RELATIVE CO-ANNIHILATORS IN LATTICE EQUALITY ALGEBRAS

Sogol Niazian, Tehran, Mona Aaly Kologani, Zahedan,
Rajab Ali Borzooei, Tehran

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Abstract. We introduce the notion of relative co-annihilator in lattice equality algebras and investigate some important properties of it. Then, we obtain some interesting relations among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Given a lattice equality algebra $\mathbb{E}$ and $\mathbb{F}$ a filter of $\mathbb{E}$, we define the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ and show that by defining some operations on it, it makes a BL-algebra.

Keywords: equality algebra; annihilator; co-annihilator; relative co-annihilator; filter

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1. Introduction

Equality algebras were introduced by Jenei in [9] and are assumed for possible algebraic semantics of fuzzy type theory (FTT). It was proved in [4], [9] that any equality algebra has a corresponding BCK-meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra. From a logical point of view, various filters have natural interpretation as various sets of provable formulas, which has a very close relationship with decision-making.

Davey studied the relationship between minimal prime ideals conditions and annihilators conditions on distributive lattices, see [5]. Turunen defined co-annihilator of a nonempty subset of a BL-algebra and proved some of its properties (see [17]). Leustean introduced the notion of co-annihilator relative to a filter F on pseudo BL-algebras (see [11]). Then Meng introduced generalized co-annihilators in BL-algebras and gave characterizations of prime and minimal prime filters (see [14]). Also, Zou et al. introduced the notion of annihilators in BL-algebras and investigated some related properties of them in [20]. Filipoiu in [6] used the notion of

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annihilator for Baer extensions of MV-algebras. In [1], [8] the notion of annihilators was studied for BCK-algebras. Leustean in [12] used the notion of co-annihilator for Baer extensions of BL-algebras. Recently, as the generalization of the co-annihilator in a BL-algebra, Saeid et al. in [13] introduced the co-annihilator of a set relative to another set in a residuated lattice, where they gave some relations between filters and co-annihilators. It is helpful for the co-annihilators to study structures and properties in algebraic systems.

In this paper, we introduce the notion of co-annihilator in equality algebras. We study basic properties of co-annihilators and investigate the relationship between them and some special types of filters. Also, we obtain some interesting relations among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Moreover, we define the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ and show that by defining some operations on it, it makes a BL-algebra.

The paper is organized as follows: In Section 2, we gather the basic notions and results on equality algebras. In Section 3, we introduce the notion of co-annihilator relative to a filter in equality algebras and get some interesting results about them. Then, we study the relation among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Finally, we prove that the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ can make a BL-algebra.

2. Preliminaries

In this section, we gather some basic notions and results relevant to the equality algebra, which will be needed in the next sections. For a survey of equality algebras we refer to [19].

Definition 2.1 ([9]). An algebraic structure $\mathbb{E} = (\mathbb{E}; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if for all $\alpha, \gamma, \eta \in \mathbb{E}$ it satisfies the following conditions:
(E1) $(\mathbb{E}, \wedge, 1)$ is a commutative idempotent integral monoid,
(E2) $\alpha \sim \gamma = \gamma \sim \alpha$,
(E3) $\alpha \sim \alpha = 1$,
(E4) $\alpha \sim 1 = \alpha$,
(E5) $\alpha \leqslant \gamma \leqslant \eta$ implies $\alpha \sim \eta \leqslant \gamma \sim \eta$ and $\alpha \sim \eta \leqslant \alpha \sim \gamma$,
(E6) $\alpha \sim \gamma \leqslant (\alpha \wedge \eta) \sim (\gamma \wedge \eta)$,
(E7) $\alpha \sim \gamma \leqslant (\alpha \sim \eta) \sim (\gamma \sim \eta)$.

The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation. On an equality algebra $\mathbb{E}$ we write $\alpha \leqslant \gamma$ if and only if $\alpha \wedge \gamma = \alpha$. It is easy to see that "$\leqslant$" is a partial order relation on $\mathbb{E}$. Also, other two derived operations are defined, as the following, and we call them *implication* and *equivalence*, respectively:

$$\alpha \to \gamma = \alpha \sim (\alpha \wedge \gamma) \quad \text{and} \quad \alpha \leftrightarrow \gamma = (\alpha \to \gamma) \wedge (\gamma \to \alpha).$$

An equality algebra $\mathbb{E}$ is *bounded* if there is an element $0 \in \mathbb{E}$ such that $0 \leqslant \alpha$ for all $\alpha \in \mathbb{E}$. A *lattice equality algebra* is an equality algebra which is a lattice.

Proposition 2.2 ([9], [16], [19]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra. Then for all $\alpha, \gamma, \eta \in \mathbb{E}$, the following conditions hold:*
(i) $\alpha \to \gamma = 1$ *if and only if* $\alpha \leqslant \gamma$,
(ii) $1 \to \alpha = \alpha$, $\alpha \to 1 = 1$, *and* $\alpha \to \alpha = 1$,
(iii) $\alpha \leqslant \gamma \to \alpha$,
(iv) $\alpha \leqslant (\alpha \to \gamma) \to \gamma$,
(v) $\alpha \to (\gamma \to \eta) = \gamma \to (\alpha \to \eta)$,
(vi) $\alpha \leqslant \gamma$ *implies* $\gamma \to \eta \leqslant \alpha \to \eta$ *and* $\eta \to \alpha \leqslant \eta \to \gamma$.
If $\mathbb{E}$ *is a lattice equality algebra, then*
(vii) $\alpha \to \gamma = (\alpha \vee \gamma) \to \gamma$.

Definition 2.3 ([10]). Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$. Then $\mathbb{F}$ is called a *deductive system* or *filter* of $\mathbb{E}$ if for all $\alpha, \gamma \in \mathbb{E}$ we have
(i) $\alpha \in \mathbb{F}$ and $\alpha \leqslant \gamma$ imply $\gamma \in \mathbb{F}$;
(ii) $\alpha \in \mathbb{F}$ and $\alpha \sim \gamma \in \mathbb{F}$ imply $\gamma \in \mathbb{F}$.

Proposition 2.4 ([2], [4], [10]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$. Then $\mathbb{F}$ is a filter of $\mathbb{E}$ if and only if for all $\alpha, \gamma \in \mathbb{E}$*
(i) $1 \in \mathbb{F}$,
(ii) $\alpha \in \mathbb{F}$ *and* $\alpha \to \gamma \in \mathbb{F}$ *imply* $\gamma \in \mathbb{F}$.

The set of all filters of $\mathbb{E}$ is denoted by $\mathcal{F}(\mathbb{E})$. Clearly, $1 \in \mathbb{F}$ for all $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. A filter $\mathbb{F}$ of $\mathbb{E}$ is called a *proper filter* if $\mathbb{F} \neq \mathbb{E}$. Clearly, if $\mathbb{E}$ is a bounded equality algebra, then a filter is proper if and only if it does not contain 0. A proper filter $\mathbb{F}$ of $\mathbb{E}$ is called a *prime filter* if $\alpha \to \gamma \in \mathbb{F}$ or $\gamma \to \alpha \in \mathbb{F}$ for all $\alpha, \gamma \in \mathbb{E}$. A *maximal filter* (or ultra filter) is a proper filter of $\mathbb{E}$ that is not included in any other proper filter. The set of all prime (maximal) filters of $\mathbb{E}$ is denoted by $\mathrm{Prime}(\mathbb{E})$ $(\mathrm{Max}(\mathbb{E}))$.

Definition 2.5 ([4]). Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\theta \subseteq \mathbb{E} \times \mathbb{E}$. Then θ is called a *congruence* relation of $\mathbb{E}$ if it is an equivalence relation on $\mathbb{E}$ and if $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2) \in \theta$,

$$(\alpha_1 \wedge \alpha_2, \gamma_1 \wedge \gamma_2) \in \theta, \quad (\alpha_1 \sim \alpha_2, \gamma_1 \sim \gamma_2) \in \theta$$

for all $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{E}$.

The set of all congruences of $\mathbb{E}$ is denoted by $\mathrm{Con}(\mathbb{E})$. For any $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, a binary relation $\theta_{\mathbb{F}}$ associated by defining: $\alpha\, \theta_{\mathbb{F}}\, \gamma$ if and only if $\alpha \sim \gamma \in \mathbb{F}$. In [4], it is proved that there is a one-to-one correspondence between $\mathcal{F}(\mathbb{E})$ and $\mathrm{Con}(\mathbb{E})$. Denote $\mathbb{E}/\mathbb{F} = \mathbb{E}/\theta_{\mathbb{F}} := \{[\alpha]\colon\ \alpha \in \mathbb{E}\}$, where $[\alpha] := \{\gamma \in \mathbb{E}\colon\ (\alpha, \gamma) \in \theta_{\mathbb{F}}\}$.

Theorem 2.6 ([4]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{E}/\mathbb{F}; \bar{\wedge}, \bar{\sim}, \mathbb{F})$ is an equality algebra with the following operations:*

$$[\alpha]\bar{\wedge}[\gamma] := [\alpha \wedge \gamma], \quad [\alpha] \mathrel{\bar{\sim}} [\gamma] := [\alpha \sim \gamma]$$

for all $\alpha, \gamma \in \mathbb{E}$.

Definition 2.7 ([2]). Let $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$ such that $1 \in \mathbb{F}$. Then $\mathbb{F}$ is called a *positive implicative filter* if $\alpha \to (\gamma \to \eta) \in \mathbb{F}$ and $\alpha \to \gamma \in \mathbb{F}$ imply $\alpha \to \eta \in \mathbb{F}$ for all $\alpha, \gamma, \eta \in \mathbb{E}$.

Let $\mathcal{X} \subseteq \mathbb{E}$. The smallest filter of $\mathbb{E}$ containing $\mathcal{X}$ is called the *generated filter by $\mathcal{X}$ in* $\mathbb{E}$ and is denoted by $\langle \mathcal{X} \rangle$. Indeed, $\langle \mathcal{X} \rangle = \bigcap\limits_{\mathcal{X} \subseteq \mathbb{F} \in \mathcal{F}(\mathbb{E})} F$.

Proposition 2.8 ([15]). *Let $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$. Then*

$$\begin{aligned}\langle \mathcal{X} \rangle = \{\alpha \in \mathbb{E}\colon\ & p_1 \to (p_2 \to (\ldots \to (p_n \to \alpha) \ldots)) = 1 \\ & \text{for some } n \in \mathbb{N} \text{ and } p_1, \ldots, p_n \in \mathcal{X}\}.\end{aligned}$$

In particular, for any element $p \in \mathbb{E}$ we have

$$\langle p \rangle = \{\alpha \in \mathbb{E}\colon\ p \to^n \alpha = 1 \text{ for some } n \in \mathbb{N}\},$$

where $\alpha \to^0 \gamma = \gamma$ and $\alpha \to^n \gamma = \alpha \to (\alpha \to^{n-1} \gamma)$.

If $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $p \in \mathbb{E} \setminus \mathbb{F}$, then

$$\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E}\colon\ p \to^n \alpha \in \mathbb{F} \text{ for some } n \in \mathbb{N}\}.$$

If $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$, then

$$\begin{aligned}\langle \mathbb{F} \cup \mathbb{G} \rangle &= \{\alpha \in \mathbb{E}\colon\ g \to \alpha \in \mathbb{F} \text{ for some } g \in \mathbb{G}\} \\ &= \{\alpha \in \mathbb{E}\colon\ m \to \alpha \in \mathbb{G} \text{ for some } m \in \mathbb{F}\}.\end{aligned}$$

Proposition 2.9 ([15]). *Let $\mathbb{F}$ and $\mathbb{G}$ be two proper filters of $\mathbb{E}$. Then for all $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$ and $\alpha, p, q \in \mathbb{E}$, the following statements hold:*

(i) *if $\mathcal{X} \subseteq \mathcal{Y}$, then $\langle \mathcal{X} \rangle \subseteq \langle \mathcal{Y} \rangle$;*
(ii) *if $\mathbb{F} \subseteq \mathbb{G}$, then $\langle \mathbb{F} \cup \{\alpha\} \rangle \subseteq \langle \mathbb{G} \cup \{\alpha\} \rangle$;*
(iii) *if $p \leqslant q$, then $\langle q \rangle \subseteq \langle p \rangle$;*
(iv) *if $\mathbb{F}$ is a positive implicative filter, then $\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E}\colon\ p \to \alpha \in \mathbb{F}\}$.*

Theorem 2.10 ([15]). *The algebraic structure* $(\mathcal{F}(\mathbb{E}), \subseteq, \wedge, \vee, \{1\}, \mathbb{E})$ *is a bounded distributive complete lattice, where for any* $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$,

$$\mathbb{F} \wedge \mathbb{G} := \mathbb{F} \cap \mathbb{G}, \quad \mathbb{F} \vee \mathbb{G} := \langle \mathbb{F} \cup \mathbb{G} \rangle.$$

Note. From now on, we let $(\mathbb{E}, \sim, \wedge, 0, 1)$ or $\mathbb{E}$ be a lattice equality algebra, unless otherwise stated, where for any $\alpha, \gamma \in \mathbb{E}$, the join operation $\vee$ on $\mathbb{E}$ is defined as

$$\alpha \vee \gamma := ((\alpha \to \gamma) \to \gamma) \wedge ((\gamma \to \alpha) \to \alpha).$$

Definition 2.11 ([15]). Let $\mathbb{F}$ be a proper filter of $\mathbb{E}$. Then $\mathbb{F}$ is called a $\vee$-*irreducible* filter of $\mathbb{E}$ if $\alpha \vee \gamma \in \mathbb{F}$ implies $\alpha \in \mathbb{F}$ or $\gamma \in \mathbb{F}$ for all $\alpha, \gamma \in \mathbb{E}$.

Corollary 2.12 ([15]). *Let* $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. *Then for each* $p \notin \mathbb{F}$ *there exists a* $\vee$-*irreducible filter* $\mathbb{P}$ *containing* $\mathbb{F}$ *such that* $p \notin \mathbb{P}$.

Definition 2.13 ([7]). The algebraic structure $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if the following conditions hold for all $x, y, z \in L$:
(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(BL2) $(L, \odot, 1)$ is a commutative monoid;
(BL3) $x \odot y \leqslant z$ if and only if $x \leqslant y \to z$;
(BL4) $x \wedge y = x \odot (x \to y)$;
(BL5) $(x \to y) \vee (y \to x) = 1$.

In the bounded lattice $(L, \wedge, \vee, 0, 1)$ and given a pair of elements $a, b \in L$, if $a \wedge b = 0$ and $a \vee b = 1$, then one of a and b is called a *complement* of the other. If any $a \in L$ has its complement, then L is called a *complemented* lattice. If a lattice is both complemented and distributive, then it is called a *Boolean algebra* or a *Boolean lattice* (see [3]).

3. Relative co-annihilators

In this section, we introduce the notion of relative co-annihilators on a lattice equality algebra $\mathbb{E}$ and investigate some related properties of them. Moreover, we show that for any $\mathbb{G} \in \mathcal{F}(\mathbb{E})$, its relative pseudo complement with respect to $\mathbb{F}$ is the relative co-annihilator of $\mathbb{G}$ with respect to $\mathbb{F}$.

Definition 3.1. Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathcal{X} \subseteq \mathbb{E}$. We define a *co-annihilator of* $\mathcal{X}$ *relative to* $\mathbb{F}$ as

$$\{\alpha \in \mathbb{E} \colon \alpha \vee p \in \mathbb{F} \ \forall p \in \mathcal{X}\},$$

and denote it by $(\mathbb{F} : \mathcal{X})$. When $\mathcal{X} = \{p\}$, we denote $(\mathbb{F} : \{p\})$ by $(\mathbb{F} : p)$ for short. If $\mathbb{F} = \{1\}$, then $(\{1\} : \mathcal{X}) = \mathcal{X}^{\top} = \{\alpha \in \mathbb{E} \colon \alpha \vee p = 1 \text{ for all } p \in \mathcal{X}\}$ and $(\{1\} : p) = p^{\top}$. For more details, see [15].

E x a m p l e 3.2. Let $\mathbb{E} = \{0, p, q, r, s, 1\}$ be a set, where $0 \leqslant p \leqslant s \leqslant 1$ and $0 \leqslant q \leqslant r \leqslant s \leqslant 1$. Define the operation "$\sim$" on $\mathbb{E}$ as follows:

$\sim$	0	p	q	r	s	1
0	1	r	p	p	0	0
p	r	1	0	0	p	p
q	p	0	1	s	r	q
r	p	0	s	1	r	r
s	0	p	r	r	1	s
1	0	p	q	r	s	1

$\rightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	r	1	r	r	1	1
q	p	p	1	1	1	1
r	p	p	s	1	1	1
s	0	p	r	r	1	1
1	0	p	q	r	s	1

Then $(\mathbb{E}, \sim, \wedge, 1)$ is an equality algebra. Clearly, $\mathbb{F} = \{s, 1\} \in \mathcal{F}(\mathbb{E})$. If $\mathcal{X} = \{p, s\}$ and $\mathcal{Y} = \{r\}$, then $(\mathbb{F} : \mathcal{X}) = \{q, r, s, 1\}$ and $(\mathbb{F} : r) = \{p, s, 1\}$. In addition, $\mathcal{X}^{\top} = \{1\} = r^{\top}$.

Proposition 3.3. *Let $p, q \in \mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then the following statements hold:*

(i) *If $p \leqslant q$, then $(\mathbb{F} : p) \subseteq (\mathbb{F} : q)$.*

(ii) *If $p \in \mathbb{F}$, then $(\mathbb{F} : p) = \mathbb{E}$. The converse is true when $\mathbb{E}$ is bounded.*

(iii) *$(\mathbb{F} : p) \cap (\mathbb{G} : p) = (\mathbb{F} \cap \mathbb{G} : p)$ and $(\mathbb{F} : p) \cup (\mathbb{G} : p) = (\mathbb{F} \cup \mathbb{G} : p)$.*

(iv) *$(\mathbb{F} : p \wedge q) \subseteq (\mathbb{F} : p \vee q)$.*

(v) *$(\mathbb{F} : p) \cup (\mathbb{F} : q) \subseteq (\mathbb{F} : p \vee q)$. If p, q are comparable, then the converse is true.*

(vi) *$((\mathbb{F} : p) : q) = ((\mathbb{F} : q) : p) = (\mathbb{F} : p \vee q)$.*

P r o o f. (i) Let $p \leqslant q$ and $\alpha \in (\mathbb{F} : p)$. Then $\alpha \vee p \in \mathbb{F}$ and since $\alpha \vee p \leqslant \alpha \vee q$, we get $\alpha \vee q \in \mathbb{F}$. Thus $\alpha \in (\mathbb{F} : q)$ and so, $(\mathbb{F} : p) \subseteq (\mathbb{F} : q)$.

(ii) Let $p \in \mathbb{F}$. Then for all $\alpha \in \mathbb{E}$ we have $p \leqslant p \vee \alpha$ and so, $p \vee \alpha \in \mathbb{F}$. Hence, $\alpha \in (\mathbb{F} : p)$, which means $\mathbb{E} \subseteq (\mathbb{F} : p)$. On the other hand, we always have $(\mathbb{F} : p) \subseteq \mathbb{E}$. Therefore, $(\mathbb{F} : p) = \mathbb{E}$. Now, let $\mathbb{E}$ be bounded and $\mathbb{E} = (\mathbb{F} : p)$. Then $0 \in (\mathbb{F} : p)$ and so, $p \vee 0 = p \in \mathbb{F}$.

(iii) $\alpha \in (\mathbb{F} : p) \cap (\mathbb{G} : p)$ if and only if $\alpha \vee p \in \mathbb{F} \cap \mathbb{G}$ if and only if $\alpha \in (\mathbb{F} \cap \mathbb{G} : p)$. Similarly, the next one is true.

(iv) Since $p \wedge q \leqslant p \vee q$ and (i), we have $(\mathbb{F} : p \wedge q) \subseteq (\mathbb{F} : p \vee q)$.

(v) If $\alpha \in (\mathbb{F} : p) \cup (\mathbb{F} : q)$, then $\alpha \vee p \in \mathbb{F}$ or $\alpha \vee q \in \mathbb{F}$. From $p, q \leqslant p \vee q$, we get $\alpha \vee p$, $\alpha \vee q \leqslant \alpha \vee (p \vee q)$ and by $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $\alpha \in (\mathbb{F} : p \vee q)$. Conversely, let p, q be comparable and $\alpha \in (\mathbb{F} : p \vee q)$. Since $p \leqslant q$ or $q \leqslant p$, we get $\alpha \vee q = \alpha \vee (p \vee q) \in \mathbb{F}$ or $\alpha \vee p = \alpha \vee (p \vee q) \in \mathbb{F}$. Hence, $\alpha \in (\mathbb{F} : p) \cup (\mathbb{F} : q)$.

(vi) From $\alpha \vee (p \vee q) = (\alpha \vee p) \vee q = (\alpha \vee q) \vee p$, the proof is obvious. $\square$

The other sides of inclusions of Proposition 3.3 (iv) and (v) are not true, in general.

Example 3.4. Let $(\mathbb{E}, \wedge, \sim, 1)$ be as in Example 3.2 and $\mathbb{F} = \{d, 1\}$. Then $(\mathbb{F} : a) = \{b, c, d, 1\}$, $(\mathbb{F} : b) = \{a, d, 1\}$. Since $a \wedge b = 0$ and $a \vee b = d$, we get

$$\mathbb{E} = (\mathbb{F} : d) = (\mathbb{F} : a \vee b) \nsubseteq (\mathbb{F} : a \wedge b) = (\mathbb{F} : 0) = \mathbb{F}.$$

Also, $\mathbb{E} = (\mathbb{F} : a \vee b) \nsubseteq (\mathbb{F} : a) \cup (\mathbb{F} : b) = \{b, c, d, 1\} \cup \{a, d, 1\} = \{a, b, c, d, 1\}$.

Proposition 3.5. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathcal{X}) \in \mathcal{F}(\mathbb{E})$.*

Proof. Let $p \in \mathcal{X}$. Since $1 \vee p = 1 \in \mathbb{F}$, we have $1 \in (\mathbb{F} : \mathcal{X})$. If $\alpha, \alpha \to \gamma \in (\mathbb{F} : \mathcal{X})$, then $\alpha \vee p \in \mathbb{F}$ and $(\alpha \to \gamma) \vee p \in \mathbb{F}$, for all $p \in \mathcal{X}$. Suppose $\eta := \gamma \vee p$. Since $p, \gamma \leqslant \eta$, we get $p \leqslant \eta \leqslant \alpha \to \eta$ and $\alpha \to \gamma \leqslant \alpha \to \eta$ by Proposition 2.2 (iii) and (vi), respectively. Hence, $(\alpha \to \gamma) \vee p \leqslant (\alpha \to \eta) \vee p = \alpha \to \eta$. Since $(\alpha \to \gamma) \vee p \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \to \eta \in \mathbb{F}$. Moreover, $p \leqslant \eta$, then $\alpha \vee p \leqslant \alpha \vee \eta$. From $\alpha \vee p \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \vee \eta \in \mathbb{F}$. Now, by Proposition 2.2 (vii), $\alpha \to \eta = (\alpha \vee \eta) \to \eta$. In addition, $\alpha \to \eta, \alpha \vee \eta \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we obtain $\eta \in \mathbb{F}$. Thus, $\gamma \vee p \in \mathbb{F}$, i.e., $\gamma \in (\mathbb{F} : \mathcal{X})$. Therefore $(\mathbb{F} : \mathcal{X}) \in \mathcal{F}(\mathbb{E})$. $\square$

Proposition 3.6. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then the following statements hold:*

(i) $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X})$.

(ii) $(\mathbb{F} : \mathbb{E}) = \mathbb{F}$ *and* $(\mathbb{F} : \mathbb{F}) = \mathbb{E}$.

(iii) $(\mathbb{F} : (\mathbb{F} : \mathbb{E})) = \mathbb{E}$ *and* $(\mathbb{F} : (\mathbb{F} : \mathbb{F})) = \mathbb{F}$.

(iv) *If* $\mathcal{X} \subseteq \mathcal{Y}$, *then* $(\mathbb{F} : \mathcal{Y}) \subseteq (\mathbb{F} : \mathcal{X})$.

(v) *If* $\mathbb{F} \subseteq \mathbb{G}$, *then* $(\mathbb{F} : \mathcal{X}) \subseteq (\mathbb{G} : \mathcal{X})$. *In particular,* $\mathbb{G}^{\top} \subseteq (\mathbb{F} : \mathbb{G})$.

(vi) $(\mathbb{F} : \mathcal{X}) = \mathbb{E}$ *if and only if* $\mathcal{X} \subseteq \mathbb{F}$.

(vii) $\left(\mathbb{F} : \bigcup\limits_{i \in \Delta} \mathcal{X}_i\right) = \bigcap\limits_{i \in \Delta} (\mathbb{F} : \mathcal{X}_i)$.

(viii) $(\mathbb{F} : \mathcal{X}) = \bigcap\limits_{p \in \mathcal{X}} (\mathbb{F} : p)$.

(ix) $\left(\bigcap\limits_{i \in \Delta} \mathbb{F}_i : \mathcal{X}\right) = \bigcap\limits_{i \in \Delta} (\mathbb{F}_i : \mathcal{X})$.

(x) $(\mathbb{F} : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \setminus \mathbb{F})$.

(xi) *If* $\mathbb{F} \subseteq \mathcal{X}$, *then* $\mathcal{X} \cap (\mathbb{F} : \mathcal{X}) = \mathbb{F}$.

(xii) $(\mathbb{F} : \mathcal{X}) \cap (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = \mathbb{F}$.

(xiii) $\mathcal{X} \subseteq (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$.

(xiv) $(\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathcal{X}))) = (\mathbb{F} : \mathcal{X})$.

(xv) $((\mathbb{F} : \mathcal{X}) : \mathcal{Y}) = ((\mathbb{F} : \mathcal{Y}) : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$, *where* $\mathcal{X} \vee \mathcal{Y} = \{p \vee q \colon p \in \mathcal{X}, q \in \mathcal{Y}\}$.

Proof. (i) Let $f \in F$ and $p \in \mathcal{X}$. Then $f \leqslant p \vee f$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, so $p \vee f \in \mathbb{F}$. Hence, $f \in (\mathbb{F} : \mathcal{X})$. Therefore, $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X})$.

(ii) By (i), $\mathbb{F} \subseteq (\mathbb{F} : \mathbb{E})$. On the other hand, if $\alpha \in (\mathbb{F} : \mathbb{E})$, then for all $p \in \mathbb{E}$, $\alpha \vee p \in \mathbb{F}$. Suppose $p = \alpha$, then $\alpha = \alpha \vee \alpha \in \mathbb{F}$ and so $\alpha \in \mathbb{F}$, i.e., $(\mathbb{F} : \mathbb{E}) \subseteq \mathbb{F}$.

Therefore $(\mathbb{F} : \mathbb{E}) = \mathbb{F}$. Also, for any $\alpha \in \mathbb{E}$ and $f \in \mathbb{F}$, since $f \leqslant \alpha \vee f$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ we have $\alpha \vee f \in \mathbb{F}$ and so $\alpha \in (\mathbb{F} : \mathbb{F})$. Hence, $\mathbb{E} \subseteq (\mathbb{F} : \mathbb{F}) \subseteq \mathbb{E}$. Therefore $(\mathbb{F} : \mathbb{F}) = \mathbb{E}$.

(iii) By (ii), $(\mathbb{F} : (\mathbb{F} : \mathbb{E})) = (\mathbb{F} : \mathbb{F}) = \mathbb{E}$ and $(\mathbb{F} : (\mathbb{F} : \mathbb{F})) = (\mathbb{F} : \mathbb{E}) = \mathbb{F}$.

(iv) Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\alpha \in (\mathbb{F} : \mathcal{Y})$. Then for any $q \in \mathcal{Y}$ we have $\alpha \vee q \in \mathbb{F}$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we get $\alpha \in (\mathbb{F} : \mathcal{X})$. Therefore $(\mathbb{F} : \mathcal{Y}) \subseteq (\mathbb{F} : \mathcal{X})$.

(v) Let $\alpha \in (\mathbb{F} : \mathcal{X})$ and $p \in \mathcal{X}$. Then $\alpha \vee p \in \mathbb{F} \subseteq \mathbb{G}$. Hence, $(\mathbb{F} : \mathcal{X}) \subseteq (\mathbb{G} : \mathcal{X})$. Specially, from $\{1\} \subseteq \mathbb{F}$ we have $\mathbb{G}^{\top} = (\{1\} : \mathbb{G}) \subseteq (\mathbb{F} : \mathbb{G})$.

(vi) Let $(\mathbb{F} : \mathcal{X}) = \mathbb{E}$ and $p \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathbb{E}$, clearly $p \in (\mathbb{F} : \mathcal{X})$ and so $p = p \vee p \in \mathbb{F}$. Therefore $\mathcal{X} \subseteq \mathbb{F}$. Conversely, let $\mathcal{X} \subseteq \mathbb{F}$ and $\alpha \in \mathbb{E}$. Then for all $p \in \mathcal{X}$, $p \in \mathbb{F}$ and $p \leqslant p \vee \alpha$. Since $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $p \vee \alpha \in \mathbb{F}$ and so $\alpha \in (\mathbb{F} : \mathcal{X})$. Therefore $\mathbb{E} = (\mathbb{F} : \mathcal{X})$.

(vii) Since $\mathcal{X}_i \subseteq \bigcup_{i\in\Delta} \mathcal{X}_i$ for all $i \in \Delta$ by (iv), $\left(\mathbb{F} : \bigcup_{i\in\Delta} \mathcal{X}_i\right) \subseteq (\mathbb{F} : \mathcal{X}_i)$ for all $i \in \Delta$. Hence, $\left(\mathbb{F} : \bigcup_{i\in\Delta} \mathcal{X}_i\right) \subseteq \bigcap_{i\in\Delta} (\mathbb{F} : \mathcal{X}_i)$. Conversely, let $\alpha \in \bigcap_{i\in\Delta} (\mathbb{F} : \mathcal{X}_i)$ and $p \in \bigcup_{i\in\Delta} \mathcal{X}_i$. Then there exists $j \in \Delta$ such that $p \in \mathcal{X}_j$. Thus, $p \vee \alpha \in \mathbb{F}$ and so $\alpha \in \left(\mathbb{F} : \bigcup_{i\in\Delta} \mathcal{X}_i\right)$. Therefore, $\left(\mathbb{F} : \bigcup_{i\in\Delta} \mathcal{X}_i\right) = \bigcap_{i\in\Delta} (\mathbb{F} : \mathcal{X}_i)$.

(viii) This is the result of (vii).

(ix) Since for all $i \in \Delta$, $\bigcap_{i\in\Delta} \mathbb{F}_i \subseteq \mathbb{F}_i$, by (v), we get $\left(\bigcap_{i\in\Delta} \mathbb{F}_i : \mathcal{X}\right) \subseteq (\mathbb{F}_i : \mathcal{X})$ and so $\left(\bigcap_{i\in\Delta} \mathbb{F}_i : \mathcal{X}\right) \subseteq \bigcap_{i\in\Delta} (\mathbb{F}_i : \mathcal{X})$. Conversely, let $\alpha \in \bigcap_{i\in\Delta} (\mathbb{F}_i : \mathcal{X})$ and $p \in \mathcal{X}$. Then for all $i \in \Delta$, $\alpha \vee p \in \mathbb{F}_i$ and so $\alpha \vee p \in \bigcap_{i\in\Delta} \mathbb{F}_i$. Hence, $\alpha \in \left(\bigcap_{i\in\Delta} \mathbb{F}_i : \mathcal{X}\right)$. Therefore $\left(\bigcap_{i\in\Delta} \mathbb{F}_i : \mathcal{X}\right) = \bigcap_{i\in\Delta} (\mathbb{F}_i : \mathcal{X})$.

(x) We know $\mathcal{X} = (\mathcal{X} \setminus \mathbb{F}) \cup (\mathcal{X} \cap \mathbb{F})$. Since $\mathcal{X} \cap \mathbb{F} \subseteq \mathbb{F}$, by (vi), we get $(\mathbb{F} : \mathcal{X} \cap \mathbb{F}) = \mathbb{E}$. So by (vii), we have

$$\begin{aligned}(\mathbb{F} : \mathcal{X}) &= (\mathbb{F} : (\mathcal{X} \setminus \mathbb{F}) \cup (\mathcal{X} \cap \mathbb{F})) = (\mathbb{F} : \mathcal{X} \setminus \mathbb{F}) \cap (\mathbb{F} : \mathcal{X} \cap \mathbb{F}) \\ &= (\mathbb{F} : \mathcal{X} \setminus \mathbb{F}) \cap \mathbb{E} = (\mathbb{F} : \mathcal{X} \setminus \mathbb{F}).\end{aligned}$$

(xi) Let $\mathbb{F} \subseteq \mathcal{X}$. By (i), $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X})$ and so $\mathbb{F} \subseteq \mathcal{X} \cap (\mathbb{F} : \mathcal{X})$. Conversely, let $\alpha \in \mathcal{X} \cap (\mathbb{F} : \mathcal{X})$. Then $\alpha \in \mathcal{X}$ and for all $p \in \mathcal{X}$, we have $\alpha \vee p \in \mathbb{F}$. Suppose $p = \alpha$, then $\alpha = \alpha \vee \alpha \in \mathbb{F}$. Hence, $\mathcal{X} \cap (\mathbb{F} : \mathcal{X}) \subseteq \mathbb{F}$. Therefore $\mathcal{X} \cap (\mathbb{F} : \mathcal{X}) = \mathbb{F}$.

(xii) By using (i) twice, $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X}) \cap (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$. For the other side of inclusion, let $\alpha \in (\mathbb{F} : \mathcal{X}) \cap (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$. Then $\alpha \in (\mathbb{F} : \mathcal{X})$ and from $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$ we get $\alpha \vee \gamma \in \mathbb{F}$ for all $\gamma \in (\mathbb{F} : \mathcal{X})$. In particular, when $\gamma := \alpha$, we have $\alpha = \alpha \vee \alpha \in \mathbb{F}$ and so $(\mathbb{F} : \mathcal{X}) \cap (\mathbb{F} : (\mathbb{F} : \mathcal{X})) \subseteq \mathbb{F}$.

(xiii) Let $p \in \mathcal{X}$ and $\alpha \in (\mathbb{F} : \mathcal{X})$. Then $\alpha \vee p \in \mathbb{F}$. Hence, $p \in (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$. Therefore $\mathcal{X} \subseteq (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$.

(xiv) Suppose $R = (\mathbb{F} : \mathcal{X})$. Then by (xiii), $R \subseteq (\mathbb{F} : (\mathbb{F} : R))$. Conversely, by (xiii), $\mathcal{X} \subseteq (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = (\mathbb{F} : R)$, and by (iv) we get $(\mathbb{F} : (\mathbb{F} : R)) \subseteq (\mathbb{F} : \mathcal{X}) = R$. Therefore $(\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathcal{X}))) = (\mathbb{F} : \mathcal{X})$.

(xv) Let $\alpha \in ((\mathbb{F} : \mathcal{X}) : \mathcal{Y})$. Then for all $q \in \mathcal{Y}$ and $p \in \mathcal{X}$, $(\alpha \vee b) \vee a \in \mathbb{F}$. If $\eta \in \mathcal{X} \vee \mathcal{Y}$, then there are $p \in \mathcal{X}$ and $b \in \mathcal{Y}$ such that $\eta = p \vee q$ and so $\alpha \vee \eta = \alpha \vee (p \vee q) = (\alpha \vee q) \vee p \in \mathbb{F}$. Thus $\alpha \in (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. The converse is clear. Hence, $((\mathbb{F} : \mathcal{X}) : \mathcal{Y}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. Similarly, we get $((\mathbb{F} : \mathcal{Y}) : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. Therefore the proof is complete. □

The converse of Proposition 3.6 (i) and (xiii) is not true, in general.

Example 3.7. Let $\mathbb{E}$, $\mathbb{F} = \{d, 1\}$ and $\mathcal{X} = \{a, d\}$ be as in Example 3.2. Then $\mathbb{F} \subsetneqq (\mathbb{F} : \mathcal{X}) = \{b, c, d, 1\}$. Moreover, $\mathcal{X} \subsetneqq (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = \{a, d, 1\}$.

Proposition 3.8. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \langle\mathcal{X}\rangle) = (\mathbb{F} : \mathcal{X})$.*

Proof. Since $\mathcal{X} \subseteq \langle\mathcal{X}\rangle$, by Proposition 3.6 (iv), $(\mathbb{F} : \langle\mathcal{X}\rangle) \subseteq (\mathbb{F} : \mathcal{X})$. For the converse, let $\alpha \in (\mathbb{F} : \mathcal{X})$ and $\alpha \notin (\mathbb{F} : \langle\mathcal{X}\rangle)$. Then there exists $p \in \langle\mathcal{X}\rangle$ such that $\alpha \vee p \notin \mathbb{F}$. By Proposition 2.8, there are $p_1, \ldots, p_n \in \mathcal{X}$ such that $p_1 \to (\ldots (p_n \to p) \ldots) = 1$ for some $n \in \mathbb{N}$. Moreover, since $\alpha \vee p \notin \mathbb{F}$, by Corollary 2.12 (i), there is a $\vee$-irreducible filter $\mathbb{P}$ of $\mathbb{E}$ containing $\mathbb{F}$ such that $\alpha \vee p \notin \mathbb{P}$. Also, since $\alpha \in (\mathbb{F} : \mathcal{X})$, we get $p_i \vee \alpha \in \mathbb{F} \subseteq \mathbb{P}$ for all $1 \leqslant i \leqslant n$. Hence, $\alpha \in \mathbb{P}$ or $p_i \in \mathbb{P}$ for all $1 \leqslant i \leqslant n$. If $\alpha \in \mathbb{P}$, then by $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we have $\alpha \vee p \in \mathbb{P}$, which is a contradiction. Thus, for any $1 \leqslant i \leqslant n$, $p_i \in \mathbb{P}$ and so by $p_1 \to (\ldots (p_n \to p) \ldots) = 1 \in \mathbb{P}$ and $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we get $p \in \mathbb{P}$. Since $p \leqslant p \vee \alpha$ and $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we get $p \vee \alpha \in \mathbb{P}$, which is a contradiction. Thus, $\alpha \in (\mathbb{F} : \langle\mathcal{X}\rangle)$. Therefore $(\mathbb{F} : \langle\mathcal{X}\rangle) = (\mathbb{F} : \mathcal{X})$. □

Proposition 3.9. *Let $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$. Then*
(i) $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} \subseteq \mathbb{F}$;
(ii) $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$ *if and only if* $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$.

Proof. (i) It is clear.

(ii) Let $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$ and $\alpha \in \mathbb{H}$. Since for any $g \in \mathbb{G}$, $\alpha, g \leqslant \alpha \vee g$ and $\mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \vee g \in \mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$. Hence, $\alpha \vee g \in \mathbb{F}$ and so $\alpha \in (\mathbb{F} : \mathbb{G})$. Thus, $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Conversely, let $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Then by (i), $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{G}) \subseteq \mathbb{F}$. Therefore $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$. □

Proposition 3.10. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathcal{X}) = \{\alpha \in \mathbb{E} \colon \langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}\}$.*

P r o o f. Suppose $B = \{\alpha \in \mathbb{E}\colon \langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}\}$. Let $\alpha \in B$. Then $\langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}$. By Proposition 3.9 (ii), we get $\langle\alpha\rangle \subseteq (\mathbb{F} : \langle\mathcal{X}\rangle)$. Since $\alpha \in \langle\alpha\rangle$ and by Proposition 3.8, we have $\alpha \in (\mathbb{F} : \mathcal{X})$. Hence, $B \subseteq (\mathbb{F} : \mathcal{X})$. Conversely, let $\alpha \in (\mathbb{F} : \mathcal{X})$. Then by Proposition 3.8, $\alpha \in (\mathbb{F} : \langle\mathcal{X}\rangle)$ and so $\langle\alpha\rangle \subseteq (\mathbb{F} : \langle\mathcal{X}\rangle)$. Now, by Proposition 3.9 (ii), we have $\langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}$. Hence, $(\mathbb{F} : \mathcal{X}) \subseteq B$. Therefore $(\mathbb{F} : \mathcal{X}) = B$. □

Proposition 3.11. *Let $p \in \mathbb{E}$ and $\mathbb{F}$ be a positive implicative filter of $\mathbb{E}$. Then $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle = \mathbb{F}$.*

P r o o f. We know $\mathbb{F} \subseteq \langle\mathbb{F} \cup \{p\}\rangle$ and by Proposition 3.6 (i), we get $\mathbb{F} \subseteq (\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle$. Conversely, let $\alpha \in (\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle$. Then $\alpha \vee p \in \mathbb{F}$ and by Proposition 2.9 (iv), $p \to \alpha \in \mathbb{F}$. Also, by Proposition 2.2 (vii), we have $p \to \alpha = (p \vee \alpha) \to \alpha$ and since $p \to \alpha, p \vee \alpha \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \in \mathbb{F}$. Hence, $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle \subseteq \mathbb{F}$. Therefore $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle = \mathbb{F}$. □

Proposition 3.12. *Let $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\emptyset \neq \mathbb{G} \subseteq \mathbb{E}$. If $\mathbb{G}$ is a chain such that $\mathbb{G} \nsubseteq \mathbb{F}$, then $(\mathbb{F} : \mathbb{G})$ is a $\vee$-irreducible filter of $\mathbb{E}$.*

P r o o f. Let $\mathbb{G}$ be a chain and $\mathbb{G} \nsubseteq \mathbb{F}$. Then by Proposition 3.6 (vi), we get $(\mathbb{F} : \mathbb{G}) \neq \mathbb{E}$. If $\alpha \vee \gamma \in (\mathbb{F} : \mathbb{G})$, then $(\alpha \vee \gamma) \vee g \in \mathbb{F}$ for all $g \in \mathbb{G}$. On the contrary, let $\alpha, \gamma \notin (\mathbb{F} : \mathbb{G})$. Then there are $g_1, g_2 \in \mathbb{G}$ such that $\alpha \vee g_1 \notin \mathbb{F}$ and $\gamma \vee g_2 \notin \mathbb{F}$. Suppose $g := g_1 \wedge g_2$. Since $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and it is closed under $\wedge$-operation, we get $g \in \mathbb{G}$ and so $\alpha \vee g$, $\gamma \vee g \in \mathbb{G}$. Since $\mathbb{G}$ is a chain, without loss of generality, suppose $\alpha \vee g \leqslant \gamma \vee g$. Hence, we have

$$(\alpha \vee \gamma) \vee g = (\alpha \vee g) \vee \gamma \leqslant (\gamma \vee g) \vee \gamma = \gamma \vee g \leqslant \gamma \vee g_2.$$

Since $(\alpha \vee \gamma) \vee g \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $\gamma \vee g_2 \in \mathbb{F}$, which is a contradiction. Therefore $(\mathbb{F} : \mathbb{G})$ is a $\vee$-irreducible filter of $\mathbb{E}$. □

Proposition 3.13. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$ such that $\mathbb{F} \subseteq \mathbb{P}$. Then $\mathcal{X} \nsubseteq \mathbb{P}$ implies $(\mathbb{F} : \mathcal{X}) \subseteq \mathbb{P}$ for any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$.*

P r o o f. Let $\mathcal{X} \nsubseteq \mathbb{P}$. Then there exists $p \in \mathcal{X}$ such that $p \notin \mathbb{P}$. Also, if $\alpha \in (\mathbb{F} : \mathcal{X})$, then $\alpha \vee p \in \mathbb{F} \subseteq \mathbb{P}$. Since $p \notin \mathbb{P}$ and $\mathbb{P}$ is a $\vee$-irreducible filter, we get $\alpha \in \mathbb{P}$. Hence, $(\mathbb{F} : \mathcal{X}) \subseteq \mathbb{P}$. □

Corollary 3.14. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$. Then $\mathcal{X} \nsubseteq \mathbb{P}$ implies $(\mathbb{P} : \mathcal{X}) = \mathbb{P}$ for any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$.*

P r o o f. By Proposition 3.6 (i), we have $\mathbb{P} \subseteq (\mathbb{P} : \mathcal{X})$. Then by Proposition 3.13, the proof is complete. □

Theorem 3.15. *Let $\mathbb{P} \in \mathbb{F}(\mathbb{E})$. Then $\mathbb{P}$ is a $\vee$-irreducible filter of $\mathbb{E}$ if and only if $(\mathbb{P} : \alpha) = \mathbb{P}$ for any $\alpha \notin \mathbb{P}$.*

Proof. Let $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$ and $\alpha \notin \mathbb{P}$. By Corollary 3.14, it is enough to set $\mathcal{X} = \{\alpha\}$ and so the proof is clear. Conversely, let $\alpha \vee \gamma \in \mathbb{P}$ and $\alpha \notin \mathbb{P}$. By hypothesis, $(\mathbb{P} : \alpha) = \mathbb{P}$. Moreover, since $\alpha \vee \gamma \in \mathbb{P}$, we get $\gamma \in (\mathbb{P} : \alpha) = \mathbb{P}$. Therefore $\mathbb{P}$ is a $\vee$-irreducible filter of $\mathbb{E}$. $\square$

Definition 3.16 ([18]). In a lattice L with bottom element 0, an element $x \in L$ is said to have a pseudo-complement element if there exists the greatest element $x^* \in L$, disjoint from x, with the property that $x \wedge x^* = 0$. More formally, $x^* = \max\{y \in L\colon\ x \wedge y = 0\}$. The lattice L itself is called a *pseudo-complemented lattice* if every element of L has a pseudo-complement element. A *relative pseudo-complement* of a with respect to b, is a maximal element c such that $a \wedge c \leqslant b$.

Proposition 3.17. *Let $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathbb{G})$ is a relative pseudo complement of $\mathbb{G}$ with respect to $\mathbb{F}$ in the lattice $(\mathcal{F}(\mathbb{E}), \subseteq)$, where $\mathbb{F} \wedge \mathbb{G} := \mathbb{F} \cap \mathbb{G}$, $\mathbb{F} \vee \mathbb{G} := \langle \mathbb{F} \cup \mathbb{G} \rangle$.*

Proof. By Proposition 3.9 (i), $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} \subseteq \mathbb{F}$. It is enough to show that $(\mathbb{F} : \mathbb{G})$ is the greatest one. For this, suppose that there is $\mathbb{H} \in \mathcal{F}(\mathbb{E})$ such that $\mathbb{H} \cap \mathbb{G} \subseteq \mathbb{F}$ and let $\alpha \in \mathbb{H}$. Then for all $g \in \mathbb{G}$, $\alpha, g \leqslant \alpha \vee g$ and so $\alpha \vee g \in \mathbb{H} \cap \mathbb{G} \subseteq \mathbb{F}$. Thus, $\alpha \vee g \in \mathbb{F}$ for all $g \in \mathbb{G}$, i.e., $\alpha \in (\mathbb{F} : \mathbb{G})$. Hence, $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Therefore $(\mathbb{F} : \mathbb{G})$ is a relative pseudo complement of $\mathbb{G}$ with respect to $\mathbb{F}$ in the lattice $(\mathcal{F}(\mathbb{E}), \subseteq)$. $\square$

Remark 3.18. Let $\mathbb{F}$ be a proper filter of $\mathbb{E}$ and $\mathbb{H} \in \mathcal{F}(\mathbb{E}/\mathbb{F})$. If we take $\mathbb{G} := \{x \in \mathbb{E}\colon\ [x] \in \mathbb{H}\}$, then it is easy to see that $\mathbb{F} \subseteq \mathbb{G}$ and $\mathbb{H} = \mathbb{G}/\mathbb{F}$. So, any filter of quotient equality algebra $\mathbb{E}/\mathbb{F}$ has the form $\mathbb{G}/\mathbb{F}$ such that $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{F} \subseteq \mathbb{G}$. That is

$$\mathcal{F}(\mathbb{E}/\mathbb{F}) = \{\mathbb{G}/\mathbb{F}\colon\ \mathbb{F} \subseteq \mathbb{G} \in \mathcal{F}(\mathbb{E})\}.$$

Proposition 3.19. *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(\mathbb{E})$ such that $\mathbb{F} \subseteq \mathbb{G}$. Then $(\mathbb{G} : \mathcal{X})/\mathbb{F} \in \mathbb{F}(\mathbb{E}/\mathbb{F})$.*

Proof. By Proposition 3.6 (i) and (v), we have $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X}) \subseteq (\mathbb{G} : \mathcal{X})$. Then by Remark 3.18, we get $(\mathbb{G} : \mathcal{X})/\mathbb{F} \in \mathbb{F}(\mathbb{E}/\mathbb{F})$. $\square$

Corollary 3.20. *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(\mathbb{E})$ and $\mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}$ such that $\mathbb{F} \subseteq \mathbb{G}$. Then $((\mathbb{G}/\mathbb{F}) : (\mathcal{X}/\mathbb{F})) = (\mathbb{G} : \mathcal{X})/\mathbb{F}$.*

Proof. We have

$$\left(\frac{\mathbb{G}}{\mathbb{F}}:\frac{\mathcal{X}}{\mathbb{F}}\right)=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [p]\vee[\alpha]\in\frac{\mathbb{G}}{\mathbb{F}}\ \forall[\alpha]\in\frac{\mathcal{X}}{\mathbb{F}}\right\}=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [p\vee\alpha]\in\frac{\mathbb{G}}{\mathbb{F}}\ \forall\alpha\in\mathcal{X}\right\}$$
$$=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ p\vee\alpha\in\mathbb{G}\ \forall\alpha\in\mathcal{X}\right\}=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ p\in(\mathbb{G}:\mathcal{X})\right\}=\frac{(\mathbb{G}:\mathcal{X})}{\mathbb{F}}.$$

□

Proposition 3.21. *Let* $\mathbb{F}\in\mathcal{F}(\mathbb{E})$ *and* $\emptyset\neq\mathcal{X}\subseteq\mathbb{E}$*. Then*

(i) $(\mathcal{X}/\mathbb{F})^{\top}=(\mathbb{F}:\mathcal{X})/\mathbb{F}$*, particularly,* $[\alpha]^{\top}=(\mathbb{F}:\alpha)/\mathbb{F}$ *for any* $[\alpha]\in\mathbb{E}/\mathbb{F}$*,*

(ii) $(\mathcal{X}/\mathbb{F})^{\top\top}=(\mathbb{F}:(\mathbb{F}:\mathcal{X}))/\mathbb{F}$*.*

Proof. (i)

$$\left(\frac{\mathcal{X}}{\mathbb{F}}\right)^{\top}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [\alpha]\vee[p]=[1]\ \forall[p]\in\frac{\mathcal{X}}{\mathbb{F}}\right\}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [\alpha\vee p]=\mathbb{F}\ \forall p\in\mathcal{X}\right\}$$
$$=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ \alpha\vee p\in\mathbb{F}\ \forall p\in\mathcal{X}\right\}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ \alpha\in(\mathbb{F}:\mathcal{X})\right\}=\frac{(\mathbb{F}:\mathcal{X})}{\mathbb{F}}.$$

Specially, suppose $\mathcal{X}=\{x\}$, then $[\alpha]^{\top}=(\mathbb{F}:\alpha)/\mathbb{F}$.

(ii) By (i), we have $(\mathcal{X}/\mathbb{F})^{\top\top}=((\mathcal{X}/\mathbb{F})^{\top})^{\top}=((\mathbb{F}:\mathcal{X})/\mathbb{F})^{\top}=(\mathbb{F}:(\mathbb{F}:\mathcal{X}))/\mathbb{F}$. □

Definition 3.22. Let $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$. Then $\mathbb{G}$ is called $\mathbb{F}$-*involutive* if $\mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. Also, if any $\mathbb{G}\in\mathcal{F}(\mathbb{E})$ is $\mathbb{F}$-involutive, then $\mathbb{E}$ is called an *involuntary* equality algebra relative to $\mathbb{F}$. The set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ is denoted by $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Indeed, $\mathcal{S}_{\mathbb{F}}(\mathbb{E})=\{\mathbb{G}\in\mathcal{F}(\mathbb{E})\colon\ \mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\}$.

Example 3.23. Let $\mathbb{E}$ be the equality algebra as in Example 3.2, $\mathbb{F}=\{s,1\}$ and $\mathbb{G}=\{p,s,1\}$. Obviously, $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$ and $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))=\mathbb{G}$. Thus, $\mathbb{G}$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}$.

Proposition 3.24. *Let* $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$*. If* $\mathbb{F}\subseteq\mathbb{G}$ *and* $\mathbb{G}^{\top\top}=G$*, then* $\mathbb{G}\in\mathcal{S}_{\mathbb{F}}(\mathbb{E})$*.*

Proof. By Proposition 3.6 (xiii), we have $\mathbb{G}\subseteq(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. For the converse, let $g\notin\mathbb{G}$ so $g\notin\mathbb{G}^{\top\top}$. Thus, there exists $\alpha\in\mathbb{G}^{\top}$ such that $g\vee\alpha\neq1$. Since $\alpha\leqslant\alpha\vee g$ and $\alpha\in\mathbb{G}^{\top}\in\mathcal{F}(\mathbb{E})$, then $\alpha\vee g\in\mathbb{G}^{\top}$. By Proposition 3.6 (v), $\mathbb{G}^{\top}\subseteq(\mathbb{F}:\mathbb{G})$ and so $\alpha\vee g\in(\mathbb{F}:\mathbb{G})$. Moreover, $1\neq\alpha\vee g\in\mathbb{G}^{\top}$ and from $\mathbb{G}\cap\mathbb{G}^{\top}=\{1\}$ we have $\alpha\vee g\notin\mathbb{G}$. Since $\mathbb{F}\subseteq\mathbb{G}$, we get $\alpha\vee g\notin\mathbb{F}$. Hence, $\alpha\vee g\in(\mathbb{F}:\mathbb{G})$ and $\alpha\vee g\notin\mathbb{F}=(\mathbb{F}:\mathbb{G})\cap(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$, by Proposition 3.6 (xii). Thus, $\alpha\vee g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$ and since $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\in\mathcal{F}(\mathbb{E})$, we have $g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. Indeed, from $g\notin\mathbb{G}$ we conclude $g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$, which yields $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\subseteq\mathbb{G}$. Therefore $\mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. □

Corollary 3.25. *If* $\mathbb{F}=\{1\}$*, then* $\mathbb{G}$ *is* $\mathbb{F}$*-involutive if and only if* $\mathbb{G}=\mathbb{G}^{\top\top}$*.*

Proof. By Proposition 3.24, the proof is straightforward. □

Proposition 3.26. *If* $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, *then* $\mathbb{G}/\mathbb{F} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E}/\mathbb{F})$.

Proof. By Proposition 3.21 (ii), we get $(\mathbb{G}/\mathbb{F})^{\top\top} = (\mathbb{F} : (\mathbb{F} : \mathbb{G}))/\mathbb{F} = \mathbb{G}/\mathbb{F}$. Thus, by Proposition 3.24, $\mathbb{G}/\mathbb{F}$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}/\mathbb{F}$. $\square$

Proposition 3.27.
(i) $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = \{(\mathbb{F} : \mathbb{G})\colon\ \mathbb{F} \subseteq \mathbb{G} \in \mathcal{F}(\mathbb{E})\}$.
(ii) $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = \{(\mathbb{F} : \mathcal{X})\colon\ \mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}\}$.
(iii) *If* $\mathbb{G}, \mathbb{H} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$ *such that* $\mathbb{G} \subseteq \mathbb{H}$, *then* $\mathbb{G} \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}$.

Proof. (i) Take $B := \{(\mathbb{F} : \mathbb{G})\colon\ \mathbb{F} \subseteq \mathbb{G},\ \mathbb{G} \in \mathcal{F}(\mathbb{E})\}$. Then for any $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, we have $\mathbb{G} = (\mathbb{F} : (\mathbb{F} : \mathbb{G}))$. Now, suppose $\mathbb{H} := (\mathbb{F} : \mathbb{G})$. Thus, by Propositions 3.5 and 3.6 (i), we get $\mathbb{H} \in \mathcal{F}(\mathbb{E})$ such that $\mathbb{F} \subseteq \mathbb{H}$. Hence, $\mathbb{G} = (\mathbb{F} : \mathbb{H}) \in B$ and so $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) \subseteq B$. Conversely, if $(\mathbb{F} : \mathbb{G}) \in B$, then by Proposition 3.6 (xiv), we get $(\mathbb{F} : \mathbb{G}) = (\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathbb{G})))$. Thus, $(\mathbb{F} : \mathbb{G})$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}$, i.e., $(\mathbb{F} : \mathbb{G}) \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = B$.

(ii) Suppose $C := \{(\mathbb{F} : \mathcal{X})\colon\ \mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}\}$. By (i), it is obvious that $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) \subseteq C$. Now, let $(\mathbb{F} : \mathcal{X}) \in C$ such that $\mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}$. For any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$, by Proposition 3.8, we have $(\mathbb{F} : \mathcal{X}) = (\mathbb{F} : \langle \mathcal{X} \rangle)$ such that $\mathbb{F} \subseteq \mathcal{X} \subseteq \langle \mathcal{X} \rangle \in \mathcal{F}(\mathbb{E})$ and so, $C \subseteq \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $C = \mathcal{S}_{\mathbb{F}}(\mathbb{E})$.

(iii) Since $\mathbb{G} \subseteq \mathbb{H}$, by Proposition 3.6 (iv), $(\mathbb{F} : \mathbb{H}) \subseteq (\mathbb{F} : \mathbb{G})$. By $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, Proposition 3.6 (i) and (xi), we get $\mathbb{F} \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{H}) \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{G}) = \mathbb{F}$. Therefore $\mathbb{G} \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}$. $\square$

Proposition 3.28. *Let* $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$. *Then* $(\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H})) = (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$.

Proof. Since $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{G}, \mathbb{H}$, by Proposition 3.6 (iv), $(\mathbb{F} : \mathbb{G}), (\mathbb{F} : \mathbb{H}) \subseteq (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Again by Proposition 3.6 (iv), we get $(\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H})) \subseteq (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$. Conversely, let $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$ and $\gamma \in (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Then for all $g \in \mathbb{G}$ and $h \in \mathbb{H}$, we have $g, h \leqslant g \vee h$ and by $\mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$, $g \vee h \in \mathbb{G} \cap \mathbb{H}$. Thus, $\gamma \vee (g \vee h) \in \mathbb{F}$. Since $\gamma \vee (g \vee h) \leqslant (\alpha \vee \gamma) \vee (g \vee h)$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $(\alpha \vee \gamma \vee g) \vee h \in \mathbb{F}$ for all $h \in \mathbb{H}$ and so

$$(\alpha \vee \gamma) \vee g \in (\mathbb{F} : \mathbb{H}). \tag{3.1}$$

Also, $\alpha \leqslant (\alpha \vee \gamma) \vee g$ and $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \in \mathcal{F}(\mathbb{E})$. Thus, by Proposition 3.6 (xii),

$$(\alpha \vee \gamma) \vee g \in (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}. \tag{3.2}$$

Hence, for all $g \in \mathbb{G}$, $(\alpha \vee \gamma) \vee g \in \mathbb{F}$, and so $\alpha \vee \gamma \in (\mathbb{F} : \mathbb{G})$. Moreover, by $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \in \mathcal{F}(\mathbb{E})$ and $\alpha \leqslant \alpha \vee \gamma$, we have $\alpha \vee \gamma \in (\mathbb{F} : (\mathbb{F} : \mathbb{G}))$. So by

Proposition 3.6 (xii),

$$\alpha \vee \gamma \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : \mathbb{G}) = \mathbb{F} \tag{3.3}$$

for any $\gamma \in (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Thus, $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H}))$ and so $(\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \subseteq (\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H}))$. Therefore the proof is complete. □

Lemma 3.29. *The algebraic structure $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete bounded lattice, where, for any subfamily $\{\mathbb{G}_i\}_{i\in I}$ in $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$, the operations "$\wedge$" and "$\vee$" on $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$ are defined as follows:*

$$\bigwedge_{i\in I} \mathbb{G}_i = \bigcap_{i\in I} \mathbb{G}_i, \quad \text{and} \quad \bigvee_{i\in I} \mathbb{G}_i = \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup i \in I \mathbb{G}_i\Big)\Big).$$

P r o o f. By Proposition 3.6 (iii), $\mathbb{F}$ and $\mathbb{E}$ are the least and the greatest elements of $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$, respectively. Let $\{\mathbb{G}_i\}_{i\in I} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Then by Proposition 3.28, we get $(\mathbb{F} : (\mathbb{F} : \bigcap\limits_{i\in I} \mathbb{G}_i)) = \bigcap\limits_{i\in I}(\mathbb{F} : (\mathbb{F} : \mathbb{G}_i)) = \bigcap\limits_{i\in I} \mathbb{G}_i$. Thus, $\bigwedge\limits_{i\in I} \mathbb{G}_i \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Moreover, by Proposition 3.6 (xiv), we have

$$\Big(\mathbb{F} : \Big(\mathbb{F} : \bigvee_{i\in I} \mathbb{G}_i\Big)\Big) = \Big(\mathbb{F} : \underbrace{\Big(\mathbb{F} : \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup_{i\in I} \mathbb{G}_i\Big)\Big)\Big)}\Big) = \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup_{i\in I} \mathbb{G}_i\Big)\Big) = \bigvee_{i\in I} \mathbb{G}_i.$$

Hence, $\bigvee\limits_{i\in I} \mathbb{G}_i \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete bounded lattice. □

Proposition 3.30. *The algebraic structure $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complemented lattice.*

P r o o f. Let $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Then $\mathbb{F} \subseteq \mathbb{G}$ and by Proposition 3.6 (xi), we get $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} = \mathbb{F}$. Also,

$$\begin{aligned}
(\mathbb{F} : \mathbb{G}) \vee \mathbb{G} &= (\mathbb{F} : \underbrace{(\mathbb{F} : [(\mathbb{F} : \mathbb{G}) \cup \mathbb{G}])}) && \text{by definition of } \vee\text{-operation} \\
&= (\mathbb{F} : (\underbrace{(\mathbb{F} : (\mathbb{F} : \mathbb{G}))} \cap (\mathbb{F} : \mathbb{G}))) && \text{by Proposition 3.6 (vii)} \\
&= (\mathbb{F} : (\mathbb{G} \cap (\mathbb{F} : \mathbb{G}))) && \text{since } \mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E}) \\
&= (\mathbb{F} : \mathbb{F}) && \text{by Proposition 3.6 (xi)} \\
&= \mathbb{E} && \text{Proposition 3.6 (ii).}
\end{aligned}$$

Hence, $(\mathbb{F} : \mathbb{G})$ is a complemented lattice of $\mathbb{G}$ relative to $\mathbb{F}$. Therefore

$$(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$$

is a complemented lattice. □

Theorem 3.31. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ *is a complete Boolean lattice.*

P r o o f. By Lemma 3.29 and Proposition 3.30, we have that $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete and complemented lattice. So, it is enough to show the distribution:

For this, let $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Since $\mathbb{H} \cap \mathbb{K} \subseteq \mathbb{H}, \mathbb{K}$, then it is easy to see that

$$\mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}) \subseteq (\mathbb{G} \vee \mathbb{H}) \cap (\mathbb{G} \vee \mathbb{K}). \tag{3.4}$$

For the converse, we know that

$$\mathbb{H} \cap \mathbb{K} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}), \quad \mathbb{G} \cap \mathbb{K} \subseteq \mathbb{G} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.5}$$

So, by Proposition 3.27 (iii), we get

$$(\mathbb{H} \cap \mathbb{K}) \cap \underbrace{(\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))}_{B} = \mathbb{F}, \quad (\mathbb{G} \cap \mathbb{K}) \cap \underbrace{(\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))}_{B} = \mathbb{F}. \tag{3.6}$$

Hence, $\mathbb{H} \cap (\mathbb{K} \cap B) = \mathbb{F} = \mathbb{G} \cap (\mathbb{K} \cap B)$. Since by Proposition 3.17, $(\mathbb{F} : \mathbb{H})$ and $(\mathbb{F} : \mathbb{G})$ are relative pseudo complements of $\mathbb{H}$ and $\mathbb{G}$ with respect to $\mathbb{F}$, respectively, we get $(\mathbb{K} \cap B) \subseteq (\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G})$. Now, by Proposition 3.27,

$$(\mathbb{K} \cap B) \cap \underbrace{(\mathbb{F} : ((\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G})))}_{C} = \mathbb{F}. \tag{3.7}$$

Thus, $(\mathbb{K} \cap B) \cap C = \mathbb{F}$ and so $(C \cap \mathbb{K}) \cap B = \mathbb{F}$. By Proposition 3.17, $(\mathbb{F} : B)$ is a relative pseudo complement of B with respect to $\mathbb{F}$ and so

$$(C \cap \mathbb{K}) \subseteq (\mathbb{F} : B) = (\mathbb{F} : (\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))) = \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.8}$$

Moreover, from Propositions 3.6 (vii) and 3.8, we get

$$C = (\mathbb{F} : ((\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G}))) = (\mathbb{F} : (\mathbb{F} : (\mathbb{H} \cup \mathbb{G}))) = (\mathbb{F} : (\mathbb{F} : \langle \mathbb{H} \cup \mathbb{G} \rangle)) = \mathbb{H} \vee \mathbb{G}. \tag{3.9}$$

Therefore, by (3.8) and (3.9), we get for all $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$,

$$(\mathbb{G} \vee \mathbb{H}) \cap \mathbb{K} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.10}$$

Now, we have

$$\begin{aligned}
(\mathbb{G} \vee \mathbb{H}) \cap \underbrace{(\mathbb{G} \vee \mathbb{K})} &\subseteq \mathbb{G} \vee (\mathbb{H} \cap (\mathbb{G} \vee \mathbb{K})) \quad \text{by (3.10)} \\
&\subseteq \mathbb{G} \vee (\mathbb{G} \vee (\mathbb{H} \cap \mathbb{K})) \quad \text{by (3.10)} \\
&= \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}).
\end{aligned}$$

Hence, by (3.4), $(\mathbb{G} \vee \mathbb{H}) \cap (\mathbb{G} \vee \mathbb{K}) = \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K})$. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete Boolean lattice. □

Theorem 3.32. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \subseteq, \rightarrow, \odot, \mathbb{F}, \mathbb{E})$ *is a BL-algebra, where operations "$\rightarrow$" and "$\odot$", for any* $\mathbb{G}, \mathbb{H} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, *are defined as follows:*

$$\mathbb{G} \rightarrow \mathbb{H} := \mathbb{H} \vee (\mathbb{F} : \mathbb{G}), \quad \mathbb{G} \odot \mathbb{H} := \mathbb{G} \cap \mathbb{H}.$$

Proof. (BL1) By Lemma 3.29, $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \wedge, \vee, \mathbb{F}, \mathbb{E})$ is a bounded lattice.

(BL2) According to the definition of "$\odot$", clearly $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \odot, \mathbb{E})$ is a commutative monoid.

(BL3) Let $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. If $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$, then by definition of "$\vee$", we get $\mathbb{G} \subseteq \mathbb{K} \vee (\mathbb{F} : \mathbb{H})$. Moreover,

$$\begin{aligned}
\mathbb{G} \odot \mathbb{H} = \mathbb{G} \cap \mathbb{H} &\subseteq (\mathbb{K} \vee (\mathbb{F} : \mathbb{H})) \cap \mathbb{H} && \\
&= (\mathbb{K} \cap \mathbb{H}) \vee (\underbrace{(\mathbb{F} : \mathbb{H}) \cap \mathbb{H}}) && \text{by Theorem 3.31} \\
&= (\mathbb{K} \cap \mathbb{H}) \vee \mathbb{F} && \text{by Proposition 3.27 (iii)} \\
&= \mathbb{K} \cap \mathbb{H} \subseteq \mathbb{K} && \text{by Lemma 3.29.}
\end{aligned}$$

So, $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$ implies $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$. Conversely, let $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$. Then by the definition of "$\odot$", we have $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{K}$ and so

$$\begin{aligned}
\mathbb{G} = \mathbb{G} \cap \mathbb{E} = \mathbb{G} \cap [\mathbb{H} \vee (\mathbb{F} : \mathbb{H})] && \text{by Proposition 3.30} \\
= (\mathbb{G} \cap \mathbb{H}) \vee [\mathbb{G} \cap (\mathbb{F} : \mathbb{H})] && \text{by Theorem 3.31} \\
\subseteq \mathbb{K} \vee [\mathbb{G} \cap (\mathbb{F} : \mathbb{H})] \subseteq \mathbb{K} \vee (\mathbb{F} : \mathbb{H}) && \\
= \mathbb{H} \rightarrow \mathbb{K} && \text{by definition of "}\rightarrow\text{" operation.}
\end{aligned}$$

Thus, $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$ implies $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$. Therefore (BL3) is satisfied. Moreover, we have

$$\begin{aligned}
\mathbb{G} \odot (\mathbb{G} \rightarrow \mathbb{H}) &= \mathbb{G} \cap (\mathbb{G} \rightarrow \mathbb{H}) && \\
&= \mathbb{G} \cap (\mathbb{H} \vee (\mathbb{F} : \mathbb{G})) && \\
&= (\mathbb{G} \cap \mathbb{H}) \vee [\underbrace{\mathbb{G} \cap (\mathbb{F} : \mathbb{G})}] && \text{by Theorem 3.31} \\
&= (\mathbb{G} \cap \mathbb{H}) \vee \mathbb{F} && \text{by Proposition 3.27 (iii)} \\
&= \mathbb{G} \cap \mathbb{H} && \text{by Lemma 3.29} \\
&= \mathbb{G} \odot \mathbb{H}. &&
\end{aligned}$$

Hence, (BL4) is satisfied. Also,

$$
\begin{aligned}
(\mathbb{G} \to \mathbb{H}) \vee (\mathbb{H} \to \mathbb{G}) &= [\mathbb{H} \vee (\mathbb{F} : \mathbb{G})] \vee [\mathbb{G} \vee (\mathbb{F} : \mathbb{H})] \\
&= [\mathbb{H} \vee (\mathbb{F} : \mathbb{H})] \vee [\mathbb{G} \vee (\mathbb{F} : \mathbb{G})] \quad \text{by associativity of "}\vee\text{"} \\
&= \mathbb{E} \vee \mathbb{E} = \mathbb{E} \quad \text{by Proposition 3.30.}
\end{aligned}
$$

So, (BL5) is satisfied. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \subseteq, \to, \odot, \mathbb{F}, \mathbb{E})$ is a BL-algebra. □

4. Conclusions and future works

In this paper, the notion of relative co-annihilator in lattice equality algebras was introduced. Many properties of relative co-annihilators were investigated, the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ was defined and showed that it can be made as a BL-algebra.

In our future work, we will continue our study of algebraic properties of this special sets and we will investigate the relation between relative co-annihilators and some special filters in equality algebras.

References

[1] *H. A. S. Abujabal, M. A. Obaid, M. Aslam, A. B. Thaheem*: On annihilators of BCK-algebras. Czech. Math. J. *45* (1995), 727–735. zbl MR doi

[2] *R. A. Borzooei, F. Zebardast, M. Aaly Kologani*: Some type filters in equality algebras. Categ. Gen. Algebr. Struct. Appl. *7* (2017), 33–55. zbl MR

[3] *S. Burris, H. P. Sankappanavar*: A Course in Universal Algebra. Graduate Texts in Mathematics 78. Springer, New York, 1981. zbl MR doi

[4] *L. C. Ciungu*: Internal states on equality algebras. Soft Comput. *19* (2015), 939–953. zbl doi

[5] *B. A. Davey*: Some annihilator conditions on distributive lattices. Algebra Univers. *4* (1974), 316–322. zbl MR doi

[6] *A. Filipoiu*: About Baer extensions of MV-algebras. Math. Jap. *40* (1994), 235–241. zbl MR

[7] *P. Hájek*: Metamathematics of Fuzzy Logic. Trends in Logic-Studia Logica Library 4. Kluwer Academic, Dordrecht, 1998. zbl MR doi

[8] *R. Halaš*: Annihilators in BCK-algebras. Czech. Math. J. *53* (2003), 1001–1007. zbl MR doi

[9] *S. Jenei*: Equality algebras. Stud. Log. *100* (2012), 1201–1209. zbl MR doi

[10] *S. Jenei, L. Kóródi*: On the variety of equality algebras. Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology. Atlantis Press, Amsterdam, 2011, pp. 153–155. zbl doi

[11] *L. Leurstean*: Some algebraic properties of non-commutative fuzzy structures. Stud. Inf. Control *9* (2000), 365–370.

[12] *L. Leurstean*: Baer extensions of BL-algebras. J. Mult.-Val. Log. Soft Comput. *12* (2006), 321–336. zbl MR

[13] *F. G. Maroof, A. Borumand Saeid, E. Eslami*: On co-annihilators in residuated lattices. J. Intell. Fuzzy Syst. *31* (2016), 1263–1270. zbl doi

[14] *B. L. Meng, X. L. Xin*: Generalized co-annihilator of BL-algebras. Open Math. *13* (2015), 639–654. zbl MR doi

[15] *S. Niazian, M. Aaly Kologani, S. Homayon Arya, R. A. Borzooei*: On co-annihilators in equality algebras. Miskolc Math. Notes *24* (2023), 933–951. zbl MR doi

[16] *A. Paad*: Ideals in bounded equality algebras. Filomat *33* (2019), 2113–2123. zbl MR doi

[17] *E. Turunen*: BL-algebras of basic fuzzy logic. Mathware Soft Comput. *6* (1999), 49–61. zbl MR

[18] *Y. Xu, D. Ruan, K. Qin, J. Liu*: Lattice-Valued Logic: An Alternative Approach to Treat Fuzziness and Incomparability. Studies in Fuzziness and Soft Computing 132. Springer, Berlin, 2003. zbl MR doi

[19] *F. Zebardast, R. A. Borzooei, M. Aaly Kologani*: Results on equality algebras. Inf. Sci. *381* (2017), 270–282. zbl MR doi

[20] *Y. X. Zou, X. L. Xin, P. F. He*: On annihilators in BL-algebras. Open Math. *14* (2016), 324–337. zbl MR doi

Authors' addresses: *Sogol Niazian* (corresponding author), Tehran Medical Sciences, Islamic Azad University, Tehran, Iran, e-mail: `s.niazian@iautmu.ac.ir`; *Mona Aali Kologani*, Hatef Higher Education Institute, Zahedan, Iran, e-mail: `mona4011@gmail.com`; *Rajab Ali Borzooei*, Shahid Behashti University, Tehran, Iran, e-mail: `borzooei@sbu.ac.ir`.