UNIQUENESS RESULTS FOR DIFFERENTIAL POLYNOMIALS SHARING A SET

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Received July 28, 2023. Published online June 17, 2024.
Communicated by Grigore Sălăgean

Abstract. We investigate the uniqueness results of meromorphic functions if differential polynomials of the form $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share a set counting multiplicities or ignoring multiplicities, where Q is a polynomial of one variable. We give suitable conditions on the degree of Q and on the number of zeros and the multiplicities of the zeros of Q'. The results of the paper generalize some results due to T. T. H. An and N. V. Phuong (2017) and that of N. V. Phuong (2021).

Keywords: uniqueness; differential polynomials; set sharing; small function

MSC 2020: 30D35

1. Introduction, definitions and results

Let $f(z)$ be a nonconstant meromorphic function. The term "meromorphic" indicates meromorphic in the entire complex plane $\mathbb{C}$. We denote by $S(r,f)$ any function satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set with finite measure. Here, $T(r,f)$ denotes the Nevanlinna characteristic of f, and we use the standard notations of Nevanlinna value distribution theory throughout this work (see [8], [10], [16]). A meromorphic function $\alpha(z)$ is called a small function of some function $f(z)$ if $T(r,\alpha)=S(r,f)$. We say that two meromorphic functions f, g share a function α CM (counting multiplicities) if $f-\alpha$ and $g-\alpha$ admit the same zeros with the same multiplicities, and we say that f and g share α IM (ignoring multiplicities) if we do not consider the multiplicities. Let S be either a subset of $\mathbb{C}\cup\{\infty\}$ or a subset of $S(f)\cup\{\infty\}$, where $S(f)$ denotes the set of small functions of f. We define

$$E_f(S)=\bigcup_{\alpha\in S}\{z\in\mathbb{C}\colon\ f(z)-\alpha=0\},$$

DOI: 10.21136/MB.2024.0116-23

where each zero of $f-\alpha$ CM is included in the set, i.e., $E_f(S)$ is a multi-set. In the case we do not count the multiplicities, the collection $\bigcup_{\alpha\in S}\{z\in\mathbb{C}\colon\ f(z)-\alpha=0\}$ of only distinct zeros is denoted by $\overline{E}_f(S)$. Two functions f and g are said to share the set S CM (IM) if $E_f(S)=E_g(S)$ ($\overline{E}_f(S)=\overline{E}_g(S)$). Clearly, in the case when S is singleton, set sharing coincides with value sharing or a single small function sharing.

In 1959, Hayman (see [7]) published one of his significant paper, where the zero distribution of complex differential polynomials was considered, that is, if f is a transcendental meromorphic function and $n\in\mathbb{N}$, then Hayman conjectured that $f'f^n$ takes every finite nonzero value infinitely often.

Hayman conjecture has been proved completely by Hayman in [7] for the case $n\geqslant 3$, by Mues in [11] for $n=2$ and by Bergweiler and Eremenko (see [4]), Chen and Fang (see [6]) and Zalcman (see [17]) for $n=1$.

In 1997, Yang and Hua in [15] studied the unicity problem for meromorphic functions and differential monomials of the form $f'f^n$, when they share only one value.

In 2007, Bhoosnurmath and Dyavanal (see [5]) extended Yang-Hua's result to the case $(f^n)^{(k)}$.

Being inspired by Yang's problem (see [14]) that whether $f^{-1}(S)=g^{-1}(S)$ with $S=\{-1,1\}$ for the two same degree polynomials f and g implies either $f=g$ or $f=-g$, An and Khoai (see [3]) proved a uniqueness result on the meromorphic functions f and g when $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share a finite set. In this direction, Khoai and An (see [9]) proved a uniqueness result on meromorphic functions when two differential polynomials of the form $(P(f)^n)^{(k)}$ share a set of roots of unity.

Let $Q(z)$ be a polynomial of degree q in $\mathbb{C}$ and k be a positive integer. Denote the derivative of $Q(z)$ by

$$Q'(z)=b\prod_{i=1}^{l}(z-\zeta_i)^{m_i}$$

with $b\in\mathbb{C}^*$ $(=\mathbb{C}-\{0\})$, and denote by ν and h the indexes such that $1\leqslant\nu\leqslant h\leqslant l$, and

$$m_1\geqslant m_2\geqslant\ldots\geqslant m_\nu>k\geqslant m_{\nu+1}\geqslant\ldots\geqslant m_l,$$

$$m_1\geqslant m_2\geqslant\ldots\geqslant m_h\geqslant k>m_{h+1}\geqslant\ldots\geqslant m_l.$$

In 2017, An and Phuong (see [1]) proved a uniqueness result on meromorphic functions when $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share a small function α CM. Their result is as follows:

Theorem A. *Let f and g be two nonconstant meromorphic functions, and α be a nonzero small function with respect to f. Suppose that $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share α CM. If $q>k+6+2\nu(k+1)+2\sum\limits_{i=\nu+1}^{l}m_i$, then one of the following conclusions holds:*

(1) $Q(f) = Q(g) + c$ *for a constant* c*;*
(2) $[Q(f)]^{(k)}[Q(g)]^{(k)} = \alpha^2$.

The authors [1] also showed that conclusion (2) of Theorem A can be ruled out by adding more constraints on the multiple zeros of $Q'(z)$ or if f and g share ∞ IM and proved the following theorem.

Theorem B. *Let* f *and* g *be two nonconstant meromorphic functions, and* α *be a nonzero small function with respect to* f*. Assume that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* α *CM. If* $q > k + 6 + 2\nu(k+1) + 2\sum_{i=\nu+1}^{l} m_i$ *and if one of*
(1) $h \geqslant 4$;
(2) $h = 3$ *and* $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, *and* $q \neq 3m_i - 2k + 3$, *for all* $i = 1, 2, 3$; *or*
(3) $h = 2$
and f *and* g *share* ∞ *IM holds, then*

$$Q(f) = Q(g) + c \quad \text{for a constant } c.$$

In 2021, Phuong (see [12]) proved the following results for sharing the small function α IM.

Theorem C. *Let* f *and* g *be two nonconstant meromorphic functions, and* α *be a nonzero small function with respect to* f*. Suppose that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* α *IM. If* $q > 4k + 12 + \nu(5k+2) + 5\sum_{i=\nu+1}^{l} m_i$*, then one of the following conclusions holds:*
(1) $Q(f) = Q(g) + c$ *for a constant* c*;*
(2) $[Q(f)]^{(k)}[Q(g)]^{(k)} = \alpha^2$.

Theorem D. *Let* f *and* g *be two nonconstant meromorphic functions, and* α *be a nonzero small function with respect to* f*. Suppose that* $[Q(f)]^{(k)}$ *and* $[Q(g)]^{(k)}$ *share* α *IM. If* $q > 4k + 12 + \nu(5k+2) + 5\sum_{i=\nu+1}^{l} m_i$*, and if one of*
(1) $h \geqslant 4$;
(2) $h = 3$ *and* $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, *and* $q \neq 3m_i - 2k + 3$, *for all* $i = 1, 2, 3$; *or*
(3) $h = 2$
and f *and* g *share* ∞ *IM holds, then*

$$Q(f) = Q(g) + c \quad \text{for a constant } c.$$

Now the following question is inevitable.

Question 1.1. *What will happen if sharing a small function α is replaced by sharing a set $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), \dots, \omega^{d-1}\alpha(z)\}$, with $\omega^d = 1$ in Theorems* A–D*?*

In this regard, we obtain the next main results which answers the above question.

Theorem 1.1. *Let f and g be two nonconstant meromorphic functions, and α be a nonzero small function with respect to f. Let d be a positive integer such that $q > k+2+4/d+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$ and let $S = \{\alpha(z), \omega\alpha(z), \omega^2\alpha(z), \dots, \omega^{d-1}\alpha(z)\}$, where $\omega^d = 1$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set S CM, then one of the following conclusions holds:*

(1) $Q(f) = tQ(g) + c$ *for a constant c and $t^d = 1$;*
(2) $[Q(f)]^{(k)}[Q(g)]^{(k)} = t\alpha^{2/d}$ *with $t^d = 1$.*

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions, and α be a nonzero small function with respect to f. Let d, S be defined as in Theorem* 1.1 *and $q > k + 2 + 4/d + 2\nu(k + 1) + 2\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set S CM and if one of*

(1) $h \geqslant 4$;
(2) $h = 3$ *and* $q \neq 2m_1 - 2k + 2$, $q \neq (3m_1 - 2k + 3)/2$, *and* $q \neq 3m_i - 2k + 3$, *for all* $i = 1, 2, 3$; *or*
(3) $h = 2$

and f and g share ∞ IM holds, then

$$Q(f) = tQ(g) + c \quad \text{for a constant } c \text{ and } t^d = 1.$$

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions, and α be a nonzero small function with respect to f. Let d, S be defined as in Theorem* 1.1 *and $q > k + 2 + (3k + 10)/d + \nu(2k + 2 + 3k/d) + (2 + 3/d)\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set S IM, then one of the conclusions of Theorem* 1.1 *holds.*

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions, and α be a nonzero small function with respect to f. Let d, S be defined as in Theorem* 1.1 *and $q > k+2+(3k+10)/d+\nu(2k+2+3k/d)+(2+3/d)\sum_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set S IM and if one of* (1), (2) *and* (3) *of Theorem* 1.2 *holds, then*

$$Q(f) = tQ(g) + c \quad \text{for a constant } c \text{ and } t^d = 1.$$

R e m a r k 1.1. If we put $d = 1$ in Theorems 1.1–1.4, then we obtain Theorems A–D, respectively.

Definition 1.1. Let a be a finite complex number, and let p be a positive integer. We denote by $N_p(r, 1/(f-a))$ the counting function for zeros of $f-a$, where a zero of multiplicity m is counted m times if $m \leqslant p$ and p times if $m > p$.

2. Lemmas

We now present some lemmas that will be useful in the next section.

Lemma 2.1 ([13] Logarithmic derivative lemma). *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Then*

$$m\Big(r, \frac{f'}{f}\Big) = S(r, f)$$

as $r \to \infty$ outside a subset of finite measure.

Lemma 2.2 ([8], [13] First fundamental theorem). *Let f be a meromorphic function, and let c be a complex number. Then*

$$T\Big(r, \frac{1}{f-c}\Big) = T(r, f) + O(1).$$

Lemma 2.3 ([8], [13] Second fundamental theorem). *Let f be a nonconstant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ be distinct meromorphic functions on $\mathbb{C}$. Assume that a_i's are small functions with respect to f for all $i = 1, \ldots, q$. Then the inequality*

$$(q-2)T(r, f) \leqslant \sum_{j=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) + S(r, f)$$

holds for all r outside a set $E \subset (0, \infty)$ with finite Lebesgue measure.

Lemma 2.4 ([18]). *Let f be a nonconstant meromorphic function, and let p and k be two positive integers. If $f^{(k)} \not\equiv 0$, then*

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$
$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant k\overline{N}(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$

and

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leqslant k\overline{N}(r, f) + N\Big(r, \frac{1}{f}\Big) + S(r, f).$$

Lemma 2.5. *Let Q be a polynomial of degree q in $\mathbb{C}$, and let k be a positive integer. Let*

$$Q'(z) = b\prod_{i=1}^{l}(z-\zeta_i)^{m_i}$$

with $b \in \mathbb{C}^*$. Let f and g be two nonconstant meromorphic functions. Assume that $([Q(f)]^{(k)})^d = ([Q(g)]^{(k)})^d$. If $q-2l-2k-4>0$, then $Q(f) = tQ(g)+c$ for a constant c and $t^d=1$.

Proof. Since $([Q(f)]^{(k)})^d = ([Q(g)]^{(k)})^d$, we get $[Q(f)]^{(k)} = t[Q(g)]^{(k)}$ where $t^d = 1$. This gives

$$Q(f) = tQ(g)+\varphi,$$

where φ is a polynomial of degree at most $k-1$. Therefore,

$$qT(r,g) \leqslant qT(r,f)+T(r,\varphi)+O(1), \quad \text{and} \quad f'Q'(f) = tg'Q'(g)+\varphi'.$$

If $k=1$, then $\varphi = c$, a constant.

If $k \geqslant 2$, then proceeding in a similar manner as in the proof of Lemma 3.1 of [1], we can deduce that $\varphi = c$ for a constant c. □

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions, and let α be a small function with respect to f. Let d, S be defined as in Theorem 1.1 and $q > 5+1/d+\nu(k+1)+\sum\limits_{i=\nu+1}^{l} m_i$. If $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share the set S IM, then $T(r,f) = O(T(r,g))$, $T(r,g) = O(T(r,f))$, and α is a small function with respect to g.*

Proof. Let

$$F := [Q(f)]^{(k)}, \quad F_1 := Q(f), \quad \widehat{F} := F^d,$$
$$G := [Q(g)]^{(k)}, \quad G_1 := Q(g), \quad \widehat{G} := G^d.$$

It is easy to see that

$$S(r,\widehat{F}) = S(r,F) = S(r,f) \quad \text{and} \quad S(r,\widehat{G}) = S(r,G) = S(r,g).$$

Now we have

$$\begin{aligned} T(r,F_1') &= T(r,f'Q'(f)) \geqslant T\Big(r, f'Q'(f)\frac{1}{f'}\Big) - T\Big(r,\frac{1}{f'}\Big) + O(1) \\ &\geqslant T(r,Q'(f)) - 2T(r,f) + O(1) \geqslant (q-3)T(r,f)+O(1). \end{aligned} \tag{2.1}$$

Applying Lemma 2.3 to $\widehat{F}$, we obtain

$$dT(r,F)=T(r,\widehat{F})\leqslant \overline{N}(r,\widehat{F})+\overline{N}\Big(r,\frac{1}{\widehat{F}}\Big)+\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+S(r,f)$$
$$\leqslant \overline{N}(r,f)+\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+S(r,f). \tag{2.2}$$

Again by Lemma 2.4 with $(F_1')^{(k-1)}=F$, we have

$$T(r,F)\geqslant T(r,F_1')+N_2\Big(r,\frac{1}{F}\Big)-N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f). \tag{2.3}$$

From (2.1), (2.2) and (2.3) we get

$$\begin{aligned}(q-3)T(r,f)&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{F}\Big)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)-N_2\Big(r,\frac{1}{F}\Big)\\
&\quad+N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f)\\
&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{F}-\alpha}\Big)+N_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r,f)\\
&\leqslant \frac{1}{d}\overline{N}(r,f)+\frac{1}{d}\overline{N}\Big(r,\frac{1}{\widehat{G}-\alpha}\Big)+N\Big(r,\frac{1}{f'}\Big)+(k+1)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{f-\zeta_i}\Big)\\
&\quad+\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{f-\zeta_i}\Big)+S(r,f)\\
&\leqslant \Big(2+\frac{1}{d}+\nu(k+1)+\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)+q(k+1)T(r,g)+S(r,f).\end{aligned}$$

Therefore

$$\Big(q-5-\frac{1}{d}-\nu(k+1)-\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)\leqslant q(k+1)T(r,g)+S(r,f),$$

which implies $T(r,f)=O(T(r,g))$ if $q>5+1/d+\nu(k+1)+\sum_{l=\nu+1}^{l}m_i$. Similarly, it can be shown that $T(r,g)=O(T(r,f))$ and hence, α is a small function with respect to g. □

Lemma 2.7 ([2]). *Let f and g be two nonconstant meromorphic functions, and let α be a nonzero small function with respect to both f and g. If f and g share α CM, then one of the following three cases holds:*

(1) $T(r,f)\leqslant N_2(r,f)+N_2(r,g)+N_2(r,1/f)+N_2(r,1/g)+S(r,f)+S(r,g)$, *and the same inequality holds for $T(r,g)$;*
(2) $f\equiv g$;
(3) $fg\equiv\alpha^2$.

Lemma 2.8 ([12]). *Let f and g be two nonconstant meromorphic functions, and let α be a nonzero small function with respect to both f and g. If f and g share α IM, then one of the following three cases holds:*

(1) $T(r,f) \leqslant N_2(r,f) + N_2(r,g) + N_2(r,1/f) + N_2(r,1/g) + 2\overline{N}(r,f) + \overline{N}(r,g) + 2\overline{N}(r,1/f) + \overline{N}(r,1/g) + S(r,f) + S(r,g)$, *and the same inequality holds for* $T(r,g)$;

(2) $f \equiv g$;

(3) $fg \equiv \alpha^2$.

Lemma 2.9. *Let f, g be nonconstant meromorphic functions and $\alpha(\not\equiv 0,\infty)$ be a small function with respect to both f and g. If*

$$([Q(f)]^{(k)})^d([Q(g)]^{(k)})^d = \alpha^2,$$

then $h \leqslant 2$ or $h = 3$ and either $q = 2m_1 - 2k + 2$, $q = (3m_1 - 2k + 3)/2$, or $q = 3m_i - 2k + 3$, for $i = 1,2,3$. If we further assume that f and g share ∞ IM, then also $h = 1$.

Proof. From $([Q(f)]^{(k)})^d([Q(g)]^{(k)})^d = \alpha^2$ we have $[Q(f)]^{(k)}[Q(g)]^{(k)} = t\alpha^{2/d}$, where $t^d = 1$. This gives

$$[f'Q'(f)]^{(k-1)}[g'Q'(g)]^{(k-1)} = t\alpha^{2/d}.$$

Since

$$Q'(z) = b\prod_{i=1}^{l}(z-\zeta_i)^{m_i},$$

where $b \in \mathbb{C}^*$ and $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_h \geqslant k > m_{h+1} \geqslant \ldots \geqslant m_l$, we can write

$$\prod_{i=1}^{h}(f-\zeta_i)^{m_i-k+1}\prod_{i=1}^{h}(g-\zeta_i)^{m_i-k+1}R(f,f',\ldots,f^{(k)})\widetilde{R}(g,g',\ldots,g^{(k)}) = t\alpha^{2/d},$$

where $R(f,f',\ldots,f^{(k)})$ and $\widetilde{R}(g,g',\ldots,g^{(k)})$ are polynomials. Then proceeding similarly as in the proof of Lemma 3.4 in [1], we can get the required result. □

3. Proof of the theorems

Proof of Theorem 1.1. Let F, G, F_1, G_1, $\widehat{F}$ and $\widehat{G}$ be defined as in the proof of Lemma 2.6. Then it is easy to prove that

$$S(r,\widehat{F}) = S(r,F) = S(r,f) \quad \text{and} \quad S(r,\widehat{G}) = S(r,G) = S(r,g).$$

By Lemma 2.6, α is a small function with respect to g also. Since F and G share the set S CM, it follows that $\widehat{F}$ and $\widehat{G}$ share α CM. Therefore by Lemma 2.7, one of the following cases occurs:

(1) $T(r,\widehat{F}) \leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2(r,1/\widehat{F}) + N_2(r,1/\widehat{G}) + S(r,\widehat{F}) + S(r,\widehat{G})$, and the same inequality holds for $T(r,\widehat{G})$;

(2) $\widehat{F} \equiv \widehat{G}$;

(3) $\widehat{F}\widehat{G} \equiv \alpha^2$.

If Case (3) holds, then conclusion (2) of the theorem is proved. If Case (2) holds, then by Lemma 2.5, we get $Q(f) = tQ(g) + c$ for a constant c and $t^d = 1$. So conclusion (1) of the theorem is proved. Now we verify Case (1).

If Case (1) holds, then we have

$$(3.1)\quad \begin{aligned} dT(r,F) &= T(r,\widehat{F}) \\ &\leqslant N_2(r,\widehat{F}) + N_2(r,\widehat{G}) + N_2\Big(r,\frac{1}{\widehat{F}}\Big) + N_2\Big(r,\frac{1}{\widehat{G}}\Big) + S(r,\widehat{F}) + S(r,\widehat{G}) \\ &\leqslant N_2(r,F) + N_2(r,G) + dN_2\Big(r,\frac{1}{F}\Big) + dN_2\Big(r,\frac{1}{G}\Big) + S(r,f) + S(r,g). \end{aligned}$$

Now using Lemma 2.4, we have

$$(3.2)\quad N_2\Big(r,\frac{1}{G}\Big) = N_2\Big(r,\frac{1}{(G_1')^{(k-1)}}\Big) \leqslant (k-1)\overline{N}(r,G_1') + N_{k+1}\Big(r,\frac{1}{G_1'}\Big) + S(r,g).$$

Again, we can write

$$Q(z) - R(z) = a(z-\beta)Q'(z),$$

where $a \neq 0$ and β are constants and $R(z)$ is a polynomial of degree atmost $q-2$.

Applying Lemma 2.1, we have

$$\begin{aligned} m\Big(r,\frac{1}{Q(f)-R(f)}\Big) &= m\Big(r,\frac{(Q(f))'}{Q(f)-R(f)}\cdot\frac{1}{(Q(f))'}\Big) \\ &\leqslant m\Big(r,\frac{f'}{a(f-\beta)}\Big) + m\Big(r,\frac{1}{F_1'}\Big) + O(1) \leqslant m\Big(r,\frac{1}{F_1'}\Big) + S(r,f). \end{aligned}$$

From this we get

$$\begin{aligned} T(r,F_1') &= m\Big(r,\frac{1}{F_1'}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1) \\ &\geqslant T\Big(r,\frac{1}{Q(f)-R(f)}\Big) - N\Big(r,\frac{1}{Q(f)-R(f)}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1) \\ &\geqslant qT(r,f) - N\Big(r,\frac{1}{Q'(f)}\Big) - N\Big(r,\frac{1}{f-\beta}\Big) + N\Big(r,\frac{1}{F_1'}\Big) + O(1). \end{aligned}$$

Therefore, applying Lemma 2.4 to the function F_1' (with the notation $(F_1')^{(k-1)} = F$), we have

$$\begin{aligned} T(r,F) &\geqslant T(r,F_1') + N_2\Big(r,\frac{1}{F}\Big) - N_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r,f) \\ &\geqslant qT(r,f) - N\Big(r,\frac{1}{Q'(f)}\Big) - N\Big(r,\frac{1}{f-\beta}\Big) + N\Big(r,\frac{1}{F_1'}\Big) \\ &\quad + N_2\Big(r,\frac{1}{F}\Big) - N_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r,f). \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) we have

$$\begin{aligned} dqT(r,f) &\leqslant d(k-1)\overline{N}(r,G_1') + dN_{k+1}\Big(r,\frac{1}{G_1'}\Big) + N_2(r,G) \\ &\quad + N_2(r,F) + dN\Big(r,\frac{1}{Q'(f)}\Big) + dN\Big(r,\frac{1}{f-\beta}\Big) \\ &\quad - dN\Big(r,\frac{1}{F_1'}\Big) + dN_{k+1}\Big(r,\frac{1}{F_1'}\Big) + S(r) \\ &\leqslant (d(k-1)+2)\overline{N}(r,g) + d(k+1)\sum_{i=1}^{\nu} N\Big(r,\frac{1}{g-\zeta_i}\Big) \\ &\quad + dN\Big(r,\frac{1}{g'}\Big) + d\sum_{i=\nu+1}^{l} m_i N\Big(r,\frac{1}{g-\zeta_i}\Big) + 2\overline{N}(r,f) \\ &\quad + d(k+1)\sum_{i=1}^{\nu} N\Big(r,\frac{1}{f-\zeta_i}\Big) + d\sum_{i=\nu+1}^{l} m_i N\Big(r,\frac{1}{f-\zeta_i}\Big) \\ &\quad + dN\Big(r,\frac{1}{f-\beta}\Big) + S(r) \\ &\leqslant \Big(d(k+1) + 2 + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,g) \\ &\quad + \Big(2 + d + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,f) + S(r). \end{aligned}$$

This implies

$$\begin{aligned} &\Big(dq - 2 - d - d\nu(k+1) - d\sum_{i=\nu+1}^{l} m_i\Big)T(r,f) \\ &\quad \leqslant \Big(d(k+1) + 2 + d\nu(k+1) + d\sum_{i=\nu+1}^{l} m_i\Big)T(r,g) + S(r). \end{aligned} \tag{3.4}$$

Similarly, it can be shown that

$$(3.5)\qquad \Big(dq-2-d-d\nu(k+1)-d\sum_{i=\nu+1}^{l} m_i\Big)T(r,g) \leqslant \Big(d(k+1)+2+d\nu(k+1)+d\sum_{i=\nu+1}^{l} m_i\Big)T(r,f)+S(r).$$

Combining (3.4) and (3.5), we get

$$\Big(dq-4-d(k+2)-2d\nu(k+1)-2d\sum_{i=\nu+1}^{l} m_i\Big)(T(r,g)+T(r,f))\leqslant S(r).$$

Thus, we have $q > k+2+4/d+2\nu(k+1)+2\sum_{i=\nu+1}^{l} m_i$, which is a contradiction. This proves the theorem. □

P r o o f of Theorem 1.2. The proof of this theorem follows from Theorem 1.1 and Lemma 2.9. □

P r o o f of Theorem 1.3. The notations F, G, F_1, G_1, $\widehat{F}$ and $\widehat{G}$ are the same as defined in the proof of Lemma 2.6. By Lemma 2.6, α is a small function with respect to g also. Since F and G share the set S IM, $\widehat{F}$ and $\widehat{G}$ share α IM. Therefore by Lemma 2.8, one of the following cases occurs:

(1) $T(r,\widehat{F})\leqslant N_2(r,\widehat{F})+N_2(r,\widehat{G})+N_2(r,1/\widehat{F})+N_2(r,1/\widehat{G})+2\overline{N}(r,\widehat{F})+\overline{N}(r,\widehat{G})+2\overline{N}(r,1/\widehat{F})+\overline{N}(r,1/\widehat{G})+S(r,\widehat{F})+S(r,\widehat{G})$, and the same inequality holds for $T(r,\widehat{G})$;

(2) $\widehat{F}\equiv\widehat{G}$;

(3) $\widehat{F}\widehat{G}\equiv\alpha^2$.

Conclusions (1) and (2) of the theorem hold preciously from cases (2) and (3), respectively. Next we assume that Case (1) holds. Then

$$(3.6)\qquad \begin{aligned} dT(r,F) &= T(r,\widehat{F})\\ &\leqslant N_2(r,\widehat{F})+N_2(r,\widehat{G})+N_2\Big(r,\frac{1}{\widehat{F}}\Big)+N_2\Big(r,\frac{1}{\widehat{G}}\Big)+2\overline{N}(r,\widehat{F})\\ &\quad+\overline{N}(r,\widehat{G})+2\overline{N}\Big(r,\frac{1}{\widehat{F}}\Big)+\overline{N}\Big(r,\frac{1}{\widehat{G}}\Big)+S(r,\widehat{F})+S(r,\widehat{G})\\ &\leqslant N_2(r,F)+N_2(r,G)+dN_2\Big(r,\frac{1}{F}\Big)+dN_2\Big(r,\frac{1}{G}\Big)+2\overline{N}(r,F)\\ &\quad+\overline{N}(r,G)+2\overline{N}\Big(r,\frac{1}{F}\Big)+\overline{N}\Big(r,\frac{1}{G}\Big)+S(r,f)+S(r,g).\end{aligned}$$

Now using Lemma 2.4, we have

$$\overline{N}\Big(r,\frac{1}{F}\Big)=N_1\Big(r,\frac{1}{(F_1')^{(k-1)}}\Big)\leqslant (k-1)\overline{N}(r,F_1')+N_k\Big(r,\frac{1}{F_1'}\Big)+S(r,f) \tag{3.7}$$

and

$$\overline{N}\Big(r,\frac{1}{G}\Big)\leqslant (k-1)\overline{N}(r,G_1')+N_k\Big(r,\frac{1}{G_1'}\Big)+S(r,g). \tag{3.8}$$

Again, by similar arguments as in the proof of Theorem 1.1, we can get the inequalities (3.2) and (3.3).

From (3.2), (3.3), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}
dqT(r,f)\leqslant{}& d(k-1)\overline{N}(r,G_1')+dN_{k+1}\Big(r,\frac{1}{G_1'}\Big)+N_2(r,F)+N_2(r,G)\\
&+2\overline{N}(r,F)+\overline{N}(r,G)+2(k-1)N(r,F_1')+2N_k\Big(r,\frac{1}{F_1'}\Big)\\
&+(k-1)\overline{N}(r,G_1')+N_k\Big(r,\frac{1}{G_1'}\Big)+dN\Big(r,\frac{1}{Q'(f)}\Big)\\
&+dN\Big(r,\frac{1}{f-\beta}\Big)-dN\Big(r,\frac{1}{F_1'}\Big)+dN_{k+1}\Big(r,\frac{1}{F_1'}\Big)+S(r)\\
\leqslant{}&(d(k-1)+k+2)\overline{N}(r,g)+(d+1)N\Big(r,\frac{1}{g'}\Big)\\
&+(d(k+1)+k)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{g-\zeta_i}\Big)+(d+1)\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{g-\zeta_i}\Big)\\
&+(2k+2)\overline{N}(r,f)+2N\Big(r,\frac{1}{f'}\Big)+(d(k+1)+2k)\sum_{i=1}^{\nu}N\Big(r,\frac{1}{f-\zeta_i}\Big)\\
&+(d+2)\sum_{i=\nu+1}^{l}m_iN\Big(r,\frac{1}{f-\zeta_i}\Big)+dN\Big(r,\frac{1}{f-\beta}\Big)+S(r)\\
\leqslant{}&\Big(d(k+1)+k+4+\nu(d(k+1)+k)+(d+1)\sum_{i=\nu+1}^{l}m_i\Big)T(r,g)\\
&+\Big(d+2k+6+\nu(d(k+1)+2k)+(d+2)\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)+S(r).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Big(dq-d-2k-6-\nu(d(k+1)+2k)-(d+2)\sum_{i=\nu+1}^{l}m_i\Big)T(r,f)\\
&\qquad\leqslant\Big(d(k+1)+k+4+\nu(d(k+1)+k)+(d+1)\sum_{i=\nu+1}^{l}m_i\Big)T(r,g)+S(r).
\end{aligned}\tag{3.9}$$

Similarly,

(3.10)

$$\left(dq - d - 2k - 6 - \nu(d(k+1)+2k) - (d+2)\sum_{i=\nu+1}^{l} m_i\right)T(r,g)$$
$$\leqslant \left(d(k+1) + k + 4 + \nu(d(k+1)+k) + (d+1)\sum_{i=\nu+1}^{l} m_i\right)T(r,f) + S(r).$$

Combining (3.9) and (3.10), we get

$$\left(dq - d(k+2) - 3k - 10 - \nu(2d(k+1)+3k) - (2d+3)\sum_{i=\nu+1}^{l} m_i\right)(T(r,f)+T(r,g)) \leqslant S(r).$$

Thus, when $q > k+2+(3k+10)/d+\nu(2k+2+3k/d)+(2+3/d)\sum_{i=\nu+1}^{l} m_i$, we have a contradiction. This proves the theorem. □

Proof of Theorem 1.4. The proof of this theorem follows from Theorem 1.3 and Lemma 2.9. □

Acknowledgements. The authors are grateful to the referee for his/her helpful remarks and suggestions towards the improvement of the paper.

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