any two ideals of a trellis T. If either of these two ideals is a standard ideal, then $\Theta[I \vee J] = \Theta[I] \vee \Theta[J]$.*

P r o o f. Clearly, $\Theta[I] \leqslant \Theta[I \vee J]$ and $\Theta[J] \leqslant \Theta[I \vee J]$. Hence $\Theta[I] \vee \Theta[J] \leqslant \Theta[I \vee J]$. To prove the reverse inequality, we prove that $a \equiv b\ (\Theta[I] \vee \Theta[J])$ for all $a, b \in I \vee J$.

Let $a, b \in I \vee J$. As either I or J is a standard ideal, by Lemma 2.1, it follows that $a = i_1 \vee j_1$ and $b = i_2 \vee j_2$ for some $i_1, i_2 \in I$ and $j_1, j_2 \in J$. Clearly, $i_1 \equiv i_2\ (\Theta[I])$ and $j_1 \equiv j_2\ (\Theta[J])$. Consequently, $a = i_1 \vee j_1 \equiv i_2 \vee j_2 = b\ (\Theta[I] \vee \Theta[J])$. □

3. Distributive ideals in trellises

Definition 3.1. An ideal D of a trellis T is said to be *'distributive'* if the relation Θ on T, defined by for $x, y \in T$, $x \equiv y\ (\Theta) \Leftrightarrow \exists\, a, b \in D$: $x \vee a = y \vee b$, is a congruence relation on T.

R e m a r k 3.1. If T is a lattice, then Definition 3.1 is equivalent to the previously existing definition of distributive ideals in lattices.

Theorem 3.1. *Let D be a distributive ideal of a trellis T. Define a relation Θ on T by: for $x, y \in T$, $x \equiv y\ (\Theta)$ if and only if $x \vee a = y \vee b$ for some $a, b \in D$. Then $\Theta = \Theta[D]$. Furthermore, D is the ideal kernel of Θ.*

P r o o f. Let $x, y \in D$. Put $a = x \vee y$. Then $a \in D$ and $x \vee a = y \vee a = x \vee y$. Hence $x \equiv y\ (\Theta)$. Consequently, $\Theta[D] \leqslant \Theta$. Next, let $x \equiv y\ (\Theta)$ for some $x, y \in T$. Then $x \vee a = y \vee b$ for some $a, b \in D$. As $a, x \wedge a \in D$, we have $x \wedge a \equiv a\ (\Theta[D])$. Hence $x \equiv x \vee a = y \vee b\ (\Theta[D])$. Thus $x \wedge y \equiv y\ (\Theta[D])$. Again, as $b, y \wedge b \in D$, we have $y \wedge b \equiv b\ (\Theta[D])$. So $y \equiv y \vee b = x \vee a\ (\Theta[D])$ and hence $x \wedge y \equiv x\ (\Theta[D])$. Thus $x \equiv y\ (\Theta[D])$. This proves that $\Theta \leqslant \Theta[D]$. Therefore $\Theta = \Theta[D]$.

Let $x \equiv y\ (\Theta)$, with $y \in D$. Then $x \vee a = y \vee b$ for some $a, b \in D$. As $y, b \in D$, we have $x \vee a = y \vee b \in D$. Thus $x \in D$, since $x \trianglelefteq x \vee a$. □

Theorem 3.2. *Let D be a distributive ideal of a trellis T. Then D is a distributive element of the ideal lattice $I(T)$.*

P r o o f. Define a relation Θ_D on $I(T)$ by: for $I, J \in I(T)$, $I \equiv J\ (\Theta_D)$ if and only if $I \vee D = J \vee D$. To prove that D is a distributive element of $I(T)$, we need to prove, by [3, Theorem III.2.2], that Θ_D is a congruence relation on $I(T)$. As D is a distributive ideal of T, the relation Θ on T, defined by for $x, y \in T$, $x \equiv y\ (\Theta)$ $\Leftrightarrow \exists\, a, b \in D$: $x \vee a = y \vee b$, is a congruence relation on T. Now, define a binary relation $\overline{\Theta}$ on $I(T)$ by: for $I, J \in I(T)$, $I \equiv J\ (\overline{\Theta})$ if and only if for each $a \in I$ there is $b \in J$ such that $a \equiv b\ (\Theta)$, and for each $b \in J$ there is $a \in I$ such that $a \equiv b\ (\Theta)$.

Then $\overline{\Theta}$ is a congruence relation on $I(T)$, by [1, Lemma 3.5]. We complete the proof by showing that $\Theta_D = \overline{\Theta}$.

Let $I, J \in I(T)$, with $I \subseteq J$ and $I \equiv J\ (\overline{\Theta})$. We have by [1, Lemma 3.4], $J \vee D = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = J \cup D$ and for $n \geqslant 1$,

$$A_n = \{x \in T\colon\ x \trianglelefteq p(z_1, \ldots, z_k) \text{ for some } k \geqslant 1, \text{ where } p \text{ is a trellis polynomial and } z_j \in A_{n-1} \text{ for } 1 \leqslant j \leqslant k\}. \tag{3.1}$$

We now prove, by induction, that $A_n \subseteq I \vee D$ for each $n \geqslant 0$. For the case $n = 0$, consider any $j \in J$. As $J \equiv I\ (\overline{\Theta})$, there is $i \in I$ such that $j \equiv i\ (\Theta)$. Hence, $j \vee a = i \vee b$ for some $a, b \in D$. Since $i \vee b \in I \vee D$, we have $j \in I \vee D$. Thus $J \subseteq I \vee D$, and hence $A_0 = J \cup D \subseteq I \vee D$. Let $n \geqslant 1$ and assume that $A_{n-1} \subseteq I \vee D$. Consider $x \in A_n$. Then $x \trianglelefteq p(z_1, \ldots, z_k)$ for some $k \geqslant 1$, where p is a trellis polynomial and $z_j \in A_{n-1}$ for $1 \leqslant j \leqslant k$. As $A_{n-1} \subseteq I \vee D$, we get $z_j \in I \vee D$ for all j. Hence $p(z_1, \ldots, z_k) \in I \vee D$, which implies that $x \in I \vee D$. Thus, $A_n \subseteq I \vee D$ for all n. Consequently, $J \vee D \subseteq I \vee D$. But as $I \subseteq J$, we get $I \vee D = J \vee D$. Thus $I \equiv J\ (\Theta_D)$.

On the other hand, suppose that $I, J \in I(T)$ with $I \subseteq J$ and $I \equiv J\ (\Theta_D)$, that is, $I \vee D = J \vee D$. We have $I \vee D = \bigcup_{n=0}^{\infty} A_n$, where $A_0 = I \cup D$ and for $n \geqslant 1$,

$$A_n = \{x \in T\colon\ x \trianglelefteq p(z_1, \ldots, z_k) \text{ for some } k \geqslant 1, \text{ where } p \text{ is a trellis polynomial and } z_j \in A_{n-1} \text{ for } 1 \leqslant j \leqslant k\}. \tag{3.2}$$

Consider any $b \in J$. We shall find $a \in I$ such that $a \equiv b\ (\Theta)$. Note that $b \in J \vee D = I \vee D = \bigcup_{n=0}^{\infty} A_n$. Suppose $b \in A_0 = I \cup D$. If $b \in I$, then we may take $a = b$. Suppose $b \in D$. Fix an $i \in I$ and put $a = b \wedge i$. Then $a \in I$ and $a \vee b = b$, where $b \in D$. Hence $a \equiv b\ (\Theta)$. Let $n \geqslant 1$ and assume that for every $z \in A_{n-1}$ there is $y \in I$ such that $y \equiv z\ (\Theta)$. Suppose $b \in A_n$. Then $b \trianglelefteq p(z_1, \ldots, z_k)$ for some $k \geqslant 1$, where p is a trellis polynomial and $z_j \in A_{n-1}$ for $1 \leqslant j \leqslant k$. By the induction assumption, for each j, there is $y_j \in I$ such that $y_j \equiv z_j\ (\Theta)$. But then $p(z_1, \ldots, z_k) \equiv p(y_1, \ldots, y_k)\ (\Theta)$. Consequently, $b = b \wedge p(z_1, \ldots, z_k) \equiv b \wedge p(y_1, \ldots, y_k)\ (\Theta)$. Put $a = b \wedge p(y_1, \ldots, y_k)$. Then $a \in I$ as $p(y_1, \ldots, y_k) \in I$, and $a \equiv b\ (\Theta)$. Hence, by induction, for every $b \in J$ there is $a \in I$ such that $a \equiv b\ (\Theta)$. As $I \subseteq J$, we conclude that $I \equiv J\ (\overline{\Theta})$. □

R e m a r k 3.2. For a trellis T, a distributive element of $I(T)$ may not be a distributive ideal of T. The ideals S_1 and S_2 of the trellis T in Figure 1 are distributive elements of $I(T)$, but they are not distributive ideals of T.

Theorem 3.3. *Let S be a standard ideal of a trellis T. Then S is also a distributive ideal of T.*

P r o o f. Define a relation Θ on T by: for $x, y \in T$, $x \equiv y\ (\Theta)$ if and only if $x \vee a = y \vee b$ for some $a, b \in S$. We prove that Θ is a congruence relation on T. As S is a standard ideal of T, the following relation Φ is a congruence relation on T, where for $x, y \in T$, we define $x \equiv y\ (\Phi)$ if and only if $x \vee y = (x \wedge y) \vee s$ for some $s \in S$. We show that $\Phi = \Theta$.

Let $x \equiv y\ (\Phi)$, where $x, y \in T$. Then $x \vee y \equiv x\ (\Phi)$ and $x \vee y \equiv y\ (\Phi)$, as Φ is a congruence relation. Hence, $x \vee y = x \vee a$ and $x \vee y = y \vee b$ for some $a, b \in S$. Thus, $x \vee a = y \vee b$, and hence $x \equiv y\ (\Theta)$.

Suppose $x, y \in T$ and $x \equiv y\ (\Theta)$. Then $x \vee a = y \vee b$ for some $a, b \in S$. As $a, b \in S$, we have $x \vee a \equiv x\ (\Phi)$ and $y \vee b \equiv y\ (\Phi)$. Hence $x \equiv y\ (\Phi)$.

Thus, $\Phi = \Theta$ and hence, Θ is a congruence relation on T, which implies that S is a distributive ideal of T. □

It is known that in a modular lattice, every distributive ideal is standard (see [1]). The following generalizes this result to trellises.

Theorem 3.4. *In a normal modular trellis T, every distributive ideal is standard.*

P r o o f. Let D be a distributive ideal of T. As T is normal, D is a normal ideal. Define a relation Φ on T by: for $x, y \in T$, $x \equiv y\ (\Phi)$ if and only if $x \vee y = (x \wedge y) \vee d$ for some $d \in D$. We need to show that Φ is a congruence relation on T. As D is a distributive ideal of T, the relation Θ on T, defined by for $x, y \in T$, $x \equiv y\ (\Theta) \Leftrightarrow \exists\, a, b \in D$: $x \vee a = y \vee b$, is a congruence relation on T. We show that $\Phi = \Theta$.

Let $x, y \in T$ with $x \equiv y\ (\Phi)$. Then $x \vee y = (x \wedge y) \vee d$ for some $d \in D$. But then $(x \vee y) \vee d = (x \wedge y) \vee d$, and hence $x \vee y \equiv x \wedge y\ (\Theta)$, as $d \in D$. Since Θ is a congruence relation, we have $x \equiv y\ (\Theta)$.

Let $x, y \in T$ with $x \equiv y\ (\Theta)$. Then $x \vee y \equiv x \wedge y\ (\Theta)$, and thus $(x \vee y) \vee a = (x \wedge y) \vee b$ for some $a, b \in D$. Hence

$$\begin{aligned} x \vee y &= (x \vee y) \wedge \big((x \vee y) \vee a\big) \\ &= (x \vee y) \wedge \big((x \wedge y) \vee b\big) \\ &= \big((x \vee y) \wedge b\big) \vee (x \wedge y) \end{aligned}$$

as $x \wedge y \trianglelefteq x \vee y$ and T is modular. Put $d = (x \vee y) \wedge b$. Then $d \in D$ as $b \in D$, and $x \vee y = (x \wedge y) \vee d$. Thus $x \equiv y\ (\Phi)$.

Therefore $\Phi = \Theta$. This completes the proof. □

The following is a generalization of [3, Theorem III.3.7], to trellises.

Theorem 3.5. *Let D be a distributive ideal of a trellis T. Then the lattice of all ideals of T/D is isomorphic to the closed interval $[D, T]$ of $I(T)$.*

P r o o f. Let $\eta\colon T \to T/D$ denote the canonical onto homomorphism, defined by $\eta(x) = [x]\Theta$ for every $x \in T$, where $\Theta = \Theta[D]$. Define a map $\Psi\colon I(T/D) \to [D, T]$ as follows: Let K be any ideal of T/D. Clearly, $\eta^{-1}(K)$ is an ideal of T and $D \subseteq \eta^{-1}(K)$. Define $\Psi(K) = \eta^{-1}(K)$. Note that the map Ψ is isotone. Also, as η is onto, we have $\eta(\eta^{-1}(K)) = K$ for any $K \in I(T/D)$, and this implies that Ψ is one-to-one.

Furthermore, Ψ is also onto. Let I be an ideal of T containing D. Note that $K = \eta(I)$ is an ideal of T/D. We prove that $\Psi(K) = I$. Clearly $I \subseteq \eta^{-1}(\eta(I))$. Let $x \in \eta^{-1}(\eta(I))$. Then $[x]\Theta = [a]\Theta$ for some $a \in I$. Then, as D is a distributive ideal, $x \vee p = a \vee q$ for some $p, q \in D$. As $a \in I$ and $q \in D \subseteq I$, we have $x \vee p = a \vee q \in I$. Consequently, $x \in I$. Thus $\Psi(K) = \eta^{-1}(K) = \eta^{-1}(\eta(I)) = I$.

Finally, note that Ψ^{-1} is also isotone. □

Corollary 3.1. *Let S be a standard ideal of a trellis T. Then the lattice of all ideals of T/S is isomorphic to the closed interval $[S, T]$ of $I(T)$.*

P r o o f. Follows from the above theorem, because any standard ideal of T is also a distributive ideal of T by Theorem 3.3. □

Theorem 3.6. *Let d be a distributive element of a trellis T. Then $D = (d]$ is a distributive ideal of T.*

P r o o f. Define a relation Θ on T by: for $x, y \in T$, $x \equiv y\ (\Theta)$ if and only if $x \vee a = y \vee b$ for some $a, b \in D$. We prove that Θ is a congruence relation on T. Define a relation Θ_d on T by: for $x, y \in T$, $x \equiv y\ (\Theta_d)$ if and only if $x \vee d = y \vee d$. Since d is a distributive element of T, by [5], Theorem 2.8, Θ_d is a congruence relation on T. We prove that $\Theta = \Theta_d$.

Let $x \equiv y\ (\Theta_d)$. Then $x \vee d = y \vee d$, where $d \in D$. Hence $x \equiv y\ (\Theta)$. On the other hand, suppose that $x \equiv y\ (\Theta)$ for some $x, y \in T$. Then $x \vee a = y \vee b$ for some $a, b \in D = (d]$. As d is $\vee$-associative, d is left-transitive too [7, Theorem 11], and hence $D = (d] = \{t \in T\colon\ t \trianglelefteq d\}$. Thus, we have $a, b \trianglelefteq d$. We have $(x \vee a) \vee d = (y \vee b) \vee d$. But as d is $\vee$-associative, we get $x \vee (a \vee d) = y \vee (b \vee d)$. Thus, $x \vee d = y \vee d$, and hence, $x \equiv y\ (\Theta_d)$. □

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