ON k-PELL NUMBERS WHICH ARE SUM OF TWO NARAYANA'S COWS NUMBERS

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Abstract. For any positive integer $k \geqslant 2$, let $(P_n^{(k)})_{n\geqslant 2-k}$ be the k-generalized Pell sequence which starts with $0,\dots,0,1$ (k terms) with the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \ldots + P_{n-k}^{(k)} \quad \text{for } n \geqslant 2.$$

Let $(N_n)_{n\geqslant 0}$ be Narayana's sequence given by

$$N_0 = N_1 = N_2 = 1 \quad \text{and} \quad N_{n+3} = N_{n+2} + N_n.$$

The purpose of this paper is to determine all k-Pell numbers which are sums of two Narayana's numbers. More precisely, we study the Diophantine equation

$$P_p^{(k)} = N_n + N_m$$

in nonnegative integers k, p, n and m.

Keywords: Diophantine equation; Narayana's cows sequence; k-Pell number; linear form in logarithms; reduction method

MSC 2020: 11B37, 11D61, 11D72, 11R04

1. Introduction

Narayana's cows sequence $(N_n)_{n\geqslant 0}$ originated from a herd of cows and calves problem, proposed by the Indian mathematician Narayana in 1996 (see [1]). It is the

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sequence A000930 in the OEIS (see [13]) satisfying the recurrence relation

$$N_{n+3} = N_{n+2} + N_n \tag{1.1}$$

for $n \geqslant 0$ with the initial terms $N_0 = 0$ and $N_1 = N_2 = 1$. The first few terms of $(N_n)_{n\geqslant 0}$ are

$$0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, \ldots$$

Further, the Fibonacci sequence $(F_m)_{m\geqslant 0}$ is given by $F_0 = 0$, $F_1 = 1$ and

$$F_{m+2} = F_{m+1} + F_m \quad \forall\, m \geqslant 0.$$

Its first few terms are given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

Let $k \geqslant 2$ be an integer. We consider a generalization of the Pell sequence $(P_n^{(k)})_{n\geqslant 2-k}$ defined as $P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \ldots + P_{n-k}^{(k)}$ for $n \geqslant 2$ with the initial conditions $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \ldots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. This sequence is called the k-generalized Pell sequence or the k-Pell sequence. We note that $P_n^{(k)}$ is the nth k-Pell number. This sequence generalizes the usual Pell sequence, which corresponds to $k = 2$. In the recent past, the study of Narayana's cows sequence has been a source of attraction for many authors. For instance, Bravo, Das and Guzmán in [4] searched for repdigits in Narayana's cows sequence. They also found all Mersenne prime numbers and numbers with distinct blocks of digits in this sequence. Recently, Bhoi and Ray (see [3]) proved that the only Fermat number in Narayana's cows sequence is $N_5 = 3$. In this paper, we are interested in solving Diophantine equations involving Narayana and k-Pell numbers. Mainly, we prove the following theorem.

Theorem 1.1. *The k-Pell numbers, which satisfy the Diophantine equation*

$$P_p^{(k)} = N_n + N_m \tag{1.2}$$

in nonnegative integers p, n, m with $0 \leqslant n \leqslant m$ and $k \geqslant 2$, are

$$1, 2, 5, 12, 13, 29, 34, 88, 89, \text{ and } 408.$$

Moreover, we have the following representations:

$$
\begin{aligned}
P_1^{(k)} &= N_0 + N_1 = N_0 + N_2 = N_0 + N_3 = 1, \quad k \geqslant 2,\\
P_2^{(k)} &= N_0 + N_4 = N_1 + N_1 = N_1 + N_2 = N_1 + N_3 = N_2 + N_2\\
&= N_2 + N_3 = N_3 + N_3 = 2, \quad k \geqslant 2,\\
P_3^{(k)} &= N_1 + N_6 = N_2 + N_6 = N_3 + N_6 = N_4 + N_5 = 5, \quad k \geqslant 2,\\
P_4^{(k)} &= N_0 + N_9 = N_6 + N_8 = 13, \quad k \geqslant 3,\\
P_5^{(k)} &= N_7 + N_{11} = 34, \quad k \geqslant 4,\\
P_6^{(k)} &= N_1 + N_{14} = N_2 + N_{14} = N_3 + N_{14} = 89, \quad k \geqslant 5,\\
P_4^{(2)} &= N_7 + N_7 = N_5 + N_8 = 12,\\
P_5^{(2)} &= N_1 + N_{11} = N_2 + N_{11} = N_3 + N_{11} = 29,\\
P_8^{(2)} &= N_4 + N_{18} = 408,\\
P_6^{(4)} &= N_{11} + N_{13} = N_0 + N_{14} = 88.
\end{aligned}
$$

Theorem 1.1 allows us to deduce the following statement.

Corollary 1.1. *All the solutions of the Diophantine equation*

$$P_p^{(k)} = N_n \tag{1.3}$$

in nonnegative integers p, n with $k \geqslant 2$ are given by

$$
\begin{gathered}
P_1^{(k)} = N_1 = N_2 = N_3 = 1,\ k \geqslant 2, \quad P_2^{(k)} = N_4 = 2,\ k \geqslant 2,\\
P_4^{(k)} = N_9 = 13,\ k \geqslant 3, \quad \textit{and} \quad P_6^{(4)} = N_{14} = 88.
\end{gathered}
$$

The proof of Theorem 1.1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [2]. Here, we use a modified version of the result due to Dujella-Pethő (see [8]). In fact, in the next section, we recall the properties to prove Theorem 1.1 completely. As we don't have in the literature the study of the Diophantine equation

$$F_p = N_n + N_m, \tag{1.4}$$

we do it in Section 3. The last section is devoted to the proof of Theorem 1.1.

2. Preliminary results

2.1. Linear form in logarithms. We use Baker's theory of linear forms in logarithms of algebraic numbers for the proof of our result. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. We denote the logarithmic height of α by

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log(\max\{|\alpha^{(i)}|, 1\})\right).$$

This height has the following properties. For any algebraic numbers α and β, we have

$$h(\alpha\beta) \leqslant h(\alpha) + h(\beta),$$
$$h(\alpha \pm \beta) \leqslant \log 2 + h(\alpha) + h(\beta).$$

Moreover, for any integer n,

$$h(\alpha^n) \leqslant |n| h(\alpha).$$

Now, let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{K}$ and $d_1, \ldots, d_l$ be nonzero integers. Let $D \geqslant \max\{|d_1|, \ldots, |d_l|\}$, and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be real numbers such that

$$A_j \geqslant \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\} \quad \text{for } j = 1, \ldots, l.$$

The first tool we need is the following result due to Bugeaud, Mignotte and Siksek (see [7], Theorem 9.4).

Theorem 2.1. *If* $\Gamma \neq 0$, *then*

$$\log |\Gamma| \geqslant -1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log D) A_1 \ldots A_l.$$

2.2. The reduction method. Using Theorem 2.1, we get an upper bound on the variable n which is too large, thus we need to reduce that bound. To do this, we have to recall a variant of the reduction method of Baker and Davenport (see [2]) due to Dujella and Pethő (see [8]).

Lemma 2.1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon = \|\mu q\| - M \cdot \|\tau q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$m \leqslant M \quad \text{and} \quad k \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following result of Guzmán and Luca (see [10]) will also be very useful.

Lemma 2.2. *If $l \geqslant 1$, $T > (4l^2)^l$ and $T > x/(\log x)^l$, then*

$$x < 2^l T(\log T)^l.$$

2.3. Properties of Narayana sequence. In this subsection, we recall some facts and properties of Narayana's sequence which will be used later. The characteristic equation of (1.1) is

$$x^3 - x^2 - 1 = 0,$$

which has roots α, β, $\gamma = \overline{\beta}$, where

$$\alpha = \frac{\sqrt[3]{116 + 12\sqrt{93}}}{6} + \frac{2}{3\sqrt[3]{116 + 12\sqrt{93}}} + \frac{1}{3}$$

and

$$\beta = -\frac{\sqrt[3]{116 + 12\sqrt{93}}}{12} - \frac{1}{3\sqrt[3]{116 + 12\sqrt{93}}} + \frac{1}{3} + \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{\sqrt[3]{116 + 12\sqrt{93}}}{6} - \frac{2}{3\sqrt[3]{116 + 12\sqrt{93}}}\right).$$

Binet's formula for Narayana's cows sequence is given by

$$N_n = C_\alpha \alpha^n + C_\beta \beta^n + C_\gamma \gamma^n \quad \text{for } n \geqslant 0, \tag{2.1}$$

where

$$C_\alpha = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}, \quad C_\beta = \frac{\beta}{(\beta - \alpha)(\beta - \gamma)}, \quad C_\gamma = \frac{\gamma}{(\gamma - \alpha)(\gamma - \beta)}.$$

Also formula (2.1) can be written in the form

$$N_n = c_\alpha \alpha^{n+2} + c_\beta \beta^{n+2} + c_\gamma \gamma^{n+2} \quad \forall\, n \geqslant 0 \tag{2.2}$$

with

$$c_\alpha = \frac{1}{\alpha^3+2}, \quad c_\beta = \frac{1}{\beta^3+2}, \quad c_\gamma = \frac{1}{\gamma^3+2}.$$

Note that the coefficient c_α has the minimal polynomial $31x^3 - 31x^2 + 10x - 1$ over $\mathbb{Z}$ and all its roots lie strictly inside the unit circle. Numerically, we have

$$\begin{aligned} 1.46 < \alpha < 1.47, \quad 0.82 < |\beta| = |\gamma| < 0.83, \\ 0.19 < c_\alpha < 0.20, \quad 0.40 < |c_\beta| = |c_\gamma| < 0.41. \end{aligned} \tag{2.3}$$

Moreover, the nth Narayana number satisfies the inequalities

$$\alpha^{n-2} \leqslant N_n \leqslant \alpha^{n-1} \tag{2.4}$$

for $n \geqslant 1$ (see [4]).

2.4. Properties of Fibonacci sequence. Recall that if k is any nonnegative integer, then

$$F_k = \frac{\varphi^k - \lambda^k}{\varphi - \lambda} = \frac{\varphi^k - \lambda^k}{\sqrt{5}} \tag{2.5}$$

where $\varphi = (1+\sqrt{5})/2$ and $\lambda = (1-\sqrt{5})/2$ are the roots of $x^2 - x - 1$. This is known as the Binet's formula for the Fibonacci sequence. It is well-known that the inequalities

$$\varphi^{k-2} \leqslant F_k \leqslant \varphi^{k-1} \tag{2.6}$$

hold for $k \geqslant 1$.

2.5. Properties of k-generalized Pell sequence. In this subsection, we recall some facts and properties of the k-Pell sequence. The characteristic polynomial of this sequence is

$$\varphi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \ldots - x - 1.$$

In [14], it is proved that $\varphi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\varrho(k)$ outside the unit circle. It is real and positive, and satisfies $\varrho(k) > 1$. The other roots are strictly inside the unit circle. Furthermore, the authors from [6] proved that

$$\varphi^2(1 - \varphi^{-k}) < \varrho(k) < \varphi^2 \quad \text{for } k \geqslant 2, \tag{2.7}$$

where $\varphi = (1+\sqrt{5})/2$. To simplify the notation, in general, we omit the dependence of $\varrho(k)$ on k and use ϱ. For $s \geqslant 2$, let

$$f_s(x) := \frac{x-1}{(s+1)x^2 - 3sx + s - 1} = \frac{x-1}{s(x^2-3x+1)+x^2-1}. \tag{2.8}$$

In [5], it is also proved that the inequalities

$$0.276 < f_k(\varrho) < 0.5 \quad \text{and} \quad |f_k(\varrho^{(i)})| < 1 \quad \text{with } 2 \leqslant i \leqslant k \tag{2.9}$$

hold, where $\varrho := \varrho^{(1)}, \ldots, \varrho^{(k)}$ (the conjugates of ϱ) are all the zeros of $\varphi_k(x)$. It was proved in [9] (see also [12]) that $f_k(\varrho)$ is not an algebraic integer. In addition, the authors of [5] proved that the logarithmic height of $f_k(\varrho)$ satisfies

$$h(f_k(\varrho)) < 4k \log \varphi + k \log(k+1) \quad \text{for } k \geqslant 2. \tag{2.10}$$

With the above notations, the authors of [6] showed that

$$P_n^{(k)} = \sum_{i=1}^{k} f_k(\varrho^{(i)}) {\varrho^{(i)}}^n \quad \text{and} \quad |P_n^{(k)} - f_k(\varrho)\varrho^n| < \frac{1}{2}, \tag{2.11}$$

which is valid for $n \geqslant 1$ and $k \geqslant 2$. So, for $n \geqslant 1$ and $k \geqslant 2$, we have

$$P_n^{(k)} = f_k(\varrho)\varrho^n + e_k(n), \quad \text{where } |e_k(n)| \leqslant \frac{1}{2}. \tag{2.12}$$

Furthermore, it was shown in [6] that

$$\varrho^{n-2} \leqslant P_n^{(k)} \leqslant \varrho^{n-1} \quad \text{for } n \geqslant 1 \text{ and } k \geqslant 2. \tag{2.13}$$

Finally, we conclude this subsection by giving the following estimate from [5]. If $k \geqslant 30$ and $n > 1$ are integers satisfying $n < \varphi^{k/2}$, then

$$f_k(\varrho)\varrho^n = \frac{\varphi^{2n}}{\varphi+2}(1+\zeta), \quad \text{where } |\zeta| < \frac{4}{\varphi^{k/2}}. \tag{2.14}$$

3. Fibonacci numbers which are sum of two Narayana numbers

In order to effectively survey the Diophantine equation (1.2), the study of Fibonacci numbers which are the sum of two Narayana numbers is inevitable. Thus, this section deals with the issue. Our result in this case is the following.

Theorem 3.1. *The Fibonacci numbers which satisfy the Diophantine equation*

$$F_p = N_n + N_m \tag{3.1}$$

in nonnegative integers p, n and m with $0 \leqslant n \leqslant m$ are 0, 1, 2, 3, 5, 8, 13, 21, 34 *and* 89. *Moreover, we have the following representations:*

$$\begin{aligned}
F_0 &= N_0 + N_0 = 0,\\
F_1 &= F_2 = N_0 + N_1 = N_0 + N_2 = N_0 + N_3 = 1,\\
F_3 &= N_0 + N_4 = N_1 + N_1 = N_1 + N_2 = N_1 + N_3\\
&= N_2 + N_2 = N_2 + N_3 = N_3 + N_3 = 2,\\
F_4 &= N_0 + N_5 = N_1 + N_4 = N_2 + N_4 = N_3 + N_4 = 3,\\
F_5 &= N_1 + N_6 = N_2 + N_6 = N_3 + N_6 = N_4 + N_5 = 5,\\
F_6 &= N_4 + N_7 = N_6 + N_6 = 8,\\
F_7 &= N_0 + N_9 = N_6 + N_8 = 13,\\
F_8 &= N_4 + N_{10} = 21,\\
F_9 &= N_7 + N_{11} = 34,\\
F_{11} &= N_1 + N_{14} = N_2 + N_{14} = N_3 + N_{14} = 89.
\end{aligned}$$

The above result implies the following corollary.

Corollary 3.1. *The solutions of the Diophantine equation*

$$F_p = N_n \tag{3.2}$$

in nonnegative integers p, n are given by

$$F_0 = N_0 = 0, \quad F_1 = F_2 = N_1 = N_2 = N_3 = 1,$$

$$F_3 = N_4 = 2, \quad F_4 = N_5 = 3 \quad \textit{and} \quad F_7 = N_9 = 13.$$

Note that if $p = 0$ in the equation (3.1), then $n = m = 0$. So we can now assume that $p \geqslant 1$. First we have the following result.

Lemma 3.1. *If the integers p, n and m with $n \leqslant m$ satisfy the Diophantine equation* (3.1), *then the inequality*

$$1.24p - 3.48 < m < 1.28p + 0.72 \tag{3.3}$$

holds.

Proof. Using the inequalities (2.4), (2.6) and equation (3.1), we obtain

$$\alpha^{m-2} \leqslant N_m \leqslant N_n + N_m = F_p \leqslant \varphi^{p-1}$$

and

$$\varphi^{p-2} \leqslant F_p = N_n + N_m \leqslant 2N_m \leqslant 2\alpha^{m-1} \leqslant \alpha^{m+1},$$

where we use the fact that $2 < \alpha^2$. Hence, we get

$$(p-2)\frac{\log \varphi}{\log \alpha} - 1 \leqslant m \leqslant (p-1)\frac{\log \varphi}{\log \alpha} + 2. \tag{3.4}$$

Thus, using the fact that $1.24 < \log \varphi / \log \alpha < 1.28$, we deduce that

$$1.24p - 3.48 < m < 1.28p + 0.72. \tag{3.5}$$

□

If $m \leqslant 250$, then $p \leqslant 204$ by Lemma 3.1. A quick computation with Maple reveals that the solutions of the Diophantine equation (3.1) in the range $m \leqslant 250$ are those listed in the statement of Theorem 3.1. We prove that these are all of them. From now on, we assume that $m > 250$. By Lemma 3.1, we obtain $p > 194$ and also $m + 2 \geqslant p$. We rewrite the equation (3.1) as

$$c_\alpha \alpha^{m+2} - \frac{\varphi^p}{\sqrt{5}} = -c_\alpha \alpha^{n+2} - c_\beta \beta^{n+2} - c_\gamma \gamma^{n+2} - c_\beta \beta^{m+2} - c_\gamma \gamma^{m+2} - \frac{\lambda^p}{\sqrt{5}}$$

using the formulas (2.2) and (2.5). Thus, we obtain

$$\begin{aligned}\left|c_\alpha \alpha^{m+2} - \frac{\varphi^p}{\sqrt{5}}\right| &= \left|c_\alpha \alpha^{n+2} + c_\beta \beta^{n+2} + c_\gamma \gamma^{n+2} + c_\beta \beta^{m+2} + c_\gamma \gamma^{m+2} + \frac{\lambda^p}{\sqrt{5}}\right| \\ &\leqslant |c_\alpha||\alpha|^{n+2} + 2|c_\beta||\beta|^{n+2} + 2|c_\beta||\beta|^{m+2} + \frac{|\lambda|^p}{\sqrt{5}} \\ &< \alpha^{n+2} + 3 < \alpha^{n+2} + \alpha^3 < \alpha^{n+8}.\end{aligned}$$

Multiplying through by $\sqrt{5}\varphi^{-p}$ and using the facts that $\alpha^{m-2} \leqslant \varphi^{p-1}$ and $\sqrt{5}/\varphi < \alpha$, we get

$$|(c_\alpha\sqrt{5})\alpha^{m+2}\varphi^{-p} - 1| < \frac{\sqrt{5}\alpha^{n+8}}{\varphi^p} < \frac{1}{\alpha^{m-n-11}}. \tag{3.6}$$

Let Γ_1 be the expression inside the absolute value in the left-hand side of (3.6). Observe that $\Gamma_1 \neq 0$. To see this, we consider the $\mathbb{Q}$-automorphism of the Galois extension $\mathbb{Q}(\varphi,\alpha,\beta)$ over $\mathbb{Q}$ given by $\sigma(\alpha) = \beta$ and $\sigma(\beta) = \alpha$. Assume that $\Gamma_1 = 0$, so we get

$$\varphi^p = (c_\alpha\sqrt{5})\alpha^{m+2}. \tag{3.7}$$

Conjugating the above relation using the $\mathbb{Q}$-automorphism of Galois σ and taking the absolute value we obtain

$$1 < \varphi^p = \sqrt{5}|c_\beta||\beta|^{m+2} < 0.77,$$

which is a contradiction. Hence, $\Gamma_1 \neq 0$ and then Theorem 2.1 can be applied to it. To do this, we consider

$$\eta_1 = \sqrt{5}c_\alpha, \quad \eta_2 = \alpha, \quad \eta_3 = \varphi, \quad d_1 = 1, \quad d_2 = m+2, \quad d_3 = -p.$$

The algebraic numbers η_1, η_2 and η_3 are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha,\varphi)$ and $d_\mathbb{K} = 6$. We have

$$h(\eta_2) = \frac{\log\alpha}{3} \quad \text{and} \quad h(\eta_3) = \frac{\log\varphi}{2}.$$

Thus, we can take

$$\max\{6h(\eta_2), |\log\eta_2|, 0.16\} < 0.77 = A_2,$$

and

$$\max\{6h(\eta_3), |\log\eta_3|, 0.16\} < 1.45 = A_3.$$

Using the properties of the logarithmic height, we obtain

$$h(\eta_1) \leqslant h(\sqrt{5}) + h(c_\alpha) = \log\sqrt{5} + \frac{\log 31}{3} < 1.95.$$

So we can take

$$\max\{6h(\eta_1), |\log\eta_1|, 0.16\} < 11.7 = A_1.$$

Finally, from Lemma 3.1 we can choose $D := m+2 = \max\{1, m+2, p\}$. Thus Theorem 2.1 tells us that

$$\begin{aligned}\log|\Gamma_1| &\geqslant -1.4\cdot 30^6\cdot 3^{4.5}\cdot 6^2(1+\log 6)(1+\log(m+2))\cdot 11.7\cdot 0.77\cdot 1.45\\ &> -1.88\cdot 10^{14}\cdot(1+\log(m+2)).\end{aligned}$$

By the fact $1 + \log(m+2) < 1.7\log(m+2)$, which holds for all $m \geqslant 3$, we obtain

$$\log|\Gamma_1| > -3.2 \cdot 10^{14} \log(m+2). \tag{3.8}$$

Combining this with (3.6), we get

$$(m-n)\log\alpha < 3.21 \cdot 10^{14} \log(m+2). \tag{3.9}$$

We rewrite once again the equation (3.1) using formulas (2.2) and (2.5) to get

$$\begin{aligned}\left|c_\alpha(\alpha^{m-n}+1)\alpha^{n+2} - \frac{\varphi^p}{\sqrt{5}}\right| &= \left|c_\beta\beta^{n+2} + c_\gamma\gamma^{n+2} + c_\beta\beta^{m+2} + c_\gamma\gamma^{m+2} + \frac{\lambda^p}{\sqrt{5}}\right| \\ &\leqslant 2|c_\beta||\beta|^{n+2} + 2|c_\beta||\beta|^{m+2} + \frac{|\lambda|^p}{\sqrt{5}} < 3 < \alpha^3.\end{aligned}$$

Multiplying through by $\sqrt{5}\varphi^{-p}$, we get

$$|(\sqrt{5}c_\alpha(\alpha^{m-n}+1))\alpha^{n+2}\varphi^{-p} - 1| < \frac{\sqrt{5}\alpha^3}{\varphi^p} < \frac{1}{\alpha^{m-6}}, \tag{3.10}$$

where we use $\alpha^{m-2} \leqslant \varphi^{p-1}$ and $\sqrt{5}/\varphi < \alpha$. Let Γ_2 be the expression inside the absolute value in the left-hand side of (3.10). Note that with an argument similar to the above one, it can be proved that $\Gamma_2 \neq 0$. So, we can apply Theorem 2.1 to it. We consider

$$\eta_1 = \sqrt{5}c_\alpha(\alpha^{m-n}+1), \quad \eta_2 = \alpha, \quad \eta_3 = \varphi, \quad d_1 = 1, \quad d_2 = n+2, \quad d_3 = -p.$$

Thus, we can choose $D = m+2$ because $n \leqslant m$. The heights of η_2 and η_3 have already been calculated. From the properties of the heights, we get

$$\begin{aligned}h(\eta_1) &\leqslant h(\sqrt{5}c_\alpha) + h(\alpha^{m-n}+1) \leqslant 1.95 + (m-n)\frac{\log\alpha}{3} + \log 2 \\ &< 1.08 \cdot 10^{14} \log(m+2)\end{aligned}$$

where we have used the inequality (3.9). We choose

$$\max\{6h(\eta_1), |\log\eta_1|, 0.16\} < 6.48 \cdot 10^{14} \log(m+2) = A_1,$$

and A_2, A_3 as above. Therefore, by Theorem 2.1, we obtain

$$\begin{aligned}\log|\Gamma_2| \geqslant & -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1+\log 6)(1+\log(m+2)) \\ & \times 6.48 \cdot 10^{14} \log(m+2) \cdot 0.77 \cdot 1.45\end{aligned}$$

and then

$$\log|\Gamma_2| > -1.79 \cdot 10^{28} \log^2(m+2) \tag{3.11}$$

follows where we used the inequality $1 + \log(m + 2) < 1.7\log(m + 2)$ for $m \geqslant 3$. Combining (3.10) and (3.11), we get

$$m + 2 < \frac{1.79 \cdot 10^{28} \log^2(m + 2)}{\log \alpha} + 8 < 4.73 \cdot 10^{28} \log^2(m + 2).$$

Therefore, we obtain

$$m + 2 < 2.64 \cdot 10^{32}. \tag{3.12}$$

So, to reduce the above bound on m, we first set

$$\Lambda_1 := p \log \varphi - (m + 2) \log \alpha + \log \frac{1}{\sqrt{5} c_\alpha}.$$

Note that $\mathrm{e}^{-\Lambda_1} - 1 = \Gamma_1 \neq 0$. Thus, $\Lambda_1 \neq 0$. If $\Lambda_1 < 0$, then

$$0 < |\Lambda_1| < \mathrm{e}^{|\Lambda_1|} - 1 = |\Gamma_1| < \frac{1}{\alpha^{m-n-11}}$$

according to the inequality (3.6). If $\Lambda_1 > 0$ we have $1 - \mathrm{e}^{-\Lambda_1} = |\mathrm{e}^{-\Lambda_1} - 1| < 1/2$. Hence, $\mathrm{e}^{\Lambda_1} < 2$. Thus, we get

$$0 < \Lambda_1 < \mathrm{e}^{\Lambda_1} - 1 = \mathrm{e}^{\Lambda_1}|\Gamma_1| < \frac{2}{\alpha^{m-n-11}}.$$

So, in both cases we have

$$0 < |\Lambda_1| < \frac{2}{\alpha^{m-n-11}}.$$

Dividing the above inequality by $\log \alpha$, we get

$$|p\tau - (m + 2) + \mu| < \frac{355}{\alpha^{m-n}}, \tag{3.13}$$

where

$$\tau := \frac{\log \varphi}{\log \alpha} \quad \text{and} \quad \mu := \frac{\log(1/\sqrt{5} c_\alpha)}{\log \alpha}.$$

Now, we apply Lemma 2.1. Since $p < m+2$, from (3.12) we can take $M = 2.64 \cdot 10^{32}$, $A = 355$ and $B = \alpha$. A quick computation with Maple reveals that the convergent

$$\frac{p_{70}}{q_{70}} = \frac{402102501968503714214750568613693 9}{319405503724697815795225792656063 6}$$

of τ is the first such that $q_{70} > 6M$ and $\varepsilon > 0$. Therefore, we obtain

$$m - n < 223.$$

Next, we put

$$\Lambda_2 = p\log\varphi - (m+2)\log\alpha + \log\frac{1}{\sqrt{5}c_\alpha(\alpha^{m-n}+1)}$$

with $0 \leqslant m-n \leqslant 222$. From the inequality (3.10), we can see that

$$0 < |\Lambda_2| < \frac{2}{\alpha^{m-6}}.$$

Put $a = m-n$. Dividing the above inequality by $\log\alpha$, we obtain

$$|p\tau - (m+2) + \mu_a| < \frac{53}{\alpha^m}, \tag{3.14}$$

where

$$\tau = \frac{\log\varphi}{\log\alpha} \quad \text{and} \quad \mu_a = \frac{\log(1/\sqrt{5}c_\alpha(\alpha^a+1))}{\log\alpha} \quad \text{with } 0 \leqslant a \leqslant 222.$$

Now we apply Lemma 2.1. Here we can take $M = 2.64\cdot 10^{32}$, $A = 53$ and $B = \alpha$. A quick computation with Maple proves that the convergent

$$\frac{p_{71}}{q_{71}} = \frac{37417183036250693833016580755802629}{29721909555760487844132538948692737}$$

of τ is the first satisfying $q_{71} > 6M$ and $\varepsilon > 0$ with $0 \leqslant a \leqslant 222$. Moreover, we get $m \leqslant 236$ which is a contradiction. Hence, Theorem 3.1 is proved.

4. Proof of Theorem 1.1

The purpose of this section is to give all details about the proof of Theorem 1.1. For this, many cases will be considered according to the values of p.

4.1. The case $1 \leqslant p \leqslant k+1$. It is known from [11] that for $1 \leqslant p \leqslant k+1$, we have

$$P_p^{(k)} = F_{2p-1}.$$

Using Theorem 3.1, we deduce that the solutions of equation (1.2) for $1 \leqslant p \leqslant k+1$ are

$$\begin{aligned}
P_1^{(k)} &= N_0 + N_1 = N_0 + N_2 = N_0 + N_3 = 1, \quad k \geqslant 2,\\
P_2^{(k)} &= N_0 + N_4 = N_1 + N_1 = N_1 + N_2 = N_1 + N_3 = N_2 + N_2\\
&= N_2 + N_3 = N_3 + N_3 = 2, \quad k \geqslant 2,\\
P_3^{(k)} &= N_1 + N_6 = N_2 + N_6 = N_3 + N_6 = N_4 + N_5 = 5, \quad k \geqslant 2,\\
P_4^{(k)} &= N_0 + N_9 = N_6 + N_8 = 13, \quad k \geqslant 3,\\
P_5^{(k)} &= N_7 + N_{11} = 34, \quad k \geqslant 4,\\
P_6^{(k)} &= N_1 + N_{14} = N_2 + N_{14} = N_3 + N_{14} = 89, \quad k \geqslant 5.
\end{aligned}$$

4.2. The case $p \geqslant k+2$. We start this subsection by assuming that $p \geqslant k+2$. We have the following result which gives us the bounds of m in terms of p.

Lemma 4.1. *If the integers p, n and m with $n \leqslant m$ satisfy the Diophantine equation* (1.2), *then we have the inequalities*

$$1.24p - 1.48 < m + 2 < 2.55p + 1.45. \tag{4.1}$$

P r o o f. Combining inequalities (2.4) and (2.13) with the equation (1.2), we have

$$\alpha^{m-2} \leqslant N_m \leqslant N_n + N_m = P_p^{(k)} \leqslant \varrho^{p-1}$$

and

$$\varrho^{p-2} \leqslant P_p^{(k)} = N_n + N_m \leqslant 2N_m \leqslant 2\alpha^{m-1} < \alpha^{m+1},$$

where we use $2 < \alpha^2$. Hence, we get

$$(p-2)\frac{\log \varrho}{\log \alpha} - 1 \leqslant m \leqslant (p-1)\frac{\log \varrho}{\log \alpha} + 2.$$

Since $1.46 < \alpha < 1.47$ and $\varphi^2(1-\varphi^{-2}) < \varrho < \varphi^2$ for $k \geqslant 2$, we deduce that $1.24p - 1.48 < m + 2 < 2.55p + 1.45$. This finishes the proof. $\square$

By Lemma 4.1, we deduce that $m \geqslant 2$. Next, we get the following result which gives an upper bound of m and p in terms of k.

Lemma 4.2. *If the integers p, n and m satisfy the Diophantine equation* (1.2), *then we have the estimate*

$$p < 1.59 \cdot 10^{32} k^9 \log^5 k.$$

P r o o f. Using Binet's formulas (2.2) and (2.12), we rewrite the equation (1.2) as

$$c_\alpha \alpha^{m+2} - f_k(\varrho)\varrho^p = -c_\alpha \alpha^{n+2} - c_\beta \beta^{n+2} - c_\gamma \gamma^{n+2} - c_\beta \beta^{m+2} - c_\gamma \gamma^{m+2} + e_k(p).$$

Thus, we obtain

$$|c_\alpha \alpha^{m+2} - f_k(\varrho)\varrho^p| = |c_\alpha \alpha^{n+2} + c_\beta \beta^{n+2} + c_\gamma \gamma^{n+2} + c_\beta \beta^{m+2} + c_\gamma \gamma^{m+2}| + |e_k(p)| < \alpha^{n+8}. \tag{4.2}$$

Dividing the above inequality by $f_k(\varrho)\varrho^p$ and using the facts that $\alpha^{m-2} \leqslant \varrho^{p-1}$ and $1/(f_k(\varrho)\varrho) < \alpha^3$, we get

$$|\Gamma_3| < \frac{\alpha^{n+8}}{f_k(\varrho)\varrho^p} < \frac{1}{\alpha^{m-n-13}}, \tag{4.3}$$

where

$$\Gamma_3 := \frac{c_\alpha}{f_k(\varrho)}\alpha^{m+2}\varrho^{-p} - 1.$$

Note that $\Gamma_3 \neq 0$. If $\Gamma_3 = 0$, then $f_k(\varrho) = c_\alpha \alpha^{m+2}\varrho^{-p}$ and so $f_k(\varrho)$ is an algebraic integer. This is a contradiction to the fact that $f_k(\varrho)$ is not an algebraic number. Thus $\Gamma_3 \neq 0$ and we can apply Theorem 2.1. Let us consider

$$\eta_1 = \frac{c_\alpha}{f_k(\varrho)}, \quad \eta_2 = \alpha, \quad \eta_3 = \varrho, \quad d_1 = 1, \quad d_2 = m+2, \quad d_3 = -p.$$

Since η_1, η_2, η_3 are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha, \varrho)$ and $d_{\mathbb{K}} \leqslant 3k$ we have

$$h(\eta_2) = \frac{\log\alpha}{3} \quad \text{and} \quad h(\eta_3) = \frac{\log\varrho}{k} < \frac{2\log\varphi}{k}.$$

Moreover,

$$\max\{3kh(\eta_2), |\log\eta_2|, 0.16\} < 0.39k = A_2,$$

and

$$\max\{3kh(\eta_3), |\log\eta_3|, 0.16\} < 2.89 = A_3.$$

Using the properties of the logarithmic height, we obtain

$$h(\eta_1) \leqslant h(f_k(\varrho)) + h(c_\alpha) < 4k\log\varphi + k\log(k+1) + \frac{\log 31}{3} < 5.3k\log k$$

for $k \geqslant 2$. So we can take

$$\max\{3kh(\eta_1), |\log\eta_1|, 0.16\} < 15.9k^2\log k = A_1.$$

Lemma 4.1 gives that $D = m+2 = \max\{1, m+2, p\}$. We have the inequality

$$\log|\Gamma_3| \geqslant -2.31\cdot 10^{13}k^5\log k\cdot(1+\log(3k))(1+\log(m+2))$$

by Theorem 2.1. By the facts $1+\log(m+2) < 1.8\log(m+2)$ and $1+\log(3k) < 4.1\log k$, which hold for all $m \geqslant 2$ and $k \geqslant 2$, we obtain

$$\log|\Gamma_3| > -1.71\cdot 10^{14}k^5\log^2 k\cdot\log(m+2). \tag{4.4}$$

Combining this with (4.3), we get

$$(m-n)\log\alpha < 1.72\cdot 10^{14}k^5\log^2 k\cdot\log(m+2). \tag{4.5}$$

To find an upper bound on m, we have to rewrite the Diophantine equation (1.2) using formulas (2.2) and (2.12) as

$$|c_\alpha(\alpha^{m-n}+1)\alpha^{n+2}-f_k(\varrho)\varrho^p| = |c_\beta\beta^{n+2}+c_\gamma\gamma^{n+2}+c_\beta\beta^{m+2}+c_\gamma\gamma^{m+2}|+|e_k(p)| < \alpha^2.$$

Dividing through by $f_k(\varrho)\varrho^p$, we obtain

$$|\Gamma_4| < \frac{\alpha^2}{f_k(\varrho)\varrho^p} < \frac{1}{\alpha^{m-7}} \tag{4.6}$$

with

$$\Gamma_4 := \frac{c_\alpha(\alpha^{m-n}+1)}{f_k(\varrho)}\alpha^{n+2}\varrho^{-p} - 1.$$

Assume that $\Gamma_4 = 0$, thus $f_k(\varrho) = c_\alpha(\alpha^{m-n}+1)\alpha^{n+2}\varrho^{-p}$, hence $f_k(\varrho)$ is an algebraic integer, which is a contradiction. Thus $\Gamma_4 \neq 0$ and we can apply Theorem 2.1. To do this, we consider,

$$\eta_1 = \frac{c_\alpha(\alpha^{m-n}+1)}{f_k(\varrho)}, \quad \eta_2 = \alpha, \quad \eta_3 = \varrho, \quad d_1 = 1, \quad d_2 = n+2, \quad d_3 = -p.$$

As η_1, η_2, η_3 are elements of the field $\mathbb{K} := \mathbb{Q}(\alpha, \varrho)$, then $d_{\mathbb{K}} \leqslant 3k$. We can take $D = m+2$. Using the properties of the heights and the inequality (4.5), we get

$$\begin{aligned} h(\eta_1) &\leqslant h\Big(\frac{c_\alpha}{f_k(\varrho)}\Big) + h(\alpha^{m-n}+1) < 5.3k\log k + (m-n)\frac{\log\alpha}{3} + \log 2 \\ &< 5.74\cdot 10^{13}k^5\log^2 k\cdot\log(m+2). \end{aligned}$$

Thus, we have

$$\max\{3kh(\eta_1), |\log\eta_1|, 0.16\} < 1.73\cdot 10^{14}k^6\log^2 k\cdot\log(m+2) = A_1,$$

and A_2, A_3 as above. Therefore, from Theorem 2.1 we obtain

$$\log|\Gamma_4| \geqslant -2.52\cdot 10^{26}k^9\log^2 k\cdot\log(m+2)\cdot(1+\log(3k))(1+\log(m+2))$$

which leads to

$$\log|\Gamma_4| > -1.86\cdot 10^{27}k^9\log^3 k\cdot\log^2(m+2), \tag{4.7}$$

where we used as above the inequalities $1+\log(m+2) < 1.8\log(m+2)$ and $1+\log(3k) < 4.1\log k$ for $m \geqslant 2$ and $k \geqslant 2$. Referring to inequalities (4.6) and (4.7), we obtain

$$\frac{m+2}{\log^2(m+2)} < 4.92\cdot 10^{27}k^9\log^3 k.$$

Now, we apply Lemma 2.2 with $T = 4.92 \cdot 10^{27} k^9 \log^3 k$, $x = m + 2$ and $\ell = 2$. So, we have

$$\begin{aligned} m + 2 &< 4(4.92 \cdot 10^{27} k^9 \log^3 k)(\log(4.92 \cdot 10^{27} k^9 \log^3 k))^2 \\ &< (1.97 \cdot 10^{28} k^9 \log^3 k)(63.8 + 9 \log k + 3 \log\log k)^2 \\ &< 1.97 \cdot 10^{32} k^9 \log^5 k. \end{aligned}$$

In the above we have used the fact that $63.8 + 9\log k + 3\log(\log k) < 100 \log k$ which holds for $k \geqslant 2$. Using the above inequality and Lemma 4.1, we obtain

$$p < 1.59 \cdot 10^{32} \cdot k^9 \cdot \log^5 k.$$

This completes the proof of Lemma 4.2. □

4.2.1. The case $2 \leqslant k \leqslant 825$. To reduce the above bound on m, we first set

$$\Lambda_3 := p \log \varrho - (m+2) \log \alpha + \log \frac{f_k(\varrho)}{c_\alpha}.$$

Since $\Gamma_3 \neq 0$, then $\mathrm{e}^{-\Lambda_3} - 1 = \Gamma_3$ gives that $\Lambda_3 \neq 0$. If $\Lambda_3 < 0$, then

$$0 < |\Lambda_3| < \mathrm{e}^{|\Lambda_3|} - 1 = |\Gamma_3| < \frac{1}{\alpha^{m-n-13}}$$

according to the inequality (4.3). If $\Lambda_3 > 0$, then we have $1 - \mathrm{e}^{-\Lambda_3} = |\mathrm{e}^{-\Lambda_3} - 1| < 1/2$. Thus, we get

$$0 < \Lambda_3 < \mathrm{e}^{\Lambda_3} - 1 = \mathrm{e}^{\Lambda_3} |\Gamma_3| < \frac{2}{\alpha^{m-n-13}}.$$

In any cases, we have

$$0 < |\Lambda_3| < \frac{2}{\alpha^{m-n-13}}.$$

Dividing the above inequality by $\log \alpha$, we get

$$|p\tau - (m+2) + \mu| < \frac{761}{\alpha^{m-n}} \tag{4.8}$$

where

$$\tau = \frac{\log \varphi}{\log \alpha} \quad \text{and} \quad \mu = \frac{\log(f_k(\varrho)/c_\alpha)}{\log \alpha}.$$

Now, we apply Lemma 2.1 to (4.8) for $2 \leqslant k \leqslant 825$ by putting

$$M = M_k := \lfloor 1.59 \cdot 10^{32} k^9 \log^5 k \rfloor, \quad A = 761, \quad \text{and} \quad B = \alpha.$$

A quick computation with Mathematica reveals that $m - n \leqslant 408$. Now, we put

$$\Lambda_4 := p \log \varrho - (m+2) \log \alpha + \log \frac{f_k(\varrho)}{c_\alpha(\alpha^{m-n} + 1)}.$$

Using inequality (4.6), we get

$$0 < |\Lambda_4| < \frac{2}{\alpha^{m-7}},$$

which leads to

$$|p\tau - (m+2) + \mu_a| < \frac{77}{\alpha^m}, \tag{4.9}$$

where

$$\tau := \frac{\log\varphi}{\log\alpha}, \quad \mu_a := \frac{\log(f_k(\varrho)/(c_\alpha(\alpha^a+1)))}{\log\alpha}, \quad \text{and} \quad 0 \leqslant a \leqslant 408.$$

Now we apply Lemma 2.1 to (4.9) by taking for $2 \leqslant k \leqslant 825$,

$$M = M_k := \lfloor 1.59\cdot 10^{32}k^9\log^5 k\rfloor, \quad A := 77 \quad \text{and} \quad B := \alpha.$$

We follow the algorithm of Lemma 2.1 using Mathematica and we see that $0 \leqslant n \leqslant m \leqslant 428$. Therefore, by Lemma 4.1, we deduce that $p < (m+0.52)/1.24 < 346$. Finally, we write a program in Maple to compare $P_p^{(k)}$ and $N_n + N_m$ for $k \in [2, 825]$, $p \in [4, 346]$ and $0 \leqslant n \leqslant m \leqslant 428$ with $p \geqslant k+2$ and we get the other solutions mentioned in Theorem 1.1.

4.2.2. The case $k > 825$. In this case, we need to show that the Diophantine equation (1.2) has no solution. We have the following lemma.

Lemma 4.3. *If the integers p, k, n and m satisfy the Diophantine equation* (1.2) *with $0 \leqslant n \leqslant m$, $k > 825$ and $p \geqslant k+2$, then the inequalities*

$$k < 1.25\cdot 10^{36} \quad \textit{and} \quad p < 4.7\cdot 10^{366}$$

hold.

Proof. Together with the inequalities (4.1) and Lemma 4.2, we have

$$0.39m < p < 1.59\cdot 10^{32}k^9\log^5 k < \varphi^{k/2} \quad \text{for } k > 825.$$

Thus, from (2.14) and (4.2), we have

$$\begin{aligned}\Big|c_\alpha\alpha^{m+2} - \frac{\varphi^{2p}}{\varphi+2}\Big| &= \Big|c_\alpha\alpha^{m+2} - f_k(\varrho)\varrho^p + f_k(\varrho)\varrho^p - \frac{\varphi^{2p}}{\varphi+2}\Big| \\ &\leqslant |c_\alpha\alpha^{m+2} - f_k(\varrho)\varrho^p| + \Big|f_k(\varrho)\varrho^p - \frac{\varphi^{2p}}{\varphi+2}\Big| \\ &< |c_\alpha\alpha^{m+2} - f_k(\varrho)\varrho^p| + \frac{\varphi^{2p}|\zeta|}{\varphi+2} < \alpha^{n+8} + \frac{\varphi^{2p}}{\varphi+2}\frac{4}{\varphi^{k/2}}.\end{aligned}$$

Dividing through by $c_\alpha \alpha^{m+2}$, we obtain

$$\left|\frac{1}{c_\alpha(\varphi+2)}\alpha^{-(m+2)}\varphi^{2p}-1\right| < \frac{\alpha^6}{c_\alpha \alpha^{m-n}} + \frac{\varphi^{2p}}{c_\alpha \alpha^{m+2}(\varphi+2)}\frac{4}{\varphi^{k/2}}. \tag{4.10}$$

Moreover, by (2.12) and (2.14), we get

$$\frac{\varphi^{2p}}{\varphi+2}\Big(1-\frac{4}{\varphi^{k/2}}\Big)-\frac{1}{2} < f_k(\varrho)\varrho^p - \frac{1}{2} < f_k(\varrho)\varrho^p + e_k(p) = P_p^{(k)}.$$

Since $P_p^{(k)} = N_n + N_m < \alpha^{m+1}$ and $k > 825$, then we get

$$\frac{1}{c_\alpha \alpha^{m+2}}\Big(\frac{\varphi^{2p}}{\varphi+2}\cdot 0.99 - \frac{1}{2}\Big) < \frac{P_p^{(k)}}{c_\alpha \alpha^{m+2}} = \frac{N_n+N_m}{c_\alpha \alpha^{m+2}} < \frac{\alpha^{m+1}}{c_\alpha \alpha^{m+2}} < 4$$

and therefore

$$\frac{\varphi^{2p}}{c_\alpha \alpha^{m+2}(\varphi+2)} < 7 \quad \forall\, m \geqslant 0.$$

Now, we return to the inequality (4.10). Then, we get

$$|\Gamma_5| < \frac{54}{\alpha^{m-n}} + \frac{28}{\varphi^{k/2}} < \frac{82}{\alpha^{\min\{m-n,k/2\}}} \tag{4.11}$$

where

$$\Gamma_5 := \frac{1}{c_\alpha(\varphi+2)}\alpha^{-(m+2)}\varphi^{2p} - 1.$$

To see that $\Gamma_5 \neq 0$, assume the contrary, i.e., $\Gamma_5 = 0$. We get $\varphi^{2p}/(\varphi+2) = c_\alpha \alpha^{m+2}$. Using the $\mathbb{Q}$-automorphism $(\alpha\beta)$ of the Galois extension $\mathbb{Q}(\varphi,\alpha,\beta)$ over $\mathbb{Q}$ we obtain

$$50 < \frac{\varphi^{2p}}{\varphi+2} = |c_\beta||\beta|^{m+2} < 1,$$

which is impossible. We apply again Theorem 2.1 to Γ_5 with

$$\eta_1 = \frac{1}{c_\alpha(\varphi+2)}, \quad \eta_2 = \alpha, \quad \eta_3 = \varphi, \quad d_1 = 1, \quad d_2 = -(m+2), \quad d_3 = 2p.$$

Since $\mathbb{K} := \mathbb{Q}(\eta_1,\eta_2,\eta_3) = \mathbb{Q}(\alpha,\varphi)$, then $d_{\mathbb{K}} = 6$. Also, we have

$$h(\eta_2) = \frac{\log\alpha}{3} \quad \text{and} \quad h(\eta_3) = \frac{\log\varphi}{2}.$$

Thus, we can take

$$A_2 = 0.77 \quad \text{and} \quad A_3 = 1.45.$$

Furthermore, we obtain

$$h(\eta_1) = h(c_\alpha(\varphi+2)) \leqslant h(c_\alpha) + h(\varphi) + h(2) + \log 2 \leqslant \frac{\log 31}{3} + \frac{\log\varphi}{2} + 2\log 2 < 2.78.$$

So we can take $A_1 := 16.68$. Because $m+2 < 2.55p + 1.45 < 3p$ for $p \geqslant 4$, we can choose $D := 3p$. Thus, by using Theorem 2.1, we obtain

$$\log|\Gamma_5| > -7 \cdot 10^{14} \log p \tag{4.12}$$

where we used the fact that $1 + \log(3p) < 2.6 \log p$, which is valid for $p \geqslant 4$. Combining (4.11) and (4.12), we can see that

$$\begin{aligned} \min\{m-n, k/2\} &< 1.85 \cdot 10^{15} \log p < 1.85 \cdot 10^{15} \log(1.59 \cdot 10^{32} k^9 \log^5 k) \\ &< 4.09 \cdot 10^{16} \log k \end{aligned} \tag{4.13}$$

where we used

$$\log p < \log(1.59 \cdot 10^{32} k^9 \log^5 k) < 22.1 \log k \quad \text{for } k \geqslant 825.$$

Now, we have to consider the following two cases according to the values of $\min\{m-n, k/2\}$.

Case $\min\{m-n, k/2\} = k/2$. In this case, by combining (4.13) and Lemma 4.2 it is easy to see the bounds

$$k < 6.38 \cdot 10^{18} \quad \text{and} \quad p < 4.24 \cdot 10^{209}.$$

Case $\min\{m-n, k/2\} = m-n$. By the inequality (4.13), we get

$$m - n < 4.09 \cdot 10^{16} \log k. \tag{4.14}$$

We rewrite the equation (1.2) as

$$\begin{aligned} &\Big|c_\alpha(\alpha^{m-n}+1)\alpha^{n+2} - \frac{\varphi^{2p}}{\varphi+2}\Big| \\ &\quad\leqslant \Big|\frac{\varphi^{2p}}{\varphi+2}\zeta + e_k(p)\Big| + |c_\beta\beta^{n+2} + c_\gamma\gamma^{n+2} + c_\beta\beta^{m+2} + c_\gamma\gamma^{m+2}| \\ &\quad\leqslant \frac{\varphi^{2p}}{\varphi+2}|\zeta| + |e_k(p)| + 2|c_\beta||\beta|^{n+2} + 2|c_\beta||\beta|^{m+2} < \frac{\varphi^{2p}}{\varphi+2}\frac{4}{\varphi^{k/2}} + \frac{5}{2}. \end{aligned}$$

Multiplying through by $(\varphi+2)/\varphi^{2p}$ and using the fact that $p \geqslant k+2$, we obtain

$$|\Gamma_6| < \frac{4}{\varphi^{k/2}} + \frac{5}{2}\frac{\varphi+2}{\varphi^{2p}} < \frac{14}{\varphi^{k/2}}, \tag{4.15}$$

where $\Gamma_6 := c_\alpha(\alpha^{m-n}+1)(\varphi+2)\alpha^{n+2}\varphi^{-2p} - 1$. To see that $\Gamma_6 \neq 0$, assume that $\Gamma_6 = 0$. We get $\varphi^{2p}/(\varphi+2) = c_\alpha\alpha^{m+2}$. Using the $\mathbb{Q}$-automorphism $(\alpha\beta)$ of the Galois extension $\mathbb{Q}(\varphi, \alpha, \beta)$ over $\mathbb{Q}$ we obtain

$$50 < \frac{\varphi^{2p}}{\varphi+2} = |c_\beta||\beta^{m-n}+1||\beta|^{n+2} < |c_\beta|(|\beta|^{m-n}+1)|\beta|^{n+2} < 2,$$

which is a contradiction. So we can apply Theorem 2.1 to Γ_6 with

$$\eta_1 = c_\alpha(\alpha^{m-n}+1)(\varphi+2), \quad \eta_2 = \alpha, \quad \eta_3 = \varphi, \quad d_1 = 1, \quad d_2 = n+2, \quad \text{and} \quad d_3 = -2p.$$

Here again, we can take $d_{\mathbb{K}} = 6$, $D = 3p$, $A_2 = 0.77$ and $A_3 = 1.45$. Moreover, one sees that

$$\begin{aligned} h(\eta_1) &\leqslant h(c_\alpha) + (m-n)h(\alpha) + h(\varphi) + h(2) + 2\log 2 \\ &\leqslant \frac{\log 31}{3} + (m-n)\frac{\log\alpha}{3} + \frac{\log\varphi}{2} + 3\log 2 < 5.22 \cdot 10^{15}\log k \end{aligned}$$

follows. Thus, we can take

$$A_1 := 3.14 \cdot 10^{16} \log k.$$

Using the fact that $1 + \log(3p) < 2.6\log p$ for $p \geqslant 4$, we get from Theorem 2.1 that

$$\log|\Gamma_6| > -2.91 \cdot 10^{31}\log^2 k. \tag{4.16}$$

Next, we put the inequalities (4.15) and (4.16) together to get

$$k < 1.25 \cdot 10^{36} \quad \text{and} \quad p < 4.7 \cdot 10^{366}.$$

This completes the proof of Lemma 4.3. □

To reduce the bound of k, we take

$$\Lambda_5 = \log(c_\alpha(\varphi+2)) + (m+2)\log\alpha - 2p\log\varphi.$$

Moreover, as $\Gamma_5 \neq 0$ one sees that $\Lambda_5 \neq 0$ and we get from (4.11) that

$$0 < |\Lambda_5| < \frac{164}{\alpha^{\min\{m-n,k/2\}}}.$$

Dividing by $\log\varphi$, we get

$$|(m+2)\tau - 2p + \mu| < \frac{341}{\alpha^{\min\{m-n,k/2\}}}, \tag{4.17}$$

where

$$\tau = \frac{\log\alpha}{\log\varphi} \quad \text{and} \quad \mu = \frac{\log(c_\alpha(\varphi+2))}{\log\varphi}.$$

With Lemma 4.3 and the fact that

$$m + 2 < 2.55p + 1.45 < 3p < 1.41 \cdot 10^{367},$$

we can apply Lemma 2.1 to (4.17) with $M = 1.41 \cdot 10^{367}$, $A = 341$ and $B = \alpha$. Thus we get

$$\min\{m-n, k/2\} \leqslant 2235. \tag{4.18}$$

▷ If $\min\{m-n, k/2\} = k/2$, then by combining (4.18) and Lemma 4.2 we obtain the bounds

$$k \leqslant 4470 \quad \text{and} \quad p < 4.76 \cdot 10^{69}.$$

▷ If $\min\{m-n, k/2\} = m-n$, then we have

$$m - n \leqslant 2235. \tag{4.19}$$

Put

$$\Lambda_6 := 2p \log \varphi - (n+2) \log \alpha + \log \frac{1}{c_\alpha(\alpha^{m-n}+1)(\varphi+2)}.$$

From (4.15), it is easy to see that

$$|2p\tau - (n+2) + \mu_{m,n}| < \frac{74}{\varphi^{k/2}}, \tag{4.20}$$

where

$$\tau = \frac{\log \varphi}{\log \alpha}, \quad \mu_{m,n} = \frac{-\log(c_\alpha(\alpha^{m-n}+1)(\varphi+2))}{\log \alpha} \quad \text{for } 0 \leqslant m-n \leqslant 2235.$$

Applying Lemma 2.1 with $A := 74$, $B := \varphi$ and $M := 1.41 \cdot 10^{367}$, we obtain

$$k \leqslant 3644 \quad \text{and} \quad p < 6.69 \cdot 10^{68}.$$

Therefore, in both cases according to $\min\{m-n, k/2\}$, we need to consider

$$k \leqslant 4470 \quad \text{and} \quad p < 4.76 \cdot 10^{69}. \tag{4.21}$$

We apply again Lemma 2.1 using the bounds from (4.21) and we get $k \leqslant 814$, which contradicts the fact that $k > 825$. This completes the proof of Theorem 1.1.

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