ON FORBIDDEN CONFIGURATION
OF PSEUDOMODULAR LATTICES

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Abstract. We characterize the pseudomodular lattices by means of a forbidden configuration.

Keywords: forbidden configuration; pseudomodular lattice; semimodular lattice

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1. INTRODUCTION AND PRELIMINARIES

Dress and Lovász in [4] studied full algebraic matroids of finite ranks for modularity of the flats. They characterized the existence of flats using the rank function and quasi-intersection. Björner and Lovász in [2] introduced a class of pseudomodular lattices as a generalization of modular lattices to contain full algebraic combinatorial geometries; see also [3]. A semimodular lattice $L$ of finite length is said to be pseudomodular if every pair of elements of $L$ has a pseudointersection. The class of pseudomodular lattices forms a subclass of the class of semimodular lattices and contains all modular lattices of finite length. Characterizations of classes of lattices by means of the non-existence of certain sublattices called forbidden configurations are available in the literature, such as the classes of distributive lattices, modular lattices, semimodular lattices, etc. In this paper, we establish a characterization by means of a forbidden sublattice for the class of pseudomodular lattices.

We give here some definitions and notations for ready reference; see Birkhoff [1], Grätzer [5], Haskins and Gudder [6], Stern [7], etc.

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Let $P$ be a nonempty poset and $x, y \in P$. If $x \leq y$, then the length of an interval $[x, y]$, denoted by $\text{lt}[x, y]$, is the supremum of the lengths of the chains in $[x, y]$. The height or rank $r(x)$ of an element $x$ of a poset $P$ bounded below is the length of the interval $[0, x]$.

A lattice $L$ is (upper) semimodular if $a \land b$ is a lower cover of $a$. Then $b$ is a lower cover of $a \lor b$, for $a, b \in L$. A lattice $L$ is said to be modular if the following condition (M) holds.

(M): $c \lor (a \land b) = (c \land a) \lor b$ for all $a, b, c \in L$ with $c \leq b$.

**Definition 1.1** ([1]). The graded poset is defined as a poset $P$ with a function $g: P \to \mathbb{Z}$ such that:

(i) $x > y$ implies $g[x] > g[y]$ and

(ii) if $x < y$, then $g[x] = g[y] + 1$.

Note that any semimodular lattice of finite length is graded by its rank function. Following is the definition due to Björner and Lovász [2], see also [3].

**Definition 1.2** ([2]). Let $L$ be a semimodular lattice of finite length, and denote by $r(x)$ the rank function (height function) of $L$. For each $x, y \in L$ let $P_{x,y} = \{ z \leq y : r(x \lor z) - r(z) = r(x \lor y) - r(y) \}$. If the set $P_{x,y}$ has a unique least element, then we call this the pseudointersection of $x$ and $y$ and denote it by $x \uparrow y$.

A semimodular lattice of finite length is called pseudomodular if every pair of its elements has the pseudointersection.

**Remark 1.3** ([2]). The set $P_{x,y}$ lies in the interval $[x \land y, y]$ and is dual order ideal in $[x \land y, y]$.

**Lemma 1.4** ([2]). For any two elements $x$ and $y$ in a semimodular lattice $L$, the following are equivalent:

(i) $x$ and $y$ form a modular pair, i.e., $r(x \lor y) + r(x \land y) = r(x) + r(y)$.

(ii) $x \uparrow y$ exists and $x \uparrow y \leq x$.

(iii) $x \uparrow y$ exists and $x \uparrow y = x$.

(iv) $x \land y \in P_{x,y}$.

**Lemma 1.5** ([2]). For any two elements $x$ and $y$ of a semimodular lattice $L$, the following are equivalent:

(i) $x \uparrow y$ exists, i.e., $P_{x,y}$ has a unique least element.

(ii) $P_{x,y}$ is closed under meets.

(iii) If $u, v, z \in P_{x,y}$ and $z$ covers $u$ and $v$, then $u \land v \in P_{x,y}$.

A subset $I$ of a poset $P$ is an order ideal if $x \in I$ and $y \leq x$ imply $y \in I$.  

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2. Main result

We have a forbidden characterization of pseudomodular lattices in the following theorem. In what follows, a sublattice $S$ of a lattice $L$ is said to be cover-preserving in $L$ if for $a, b \in S$, $a \prec b$ in $S$ implies $a \prec b$ in $L$, see [7].

**Theorem 2.1.** Let $L$ be a semimodular lattice of finite length. Then $L$ is pseudomodular if and only if $L$ does not contain a cover preserving sublattice isomorphic to the lattice as depicted in Figure 1.

![Figure 1](image)

**Proof of Theorem 2.1.** Let $L$ be a semimodular lattice of finite length. If $L$ contains a sublattice isomorphic to the lattice as depicted in Figure 1, then the elements $c_4, b_4, b_6$ belong to $P_{b_1,c_4}$. Thus $r(b_1 \lor c_4) - r(c_4) = r(b_1 \lor b_4) - r(b_4) = r(b_1 \lor b_6) - r(b_6)$, but $r(b_1 \lor (b_4 \land b_6)) - r(b_4 \land b_6) \neq r(b_1 \lor b_6) - r(b_6)$. Therefore $b_4 \land b_6 \notin P_{b_1,c_4}$, which implies that $P_{b_1,c_4}$ does not have the least element and so the pseudointersection of $b_1$ and $c_4$ does not exist. Therefore, $L$ is not a pseudomodular lattice.

Conversely, suppose that $L$ is a semimodular lattice of finite length which is not pseudomodular. Then there exists a pair of elements $x, y \in L$ such that $P_{x,y} = \{z \leq y: r(x \lor z) - r(z) = r(x \lor y) - r(y)\}$ does not have the least element. Or equivalently, we have a pair $x, y$ in $L$ whose meet does not belong to $P_{x,y}$.

Consider a pair $z_1, z_2$ in $P_{x,y}$ with minimal height whose meet does not belong to $P_{x,y}$. Since $y, z_1, z_2 \in P_{x,y}$, we have $r(x \lor y) - r(y) = r(x \lor z_1) - r(z_1) = r(x \lor z_2) - r(z_2)$. Without loss of generality we assume that $lt[x \land y, x \lor y]$ is minimum, i.e., for $u, v \in L$, if $lt[u \land v, u \lor v] < lt[x \land y, x \lor y]$, then $P_{u,v}$ has the least element. We also assume that $x, y \in L$ is a pair such that for $x \land y < u < x$, the set $P_{u,y}$ has the least element.
If $x \leq y$, then $x, y$ becomes a modular pair and by Lemma 1.4, $x \land y \in P_{x,y}$, which is nothing but the pseudointersection of $x$ and $y$, a contradiction to the assumption.

Similarly, if $x > y$, then also by Lemma 1.4, we have a contradiction. Consequently, we must have $x \parallel y$.

If $x \land y < x$, then by semimodularity we have $y < x \lor y$ and thus $r(x) - r(x \land y) = 1 = r(x \lor y) - r(y)$ and $P_{x,y} = \{z \leq y: r(x \lor z) - r(z) = r(x \lor y) - r(y)\} = \{z \leq y: r(x \lor z) - r(z) = 1\}$. Also, as $x \land y \leq y$ and $r(x \lor (x \land y)) - r(x \land y) = r(x) - r(x \land y) = 1$, we have $x \land y \in P_{x,y}$ and so $x \land y$ is the least element of $P_{x,y}$, a contradiction, and therefore we must have $x \land y \not< x$.

Consider an element $q$ such that $x \land y < q < x$ and without loss of generality, we consider $x \land y < q < x$. If $x \land y < z_1$, then by semimodularity we have $q < q \lor z_1$ and also $x < x \lor z_1$. Therefore, $r(x \lor (x \land y)) - r(x \land y) = r(x \lor z_1) - r(z_1)$ and consequently, $x \land y \in P_{x,y}$, a contradiction, and so, we must have $x \land y \not< z_1$. Similarly, we have to have $x \land y \not< z_2$.

Now, since $x \land y \not< z_1$ and $x \land y \not< z_2$, there exist $q_1$ and $q_2$ such that $z_1 \land z_2 < q_1 < z_1$ and $z_1 \land z_2 < q_2 < z_2$ and without loss of generality, we consider $x \land y < q_1 < z_1$ and $x \land y < q_2 < z_2$.

Consider the set $\{x, y, x \lor y, x \land y, z_1, z_2, x \lor z_1, x \lor z_2, q, q_1, q_2, q \lor q_1, q \lor q_2, q_1 \lor q_2, q \lor q_1 \lor q_2\}$ and we contend that these elements are distinct and also the set forms a cover preserving sublattice of $L$. Note that by the choice $x \lor y, x \land y, z_1, z_2, x \lor z_1, x \lor z_2, q, q_1$ and $q_2$ are distinct elements. For the other elements, we have the following.

**Claim 2.2.** $(x \lor z_1) \land (x \lor z_2) = x$.

**Proof.** Suppose that $(x \lor z_1) \land (x \lor z_2) > x$. As $q_1 < z_1$, by semimodularity we have $q_1 \lor x < z_1 \lor x$. If $q_1 \lor x < z_1 \lor x$, then $r(q_1 \lor x) = r(z_1 \lor x) - r(z_1) = r(x \lor y) - r(y) = r(x \lor z_2) - r(z_2)$, which implies $q_1 \in P_{x,y}$, a contradiction, and so we must have $(x \lor z_1) \land (x \lor z_2) = x$. 

**Claim 2.3.** $x \land y = z_1 \land z_2$.

**Proof.** Suppose $x \land y < z_1 \land z_2$. If $x_1 = x \lor (z_1 \land z_2)$, then we have $x_1 \leq (x \lor z_1) \land (x \lor z_2)$, which gives $x \lor y = x_1 \lor y, x \land z_1 = x_1 \lor z_1$ and $x \lor z_2 = x_1 \lor z_2$. It follows that $r(x_1 \lor y) - r(y) = r(x_1 \lor z_1) - r(z_1) = r(x_1 \lor z_2) - r(z_2)$ and so $y, z_1, z_2 \in P_{x_1,y}$. Since $\ell [x \land y, x \lor y] < \ell [x \land y, x \lor y]$, we have $z_1 \land z_2 \in P_{x_1,y}$. Thus, $r(x_1 \lor (z_1 \land z_2)) - r(z_1 \land z_2) = r(x_1 \lor y) - r(y) = r(x \lor y) - r(y)$. Moreover, $z_1 \land z_2 \in P_{x,y}$, a contradiction, and so we must have $x \land y = z_1 \land z_2$. 

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Claim 2.4. $z_1 \lor z_2 = y$.

Proof. Suppose that $z_1 \lor z_2 < y$. If $y_1 = z_1 \lor z_2$, then $z_1 < y_1 < y$. Since $P_{x,y}$ is a dual order ideal, we have $y_1 \in P_{x,y}$ and so $r(x \lor y_1) - r(y_1) = r(x \lor y) - r(y)$, which gives $x \lor y \neq x \lor y_1$. Now consider the interval $[x \land y_1, x \lor y_1]$. Since $lt[x \land y_1, x \lor y_1] < lt[x \land y, x \lor y]$ and $z_1, z_2, y_1 \in P_{x,y_1}$, we have $z_1 \land z_2 \in P_{x,y_1}$. Thus $r(x \lor (z_1 \land z_2)) - r(z_1 \land z_2) = r(x \lor y_1) - r(y_1) = r(x \lor y) - r(y)$, which implies $z_1 \land z_2 \in P_{x,y_1}$, a contradiction, and so $z_1 \lor z_2 = y$. \hfill \Box

Claim 2.5. $(x \lor z_1) \lor (x \lor z_2) = x \lor y$.

Proof. Observe that $(x \lor z_1) \lor (x \lor z_2) = x \lor (z_1 \lor z_2) = x \lor y$. \hfill \Box

Claim 2.6. $q \lor y = x \lor y$.

Proof. Suppose that $q \lor y < x \lor y$. As $q < x$, by semimodularity we have $q \lor y < x \lor y$. In this case, $q \lor z_1 < x \lor z_1$ and $q \lor z_2 < x \lor z_2$. If $q \lor z_1 = x \lor z_1$ or $q \lor z_2 = x \lor z_2$, then this implies that $q \lor z_1 \lor y = x \lor z_1 \lor y = q \lor y = x \lor y$, which is not possible and so, $r(x \lor y) - r(y) = r(x \lor z_1) - r(z_1) = r(x \lor z_2) - r(z_2)$.

Also, we have $r(q \lor y) - r(y) = r(q \lor z_1) - r(z_1) = r(q \lor z_2) - r(z_2)$, which implies $y, z_1, z_2 \in P_{q,y}$. Since $lt[q \land y, q \lor y] < lt[x \land y, x \lor y]$, we have $z_1 \land z_2 \in P_{q,y}$ and so, $r(q \lor (z_1 \land z_2)) - r(z_1 \land z_2) = r(q \lor y) - r(y) = r(x \lor y) - r(y) - 1$. Consequently, we have $r(q \lor (z_1 \land z_2)) + 1 - r(z_1 \land z_2) = r(x \lor y) - r(y)$. Now, since $q < x$ and $q \lor (z_1 \land z_2) = q$, we have $r(q \lor (z_1 \land z_2)) + 1 = r(x \lor (z_1 \land z_2))$, which gives $r(x \lor (z_1 \land z_2)) - r(z_1 \land z_2) = r(x \lor y) - r(y)$. This implies that $z_1 \land z_2 \in P_{x,y}$, a contradiction, and so we must have $y \lor q = x \lor y$. \hfill \Box

Claim 2.7. $q \lor z_1 = x \lor z_1$.

Proof. Suppose that $q \lor z_1 < x \lor z_1$ and consider a chain of length $n$ in $[z_1, x \lor z_1]$: $z_1 < p_1 < p_2 < \ldots < p_{n-1} = q \lor z_1 < x \lor z_1$. By semimodularity, we have a chain in $[y, x \lor y]$: $y = z_1 \lor y < p_1 \lor y \leq p_2 \lor y \leq \ldots \leq p_n = q \lor z_1 \lor y = x \lor z_1 \lor y$ which is of length at most $n$, a contradiction to the fact that $r(q \lor y) - r(y) = r(q \lor z_1) - r(z_1)$. So we must have $q \lor z_1 = x \lor z_1$. \hfill \Box

Claim 2.8. $q \lor z_2 = x \lor z_2$.

Proof. Is similar to that of Claim 2.7. \hfill \Box

Claim 2.9. $x \lor q_1 = x \lor z_1$.

Proof. Suppose that $x \lor q_1 < x \lor z_1$. As $q_1 < z_1$, by semimodularity we have $x \lor q_1 < x \lor z_1$. This gives $r(x \lor z_1) - r(z_1) = r(x \lor q_1) - r(q_1)$. Thus, $q_1 \in P_{x,y}$, a contradiction, and so we must have $x \lor q_1 = x \lor z_1$. \hfill \Box

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Claim 2.10. $x \lor q_2 = x \lor z_2$.

Proof. Is similar to that of Claim 2.9.

Claim 2.11. $q_1 \lor z_2 = y$.

Proof. Suppose that $q_1 \lor z_2 < y$. As $q_1 < z_1$, by semimodularity we have $q_1 \lor z_2 < y$. If $y_1 = q_1 \lor z_2$, then $z_2 < y_1 < y$. Since $P_{x,y}$ is a filter, we have $y_1 \in P_{x,y}$, which implies $r(x \lor y_1) = r(y_1) = r(x \lor y) - r(y)$. Also, we have $q_1 \leq y_1 \leq x \lor y_1$ and $x \leq x \lor y_1$, therefore $x \lor q_1 \leq x \lor y_1$. Since $x \lor q_1 = x \lor z_1$, we have $z_1 \leq x \lor z_1 \leq x \lor y_1$. Also $z_2 \leq y_1 \leq x \lor y_1$ and therefore $z_1 \lor z_2 \leq x \lor y_1$. However, we have $z_1 \lor z_2 = y < x \lor y_1$, a contradiction, and so we must have $q_1 \lor z_2 = y$.

Claim 2.12. $q_2 \lor z_2 = y$.

Proof. Is similar to that of Claim 2.11.

Claim 2.13. $q \lor q_1 < x \lor z_1$.

Proof. Suppose that $q \lor q_1 = x \lor z_1$. Note that $x \land y \neq q$ and $x \land y \neq q_1$; otherwise, $q_1 < q \lor q_1$, which is not true since $q_1 < z_1 < x \lor z_1$. Therefore there exists $p_1$ such that $x \land y < p_1 < q$. We have $p_1 \parallel z_1$, $p_1 \parallel q_1$ and $p_1 \lor q_1 \parallel z_1$. Now, if $(p_1 \lor q_1) \lor z_1 = x \lor z_1$ and $z_1 < x \lor z_1$ and so $r(x \lor z_1) - r(z_1) = 1$. Also, as $q \lor y = x \lor y$, $q \lor z_1 = x \lor z_1$, $q \lor z_2 = x \lor z_2$ and $q < x$ and so, by assumption, $P_{q,y}$ must have the least element. We have $y, z_1, z_2 \in P_{q,y}$, $z_1 \land z_2 \in P_{q,y}$ but $r(q \lor (x \land y)) - r(x \land y) \neq 1$, a contradiction, and so we must have $q \lor q_1 < x \lor z_1$.

Claim 2.14. $q \lor q_2 < x \lor z_2$.

Proof. Is similar to that of Claim 2.13.
Claim 2.15. $x \land y < q$.

Proof. Suppose there exists an element $p$ such that $x \land y < p < q$. It follows that $p \lor q_1 \leq q \lor q_1$. If $p \lor q_1 = q \lor q_1$, then by semimodularity we have $q_1 < q \lor q_1$, which is not true. Therefore $p \lor q_1 < q \lor q_1$ and by semimodularity we have $q_1 < p \lor q_1$ and similarly, $q_2 < p \lor q_2$. In this case, $(p \lor q_1) \land z_1 \leq x \lor z_1$. We consider the following subcases:

(i) Suppose $(p \lor q_1) \land z_1 = x \lor z_1$. By semimodularity we have $z_1 < x \lor z_1$ and $y < x \lor y$ and therefore $r(x \lor z_1) - r(z_1) = r(x \lor y) - r(y) = 1$. Since $r(x \lor y) - r(y) = r(x \lor z_2) - r(z_2) = 1$ and hence $z < x \lor z_2$. We also have $q \lor y = x \lor y$, $q \lor z_1 = x \lor z_1$, $q \land z_2 = x \lor z_2$. Thus $y_1, z_1, z_2 \in P_{q,y}$ and by assumption, $P_{q,y}$ must have the least element, which gives $z_1 \land z_2 \in P_{q,y}$. Therefore $r(x \lor y) - r(y) = r(q \lor (z_1 \land z_2)) - r(z_1 \land z_2) = 1$, a contradiction to the fact that $r(q) - r(x \lor y) > 1$, and therefore $(p \lor q_1) \land z_1 \neq x \lor z_1$. Similarly, $(p \lor q_2) \land z_2 \neq x \lor z_2$.

(ii) Suppose $(p \lor q_1) \land z_1 < x \lor z_1$. Let $p_1 = (p \lor q_1) \land z_1$. By semimodularity we have $z_1 < p_1$. If $p_1 \lor y = x \lor y$, then $y < x \lor y$, a contradiction to the fact that $r(x \lor y) - r(y) = r(x \lor z_1) - r(z_1)$, and therefore $p_1 \lor y < x \lor y$. By semimodularity we have $y < p_1 \lor y$. Similarly, for $p_2 = (p \lor q_2) \lor z_2$, we have $y < p_2 \lor y$.

In this case, $p_1 \lor y = p_2 \lor y = (p \lor q_2 \lor z_2) \lor y = (p \lor z_2) \lor y = p \lor y$. Let $y_1 = p_1 \lor y = p_2 \lor y = p \lor y$. Then $x \lor y_1 = x \lor y$, $x \land y < x \land y_1$ and $x \land y_1 \geq p$. Therefore $l(t[x \land y_1, x \land y]) < l(t[x \land y, x \land y])$. As $r(x \lor y_1) - r(y_1) = r(x \lor p_1) - r(p_1) = r(x \lor z_1) - r(p_1) = r(x \lor z_2) - r(p_2) = r(x \lor p_2) - r(p_2)$, we have $y_1, p_1, p_2 \in P_{x,y_1}$. Hence $P_{x,y_1}$ has the least element, which gives $p_1 \land p_2 \in P_{x,y_1}$. Thus $r(x \lor y_1) - r(y_1) = r(x \lor (p_1 \land p_2)) - r(p_1 \lor p_2)$ and we have $p_1 \land p_2 \geq p$. Also, $p_1 \land p_2 \leq x \lor z_1$ and $p_1 \land p_2 \leq x \lor z_2$, which gives $p_1 \land p_2 \leq (x \lor z_1) \land (x \lor z_2)$, and so $p_1 \land p_2 \leq x$. In this case, $q \lor y = q \lor y_1 = x \lor y$, $q \lor z_1 = q \lor p_1 = x \lor z_1$, $q \lor z_2 = q \lor p_2 = x \lor z_2$ and $q \land y_1 > x \land y$, and so $y_1, p_1, p_2 \in P_{q,y_1}$. By assumption, $P_{q,y_1}$ has the least element and so $p_1 \land p_2 \in P_{q,y_1}$. Thus $r(q \lor (p_1 \land p_2)) - r(p_1 \land p_2) = r(x \lor z_1) - r(p_1)$. Since $(x \lor (p_1 \land p_2)) - r(p_1 \land p_2) = r(x \lor z_1) - r(p_1)$, which implies that $r(q \lor (p_1 \land p_2)) - r(p_1 \land p_2) = (x \lor (p_1 \land p_2)) - r(p_1 \land p_2)$, we have a contradiction as $q < x$. Thus, in each of the cases we get a contradiction and consequently we must have $x \land y < q$.

Also, since $q_1 < z_1$, $q_2 < z_2$ and $q < x$, by semimodularity we have $q \lor q_2 < x \lor z_2$ and $q \lor q_1 < x \lor z_1$.

Claim 2.16. $x \land y < q_1$.

Proof. Suppose there exists $p_1$ such that $x \land y < p_1 < q_1$. By semimodularity, we have $q < q \lor p_1$, $x < x \lor p_1$, $z_2 < p_1 \lor z_2$ and $q \lor q_2 < q \lor p_2 \lor p_1$. Let $z_2 = p_1 \lor z_2$ and we have $x \lor z_2 < z_2 \lor x$. Let $x_1 = x \lor p_1$ and we have $l(t[x_1 \land y, x_1 \lor y] < l(t[x \land y, x \lor y])$.
Since $z'_2 > z_2$ and $P_{x,y}$ is a dual order ideal, we have $z'_2 \in P_{x,y}$. Thus $r(x \vee z'_2) - r(z'_2) = 1$. Also we have $x_1 \vee y = x \vee y$, $x_1 \vee z_1 = x \vee z_1$ and $x_1 \vee z'_2 = x \vee z'_2$. It follows that $y, z_1, z'_2 \in P_{x_1,y}$ and so $z_1 \wedge z'_2 \in P_{x_1,y}$. Thus $r(x_1 \vee (z_1 \wedge z'_2)) - r(z_1 \wedge z'_2) = 1$. Since $q_1 > z_1 \wedge z'_2 \geq p_1$ and $x_1 \vee (z_1 \wedge z'_2) = x \vee z_1$, we have $r(x_1 \vee (z_1 \wedge z'_2)) - r(z_1 \wedge z'_2) > 1$, a contradiction, and so we must have $x \wedge y < q_1$. □

![Figure 3.](image)

**Claim 2.17.** $x \wedge y < q_2$.

**Proof.** Is similar to that of Claim 2.16. □

**Claim 2.18.** $q_1 \lor q_2 < y$.

**Proof.** Suppose that $q_1 \lor q_2 = y$, since $x \wedge y < q_1, q_2$ implies $x \wedge y = q_1 \wedge q_2$ and by semimodularity, $q_1 < y$ and $q_2 < y$, which is not true and so we must have $q_1 \lor q_2 < y$. □

Now, $x \wedge y < q_1, q_2$ implies $x \wedge y = q_1 \wedge q_2$ and by semimodularity, $q_1 < q_1 \lor q_2$ and $q_2 < q_1 \lor q_2$. Also, $x \wedge y < q, q_1$ implies $x \wedge y = q_1 \wedge q$ and so by semimodularity, $q < q \lor q_1$ and $q_1 < q \lor q_1$. Similarly, $x \wedge y < q, q_2$ implies $x \wedge y = q_2 \lor q$, so by semimodularity, $q < q \lor q_2$ and $q_2 < q \lor q_2$. Now, $q_1 < q_1 \lor q_2$ and $q_2 < q_1 \lor q_2$ implies $q_1 = (q_1 \lor q_2) \wedge z_1$ and $q_2 = (q_1 \lor q_2) \wedge z_2$ and by semimodularity, $z_1 < (q_1 \lor q_2) \lor z_1 = y$, $z_2 < (q_1 \lor q_2) \lor z_2 = y$ and $q_1 \lor q_2 < (q_1 \lor q_2) \lor z_2 = y$.

**Claim 2.19.** $(q_1 \lor q_2) \lor y < x \lor y$.

**Proof.** Suppose that $(q_1 \lor q_2) \lor y = x \lor y$. Since $q \wedge (q_1 \lor q_2) < q$, by semimodularity we have $q_1 \lor q_2 < q \lor (q_1 \lor q_2)$, which is not true as $q_1 \lor q_2 < y < x \lor y$. Therefore $(q_1 \lor q_2) \lor y < x \lor y$. □
Since \( q_1 \prec q \lor q_1 \), by semimodularity, \((q_1 \lor q_2) \prec (q_1 \lor q_2) \lor q\). Also, \( q_1 \prec q_1 \lor q_2 \) implies \( q_1 \lor q \prec q_1 \lor q_2 \lor q \), which further implies \( q_2 \lor q \prec q_1 \lor q_2 \lor q \) and also \( q_1 \lor q_2 \prec y \) implies \( q_1 \lor q_2 \lor q \prec x \lor y \). Hence, \( L \) contains the following cover preserving sublattice.

![Figure 4](image)

The following result is due to Teo [8].

**Corollary 2.20** ([8]). A lattice \( L \) of finite length is not semimodular if and only if \( L \) contains a subpentagon \((a \land c, a, b, c, a \lor b)\) with the properties

(i) \( a \land c \prec a, b \prec c \prec a \lor b \), or

(ii) \( a \land c \prec a, a \land c \prec b, c \prec a \lor b \).

![Figure 5](image)

**Corollary 2.21.** Let \( L \) be a lattice of finite length. Then \( L \) is a pseudomodular lattice if and only if it does not contain a sublattice isomorphic to a cover preserving lattice as depicted in Figure 4 or Figure 5 (a) or Figure 5 (b).
References


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