SAKAGUCHI TYPE FUNCTIONS DEFINED BY BALANCING POLYNOMIALS

Gunasekar Saravanan, Sudharsanan Baskaran,
Balasubramaniam Vanithakumari, Chennai,
Serap Bulut, Kocaeli

Received December 11, 2023. Published online June 10, 2024.
Communicated by Grigore Sălăgean

Abstract. The class of Sakaguchi type functions defined by balancing polynomials has been introduced as a novel subclass of bi-univalent functions. The bounds for the Fekete-Szegö inequality and the initial coefficients $|a_2|$ and $|a_3|$ have also been estimated.

Keywords: analytic function; bi-univalent function; Sakaguchi type function; balancing polynomial

MSC 2020: 30C45, 30C50

1. Introduction and preliminaries

Let $\mathcal{H}$ be the class of analytic functions in the open unit disc $\mathfrak{U} = \{z \in \mathbb{C} \colon |z| < 1\}$ and consider the classes $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{S}$ defined by

$$\begin{aligned}\mathcal{P} &= \{p \in \mathcal{H} \colon\ p(0) = 1 \text{ and } \Re(p(z)) > 0,\ z \in \mathfrak{U}\},\\ \mathcal{A} &= \{\mathfrak{f} \in \mathcal{H} \colon\ f(0) = f'(0) - 1 = 0\},\\ \mathcal{S} &= \{\mathfrak{f} \in \mathcal{A} \colon\ \mathfrak{f} \text{ is univalent in } \mathfrak{U}\},\end{aligned}$$

respectively. It is clear that the function $\mathfrak{f} \in \mathcal{A}$ can be expressed as

$$\mathfrak{f}(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathfrak{U}. \tag{1.1}$$

For two functions $\mathfrak{f}, \mathfrak{g} \in \mathcal{H}$ we say that the function $\mathfrak{f}$ is subordinate to $\mathfrak{g}$ in $\mathfrak{U}$, and write

$$\mathfrak{f}(z) \prec \mathfrak{g}(z), \quad z \in \mathfrak{U},$$

DOI: 10.21136/MB.2024.0173-23

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} \colon \ \omega(0) = 0 \text{ and } |\omega(z)| < 1, \ z \in \mathfrak{U}\}$$

such that

$$\mathfrak{f}(z) = \mathfrak{g}(\omega(z)), \quad z \in \mathfrak{U}.$$

A subclass consisting of functions $\mathfrak{f} \in \mathcal{A}$ satisfying the analytic criterion

$$\Re\Big(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z) - \mathfrak{f}(-z)}\Big) > \alpha, \quad 0 \leqslant \alpha < 1$$

was introduced by Sakaguchi [22] and these functions were named after him as Sakaguchi type functions [16], [17], [25]. Sakaguchi type functions are starlike with respect to symmetric points. Frasin [8] generalized Sakaguchi type class which had functions of the form (1.1) given by

$$\Re\Big(\frac{(s-b)z\mathfrak{f}'(z)}{\mathfrak{f}(sz) - \mathfrak{f}(bz)}\Big) > \alpha, \quad 0 \leqslant \alpha < 1, \ s, b \in \mathbb{C} \text{ with } s \neq b, \ |s| \leqslant 1, \ |b| \leqslant 1, \ z \in \mathfrak{U}.$$

There are numerous integer number sequences in literature, including the Fibonacci, Lucas, Pell, and others. A novel integer sequence called balancing numbers was recently presented by Behera and Panda [4]. Some of the characteristics of this new number sequence have been thoroughly researched during the past 25 years. There was research done, and generalizations were undoubtedly formed. The references in [7], [9], [10], [12], [13], [18], [19], [20] provide thorough information for people who are interested in balancing numbers. The balancing polynomials are a natural generalization of the balancing numbers, and [21] provides a definition of these polynomials as well as some of their intriguing characteristics.

Definition 1.1. For $x \in \mathbb{C}$ and any integer $n \geqslant 2$, the balancing polynomials are defined by the following recurrence relations:

$$B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x) \tag{1.2}$$

with the initial values

$$B_0(x) = 0 \quad \text{and} \quad B_1(x) = 1.$$

Remark 1.1. If we set $n = 2$ and $n = 3$ in (1.2), then we obtain the polynomials

$$B_2(x) = 6x \quad \text{and} \quad B_3(x) = 36x^2 - 1, \tag{1.3}$$

respectively.

In the same way as with other number polynomials, generating functions can be used to produce balancing polynomials. One such function is as follows:

Lemma 1.1 ([11]). *The ordinary generating function of balancing polynomials is given by*

$$B(x,z)=\sum_{n=0}^{\infty}B_n(x)z^n=\frac{z}{1-6xz+z^2},\quad z\in\mathfrak{U}. \tag{1.4}$$

A function $\mathfrak{f}\in\mathcal{A}$ is called bi-univalent in $\mathfrak{U}$ if $\mathfrak{f}\in\mathcal{S}$ and its inverse function has an analytic continuation to $|w|<1$. Let $\Sigma=\{\mathfrak{f}\in\mathcal{S}\colon \mathfrak{f}\text{ is bi-univalent}\}$. For the function $\mathfrak{f}\in\mathcal{A}$ given by (1.1), the inverse function $\mathfrak{g}=\mathfrak{f}^{-1}$ is of the form

$$\mathfrak{g}(w)=\mathfrak{f}^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots \tag{1.5}$$

Note that the functions

$$\mathfrak{f}_1(z)=\frac{z}{1-z},\quad \mathfrak{f}_2(z)=\frac{1}{2}\log\frac{1+z}{1-z},\quad \mathfrak{f}_3(z)=-\log(1-z)$$

with their corresponding inverses

$$\mathfrak{f}_1^{-1}(w)=\frac{w}{1+w},\quad \mathfrak{f}_2^{-1}(w)=\frac{\mathrm{e}^{2w}-1}{\mathrm{e}^{2w}+1},\quad \mathfrak{f}_3^{-1}(w)=\frac{\mathrm{e}^{w}-1}{\mathrm{e}^{w}}$$

are elements of Σ (see [24], [26], [27]). However, the functions in $\mathcal{S}$ such as

$$\frac{z}{1-z^2}\quad\text{and}\quad z-\frac{z^2}{2}$$

and the familiar Koebe function are not a member of Σ. For a brief history and interesting examples in the class Σ, see [6] (and see also [2], [3], [23], [28]).

The class Σ of analytic bi-univalent functions was first introduced by Lewin [14], where it was proved that $|a_2|<1.51$. Brannan and Clunie (see [5]) improved Lewin's result to $|a_2|\leqslant\sqrt{2}$ and later Netanyahu in [15] proved that $\max\limits_{f\in\Sigma}|a_2|=\frac{4}{3}$.

Many scholars are currently investigating bi-univalent functions related to various polynomials. As far as we know, there is little work in the literature regarding balancing polynomials related to bi-univalent functions. Motivated primarily by the work of Aktaş and Karaman (see [1]), we present a new subclass ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$ of Sakaguchi-type bi-univalent functions subordinate to the balancing polynomial and obtain bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as Fekete-Szegö functional problem for functions in this class.

Definition 1.2. The function $\mathfrak{f} \in \Sigma$ is in the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$ if

$$\frac{(s-b)z\mathfrak{F}'_{\lambda}(z)}{\mathfrak{F}_{\lambda}(sz)-\mathfrak{F}_{\lambda}(bz)} \prec \frac{B(x,z)}{z} = \mathfrak{I}(x,z), \quad z \in \mathfrak{U} \tag{1.6}$$

and

$$\frac{(s-b)w\mathfrak{G}'_{\lambda}(w)}{\mathfrak{G}_{\lambda}(sw)-\mathfrak{G}_{\lambda}(bw)} \prec \frac{B(x,w)}{w} = \mathfrak{I}(x,w), \quad w \in \mathfrak{U}, \tag{1.7}$$

where

$$\mathfrak{F}_{\lambda}(z) = (1-\lambda)\mathfrak{f}(z) + \lambda z\mathfrak{f}'(z), \quad z \in \mathfrak{U}, \tag{1.8}$$

$$\mathfrak{G}_{\lambda}(w) = (1-\lambda)\mathfrak{g}(w) + \lambda w\mathfrak{g}'(w), \quad w \in \mathfrak{U}, \tag{1.9}$$

$\mathfrak{g} = \mathfrak{f}^{-1}$ given by (1.5) and $0 \leqslant \lambda \leqslant 1$, $s, b \in \mathbb{C}$ with $s \neq b$, $|s| \leqslant 1$, $|b| \leqslant 1$.

Remark 1.2. For $s = 1$ and $b = -1$, the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$ reduces to the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{2z\mathfrak{F}'_{\lambda}(z)}{\mathfrak{F}_{\lambda}(z)-\mathfrak{F}_{\lambda}(-z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad \frac{2w\mathfrak{G}'_{\lambda}(w)}{\mathfrak{G}_{\lambda}(w)-\mathfrak{G}_{\lambda}(-w)} \prec \mathfrak{I}(x,w).$$

(i) For $\lambda = 0$ we get the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{0}(x,z) = {}_{\mathcal{B}}\mathcal{S}_{\Sigma}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{2z\mathfrak{f}'(z)}{\mathfrak{f}(z)-\mathfrak{f}(-z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad \frac{2w\mathfrak{g}'(w)}{\mathfrak{g}(w)-\mathfrak{g}(-w)} \prec \mathfrak{I}(x,w).$$

(ii) For $\lambda = 1$ we get the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{1}(x,z) = {}_{\mathcal{B}}\mathcal{K}_{\Sigma}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{2(z\mathfrak{f}'(z))'}{\mathfrak{f}'(z)+\mathfrak{f}'(z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad \frac{2(w\mathfrak{g}'(w))'}{\mathfrak{g}'(w)+\mathfrak{g}'(-w)} \prec \mathfrak{I}(x,w).$$

Remark 1.3. For $s = 1$ and $b = 0$, the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$ reduces to the class ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}^{\lambda}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{z\mathfrak{F}'_{\lambda}(z)}{\mathfrak{F}_{\lambda}(z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad \frac{w\mathfrak{G}'_{\lambda}(w)}{\mathfrak{G}_{\lambda}(w)} \prec \mathfrak{I}(x,w).$$

(i) For $\lambda = 0$, we get the class ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}^{0}(x,z) = {}_{\mathcal{B}}\mathcal{H}_{\Sigma}(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad \frac{w\mathfrak{g}'(w)}{\mathfrak{g}(w)} \prec \mathfrak{I}(x,w).$$

(ii) For $\lambda = 1$, we get the class ${}_{\mathcal{B}}\mathcal{H}^1_\Sigma(x,z) = {}_{\mathcal{B}}\mathcal{N}_\Sigma(x,z)$, which consists of functions $\mathfrak{f} \in \Sigma$ satisfying

$$1 + \frac{z\mathfrak{f}''(z)}{\mathfrak{f}'(z)} \prec \mathfrak{I}(x,z) \quad \text{and} \quad 1 + \frac{w\mathfrak{g}''(w)}{\mathfrak{g}'(w)} \prec \mathfrak{I}(x,w).$$

The classes ${}_{\mathcal{B}}\mathcal{H}_\Sigma(x,z)$ and ${}_{\mathcal{B}}\mathcal{N}_\Sigma(x,z)$ are introduced by Aktaş and Karaman in [1].

2. Coefficients estimates and Fekete-Szegö inequality

Let the function $\mathfrak{f} \in \mathcal{A}$ be given by (1.1) and $\mathfrak{F}_\lambda$ be defined by (1.8). For

$$s, b \in \mathbb{C} \text{ with } s \neq b,\ |s| \leqslant 1,\ |b| \leqslant 1,$$

we have

$$\frac{(s-b)z\mathfrak{F}'_\lambda(z)}{\mathfrak{F}_\lambda(sz) - \mathfrak{F}_\lambda(bz)} = 1 + (1+\lambda)\delta_2 a_2 z + ((1+2\lambda)\delta_3 a_3 - (1+\lambda)^2\delta_2\gamma_2 a_2^2)z^2 + \ldots, \tag{2.1}$$

where

$$\delta_n = n - \gamma_n, \quad n \in \mathbb{N} \tag{2.2}$$

and

$$\gamma_n = \frac{s^n - b^n}{s-b} = s^{n-1} + s^{n-2}b + \ldots + sb^{n-2} + b^{n-1}, \quad n \in \mathbb{N}. \tag{2.3}$$

Throughout this paper, unless otherwise stated, we assume that

$$0 \leqslant \lambda \leqslant 1, \quad s, b \in \mathbb{C} \text{ with } s \neq b,\ |s| \leqslant 1,\ |b| \leqslant 1, \quad \gamma_n \neq n$$

and for real s, b, $\gamma_n < n$, $n \in \mathbb{N} \setminus \{1\}$.

Theorem 2.1. *Let $\mathfrak{f}$ given by (1.1) be in the class ${}_{\mathcal{B}}\mathcal{S}^\lambda_\Sigma(s,b,x,z)$, and define*

$$L := 2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2 \quad \textit{and} \quad M := (1+\lambda)\delta_2. \tag{2.4}$$

Then we have

$$|a_2| \leqslant \min\Big\{\frac{6|x|}{(1+\lambda)|\delta_2|}, \gamma\Big\},$$

where

$$\gamma = \begin{cases} \dfrac{6\sqrt{6}|x|\sqrt{|x|}}{\sqrt{|36(L-M^2)x^2 + M^2|}}, & L \neq M^2 \textit{ and } x^2 \neq \dfrac{M^2}{36(M^2-L)}, \\ \dfrac{6\sqrt{6}|x|\sqrt{|x|}}{|M|}, & L = M^2, \end{cases} \tag{2.5}$$

$$|a_3| \leqslant 6|x|\Big(\frac{1}{(1+2\lambda)|\delta_3|} + \frac{6|x|}{(1+\lambda)^2|\delta_2|^2}\Big). \tag{2.6}$$

Proof. Let $\mathfrak{f} \in {}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$. Then there exist analytic functions $l(z)$ and $m(w)$ given by

$$l(z) = l_1 z + l_2 z^2 + l_3 z^3 + \dots \tag{2.7}$$

and

$$m(w) = m_1 w + m_2 w^2 + m_3 w^3 + \dots, \tag{2.8}$$

respectively, which are analytic in $\mathfrak{U}$ with $l(0) = 0$, $m(0) = 0$ and $|l(z)| < 1$, $|m(w)| < 1$, $z, w \in \mathfrak{U}$, such that

$$\frac{(s-b)z\mathfrak{F}'_{\lambda}(z)}{\mathfrak{F}_{\lambda}(sz) - \mathfrak{F}_{\lambda}(bz)} = \mathfrak{I}(x, l(z)) \tag{2.9}$$

and

$$\frac{(s-b)w\mathfrak{G}'_{\lambda}(w)}{\mathfrak{G}_{\lambda}(sw) - \mathfrak{G}_{\lambda}(bw)} = \mathfrak{I}(x, m(w)), \tag{2.10}$$

respectively. It is to be noted that since

$$|l(z)| = |l_1 z + l_2 z^2 + l_3 z^3 + \dots| < 1, \quad z \in \mathfrak{U}$$

and

$$|m(w)| = |m_1 w + m_2 w^2 + m_3 w^3 + \dots| < 1, \quad w \in \mathfrak{U},$$

then

$$|l_i| \leqslant 1 \quad \text{and} \quad |m_i| \leqslant 1, \quad i = 1, 2, 3, \dots \tag{2.11}$$

For the functions $\mathfrak{F}_{\lambda}$ and $\mathfrak{G}_{\lambda}$ defined by (1.8) and (1.9), respectively, we have the equalities (2.1) and

$$\begin{aligned}\frac{(s-b)w\mathfrak{G}'_{\lambda}(w)}{\mathfrak{G}_{\lambda}(sw) - \mathfrak{G}_{\lambda}(bw)} = 1 &- (1+\lambda)\delta_2 a_2 w - ((1+2\lambda)\delta_3 a_3 \\ &- ((1+\lambda)^2\delta_2\gamma_2 - 2(1+2\lambda)\delta_3)a_2^2)w^2 + \dots\end{aligned} \tag{2.12}$$

On the other hand, we get

$$\mathfrak{I}(x, l(z)) = B_1(x) + B_2(x)l_1 z + (B_2(x)l_2 + B_3(x)l_1^2)z^2 + \dots \tag{2.13}$$

and

$$\mathfrak{I}(x, m(w)) = B_1(x) + B_2(x)m_1 w + (B_2(x)m_2 + B_3(x)m_1^2)w^2 + \dots, \tag{2.14}$$

respectively. From equalities (2.1), (2.9), (2.13) and (2.10), (2.12), (2.14), we obtain the following equations, respectively:

$$(1+\lambda)\delta_2 a_2 = B_2(x) l_1, \tag{2.15}$$

$$(1+2\lambda)\delta_3 a_3 - (1+\lambda)^2\delta_2\gamma_2 a_2^2 = B_2(x) l_2 + B_3(x) l_1^2, \tag{2.16}$$

$$-(1+\lambda)\delta_2 a_2 = B_2(x) m_1, \tag{2.17}$$

$$(2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2) a_2^2 - (1+2\lambda)\delta_3 a_3 = B_2(x) m_2 + B_3(x) m_1^2. \tag{2.18}$$

Adding (2.15) and (2.17), we get the equation

$$l_1 = -m_1. \tag{2.19}$$

Further, squaring and adding (2.15) and (2.17), we have

$$2(1+\lambda)^2\delta_2^2 a_2^2 = B_2^2(x)(l_1^2 + m_1^2). \tag{2.20}$$

Then the addition of (2.16) and (2.18) gives

$$2(2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2) a_2^2 = B_2(x)(l_2 + m_2) + B_3(x)(l_1^2 + m_1^2). \tag{2.21}$$

From (1.3), (2.11) and (2.20) we get

$$|a_2| \leqslant \frac{6|x|}{(1+\lambda)|\delta_2|}.$$

Also using (2.20) in equation (2.21), we obtain

$$2\Big((1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2 - \frac{B_3(x)}{B_2^2(x)}(1+\lambda)^2\delta_2^2\Big) a_2^2 = B_2(x)(l_2 + m_2), \tag{2.22}$$

and then

$$a_2^2 = \frac{B_2^3(x)(l_2+m_2)}{2((2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2)B_2^2(x) - (1+\lambda)^2\delta_2^2 B_3(x))}. \tag{2.23}$$

A small computation leads to

$$|a_2| \leqslant \frac{6\sqrt{6}|x|\sqrt{|x|}}{\sqrt{|36(L-M^2)x^2 + M^2|}}, \tag{2.24}$$

where

$$L = 2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2 \quad \text{and} \quad M = (1+\lambda)\delta_2.$$

Next, in order to obtain the bound for $|a_3|$, subtracting (2.18) from (2.16) we have

$$2(1+2\lambda)\delta_3(a_3-a_2^2)=B_2(x)(l_2-m_2)+B_3(x)(l_1^2-m_1^2). \tag{2.25}$$

Using equations (2.19) and (2.20) in (2.25), we get

$$a_3=\frac{B_2^2(x)}{2(1+\lambda)^2\delta_2^2}(l_1^2+m_1^2)+\frac{B_2(x)}{2(1+2\lambda)\delta_3}(l_2-m_2). \tag{2.26}$$

Applying (1.3) and (2.11), we have the desired bound for $|a_3|$,

$$|a_3|\leqslant 6|x|\Big(\frac{1}{(1+2\lambda)|\delta_3|}+\frac{6|x|}{(1+\lambda)^2|\delta_2|^2}\Big). \tag{2.27}$$

□

Letting $s=1$ and $b=-1$ in Theorem 2.1, we get the following result.

Corollary 2.1. *Let $\mathfrak{f}$ given by* (1.1) *be in the class ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(x,z)$. Then we have*

$$|a_2|\leqslant \min\Big\{\frac{3|x|}{1+\lambda},\gamma\Big\},$$

where

$$\gamma=\frac{3\sqrt{6}|x|\sqrt{|x|}}{\sqrt{|(1+\lambda)^2-18(1+2\lambda+2\lambda^2)x^2|}},\qquad x^2\neq\frac{(1+\lambda)^2}{18(1+2\lambda+2\lambda^2)}$$

and

$$|a_3|\leqslant 3|x|\Big(\frac{1}{1+2\lambda}+\frac{3|x|}{(1+\lambda)^2}\Big).$$

Letting $s=1$ and $b=0$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let $\mathfrak{f}$ given by* (1.1) *be in the class ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}^{\lambda}(x,z)$. Then we have*

$$|a_2|\leqslant \min\Big\{\frac{6|x|}{1+\lambda},\gamma\Big\},$$

where

$$\gamma=\begin{cases}\dfrac{6\sqrt{6}|x|\sqrt{|x|}}{\sqrt{|(1+\lambda)^2-72\lambda^2x^2|}}, & 0<\lambda\leqslant 1 \textit{ and } x^2\neq\dfrac{(1+\lambda)^2}{72\lambda^2},\\ 6\sqrt{6}|x|\sqrt{|x|}, & \lambda=0\end{cases}$$

and

$$|a_3|\leqslant 6|x|\Big(\frac{1}{2(1+2\lambda)}+\frac{6|x|}{(1+\lambda)^2}\Big).$$

Letting $s=1$, $b=0$ and $\lambda=0$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}(x,z)$. *Then we have*

$$|a_2| \leqslant \begin{cases} 6|x|, & |x| \geqslant \dfrac{1}{6}, \\ 6\sqrt{6}|x|\sqrt{|x|}, & |x| \leqslant \dfrac{1}{6} \end{cases} \quad \text{and} \quad |a_3| \leqslant 3|x|(1+12|x|).$$

Letting $s=1$, $b=0$ and $\lambda=1$ in Theorem 2.1, we get the following result.

Corollary 2.4. *Let* $\mathfrak{f}$ *given by* (1.1) *be in the class* ${}_{\mathcal{B}}\mathcal{N}_{\Sigma}(x,z)$. *Then we have*

$$|a_2| \leqslant \min\{3|x|, \gamma\},$$

where

$$\gamma = \frac{3\sqrt{6}|x|\sqrt{|x|}}{\sqrt{|1-18x^2|}}, \quad x^2 \neq \frac{1}{18} \quad \text{and} \quad |a_3| \leqslant |x|(1+9|x|).$$

Remark 2.1. It is worth to note that our results improve the results of Aktaş and Karaman (see [1]).

Theorem 2.2. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$, *then for any complex number* ϱ,

$$(2.28) \qquad |a_3 - \varrho a_2^2| \leqslant \begin{cases} \dfrac{6|x|}{(1+2\lambda)|\delta_3|}, & 0 \leqslant |\psi(\varrho)| \leqslant \dfrac{1}{(1+2\lambda)|\delta_3|}, \\ 6|\psi(\varrho)||x|, & |\psi(\varrho)| \geqslant \dfrac{1}{(1+2\lambda)|\delta_3|}, \end{cases}$$

where

$$|\psi(\varrho)| = \frac{36|1-\varrho||x|^2}{|36(L-M^2)x^2 + M^2|}$$

and L *and* M *are defined by* (2.4).

Proof. From (2.19) and (2.25) we get

$$a_3 - \varrho a_2^2 = (1-\varrho)a_2^2 + \frac{B_2(x)}{2(1+2\lambda)\delta_3}(l_2 - m_2).$$

By using (2.23) in the above equality, we obtain

$$a_3 - \varrho a_2^2 = \frac{B_2(x)}{2}\Big(\Big(\psi(\varrho) + \frac{1}{(1+2\lambda)\delta_3}\Big)l_2 + \Big(\psi(\varrho) - \frac{1}{(1+2\lambda)\delta_3}\Big)m_2\Big),$$

where

$$\psi(\varrho) = \frac{(1-\varrho)B_2^2(x)}{(2(1+2\lambda)\delta_3 - (1+\lambda)^2\delta_2\gamma_2)B_2^2(x) - (1+\lambda)^2\delta_2^2 B_3(x)}.$$

Thus, we have

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} \dfrac{6|x|}{(1+2\lambda)|\delta_3|}, & 0 \leqslant |\psi(\varrho)| \leqslant \dfrac{1}{(1+2\lambda)|\delta_3|}, \\ 6|\psi(\varrho)||x|, & |\psi(\varrho)| \geqslant \dfrac{1}{(1+2\lambda)|\delta_3|}. \end{cases}$$

□

Letting $s = 1$ and $b = -1$ in Theorem 2.2, we get the following result.

Corollary 2.5. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(x, z)$, *then for any complex number* ϱ,

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} \dfrac{3|x|}{1+2\lambda}, & 0 \leqslant |\psi(\varrho)| \leqslant \dfrac{1}{2(1+2\lambda)}, \\ 6|\psi(\varrho)||x|, & |\psi(\varrho)| \geqslant \dfrac{1}{2(1+2\lambda)}, \end{cases}$$

where

$$|\psi(\varrho)| = \frac{9|1-\varrho||x|^2}{|(1+\lambda)^2 - 18(1+2\lambda+2\lambda^2)x^2|}.$$

Letting $s = 1$ and $b = 0$ in Theorem 2.2, we get the following result.

Corollary 2.6. *If the function* f *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}^{\lambda}(x, z)$, *then for any complex number* ϱ,

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} \dfrac{3|x|}{1+2\lambda}, & 0 \leqslant |\psi(\varrho)| \leqslant \dfrac{1}{2(1+2\lambda)}, \\ 6|\psi(\varrho)||x|, & |\psi(\varrho)| \geqslant \dfrac{1}{2(1+2\lambda)}, \end{cases}$$

where

$$|\psi(\varrho)| = \frac{36|1-\varrho||x|^2}{|(1+\lambda)^2 - 72\lambda^2 x^2|}.$$

Letting $s = 1$, $b = 0$ and $\lambda = 0$ in Theorem 2.2, we get the following result.

Corollary 2.7. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{H}_{\Sigma}(x, z)$, *then for any complex number* ϱ,

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} 3|x|, & 0 \leqslant |1-\varrho| \leqslant \dfrac{1}{72|x|^2}, \\ 216|1-\varrho||x|^3, & |1-\varrho| \geqslant \dfrac{1}{72|x|^2}. \end{cases}$$

Letting $s = 1$, $b = 0$ and $\lambda = 1$ in Theorem 2.2, we get the following result.

Corollary 2.8. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{N}_{\Sigma}(x,z)$, *then for any complex number* ϱ,

$$|a_3 - \varrho a_2^2| \leqslant \begin{cases} |x|, & 0 \leqslant |1-\varrho| \leqslant \dfrac{|1-18x^2|}{54|x|^2}, \\ \dfrac{54|1-\varrho||x|^3}{|1-18x^2|}, & |1-\varrho| \geqslant \dfrac{|1-18x^2|}{54|x|^2}. \end{cases}$$

Letting $\varrho = 1$ in Theorem 2.2, we get the following consequence.

Corollary 2.9. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$, *then*

$$|a_3 - a_2^2| \leqslant \frac{6|x|}{(1+2\lambda)|\delta_3|}.$$

Letting $\varrho = 0$ in Theorem 2.2, we obtain the following result.

Corollary 2.10. *If the function* $\mathfrak{f}$ *of the form* (1.1) *belongs to* ${}_{\mathcal{B}}\mathcal{S}_{\Sigma}^{\lambda}(s,b,x,z)$, *then*

$$|a_3| \leqslant \begin{cases} \dfrac{6|x|}{(1+2\lambda)|\delta_3|}, & 0 \leqslant |\psi(0)| \leqslant \dfrac{1}{(1+2\lambda)|\delta_3|}, \\ 6|\psi(0)||x|, & |\psi(0)| \geqslant \dfrac{1}{(1+2\lambda)|\delta_3|}, \end{cases}$$

where

$$|\psi(0)| = \frac{36|x|^2}{|36(L-M^2)x^2 + M^2|}$$

and L *and* M *are defined by* (2.4).

Conclusion

This paper has successfully introduced a novel subclass of bi-univalent functions, specifically the class of Sakaguchi type functions defined by balancing polynomials. The bounds for the Fekete-Szegö inequality and the initial coefficients $|a_2|$ and $|a_3|$ have been estimated, providing valuable insights into the behavior of these functions.

Furthermore, some results have been improvised, enhancing our understanding of this subclass of bi-univalent functions. This work not only contributes to the existing body of knowledge but also opens new avenues for future research.

Future work will focus on exploring other subclasses of bi-univalent functions and estimating their coefficients. Additionally, the relationship between these subclasses and the Sakaguchi type functions will be investigated. This will further deepen our understanding of bi-univalent functions and their applications.

A c k n o w l e d g e m e n t. The authors thank to the referee for his/her valuable comments.

References

[1] *İ. Aktaş, İ. Karaman*: On some new subclasses of bi-univalent functions defined by balancing polynomials. KMU J. Eng. Natur. Sci. *5* (2023), 25–32. doi

[2] *I. Aldawish, T. Al-Hawary, B. A. Frasin*: Subclasses of bi-univalent functions defined by Frasin differential operator. Mathematics *8* (2020), Article ID 783, 11 pages. doi

[3] *A. Amourah, T. Al-Hawary, B. A. Frasin*: Application of Chebyshev polynomials to certain class of bi-Bazilevič functions of order $\alpha + i\beta$. Afr. Mat. *32* (2021), 1059–1066. zbl MR doi

[4] *A. Behera, G. K. Panda*: On the square roots of triangular numbers. Fibonacci Q. *37* (1999), 98–105. zbl MR

[5] *D. A. Brannan, J. G. Clunie* (eds.): Aspects of Contemporary Complex Analysis. Academic Press, London, 1980. zbl MR

[6] *D. A. Brannan, J. Clunie, W. E. Kirwan*: Coefficient estimates for a class of star-like functions. Can. J. Math. *22* (1970), 476–485. zbl MR doi

[7] *R. K. Davala, G. K. Panda*: On sum and ratio formulas for balancing numbers. J. Indian Math. Soc., New Ser. *82* (2015), 23–32. zbl MR

[8] *B. A. Frasin*: Coefficient inequalities for certain classes of Sakaguchi type functions. Int. J. Nonlinear Sci. *10* (2010), 206–211. zbl MR

[9] *R. Frontczak*: A note on hybrid convolutions involving balancing and Lucas-balancing numbers. Appl. Math. Sci. *12* (2018), 1201–1208. doi

[10] *R. Frontczak*: Sums of balancing and Lucas-balancing numbers with binomial coefficients. Int. J. Math. Anal. *12* (2018), 585–594. doi

[11] *R. Frontczak*: On balancing polynomials. Appl. Math. Sci. *13* (2019), 57–66. doi

[12] *R. Keskin, O. Karaatlı*: Some new properties of balancing numbers and square triangular numbers. J. Integer Seq. *15* (2012), Articl ID 12.1.4, 13 pages. zbl MR

[13] *T. Komatsu, G. K. Panda*: On several kinds of sums of balancing numbers. Ars Comb. *153* (2020), 127–147. zbl MR

[14] *M. Lewin*: On a coefficient problem for bi-univalent functions. Proc. Am. Math. Soc. *18* (1967), 63–68. zbl MR doi

[15] *E. Netanyahu*: The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Arch. Ration. Mech. Anal. *32* (1969), 100–112. zbl MR doi

[16] *S. Owa, T. Sekine, R. Yamakawa*: Notes on Sakaguchi functions. Aust. J. Math. Anal. Appl. *3* (2006), Article ID 12, 7 pages. zbl MR

[17] *S. Owa, T. Sekine, R. Yamakawa*: On Sakaguchi type functions. Appl. Math. Comput. *187* (2007), 356–361. zbl MR doi

[18] *B. K. Patel, N. Irmak, P. K. Ray*: Incomplete balancing and Lucas-balancing numbers. Math. Rep., Buchar. *20* (2018), 59–72. zbl MR

[19] *P. K. Ray*: Some congruences for balancing and Lucas-balancing numbers and their applications. Integers *14* (2014), Article ID A08, 8 pages. zbl MR

[20] *P. K. Ray*: Balancing and Lucas-balancing sums by matrix methods. Math. Rep., Buchar. *17* (2015), 225–233. zbl MR

[21] *P. K. Ray*: On the properties of k-balancing numbers. Ain Shams Engin. J. *9* (2018), 395–402. doi

[22] *K. Sakaguchi*: On a certain univalent mapping. J. Math. Soc. Japan *11* (1959), 72–75. zbl MR doi

[23] *T. G. Shaba*: Subclass of bi-univalent functions satisfying subordinate conditions defined by Frasin differential operator. Turkish J. Ineq. *4* (2020), 50–58.

[24] *H. M. Srivastava, A. K. Mishra, P. Gochhayat*: Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. *23* (2010), 1188–1192. zbl MR doi

[25] *S. P. Vijayalakshmi, S. Bulut, T. V. Sudharsan*: Vandermonde determinant for a certain Sakaguchi type function in Limaçon domain. Asian-Eur. J. Math. *15* (2022), Article ID 2250212, 9 pages. zbl MR doi

[26] *Q.-H. Xu, Y.-C. Gui, H. M. Srivastava*: Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Appl. Math. Lett. *25* (2012), 990–994. zbl MR doi

[27] *Q.-H. Xu, H.-G. Xiao, H. M. Srivastava*: A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Appl. Math. Comput. *218* (2012), 11461–11465. zbl MR doi

[28] *F. Yousef, A. Amourah, B. A. Frasin, T. Bulboacă*: An avant-garde construction for subclasses of analytic bi-univalent functions. Axioms *11* (2022), Article ID 267, 8 pages. doi

Authors' addresses: *Gunasekar Saravanan*, Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Chennai, 601103, Tamil Nadu, India, e-mail: `gsaran825@yahoo.com`; *Sudharsanan Baskaran*, *Balasubramaniam Vanithakumari*, Department of Mathematics, Agurchand Manmull Jain College, Meenambakkam, Chennai 600061, Tamil Nadu, India, e-mail: `sbas9991@gmail.com`, `Vanithagft@gmail.com`; *Serap Bulut* (corresponding author), Kocaeli University, Faculty of Aviation and Space Sciences, Arslanbey Campus, 41285 Kartepe-Kocaeli, Turkey, e-mail: `serap.bulut@kocaeli.edu.tr`.