2024 MATHEMATICA BOHEMICA 22 pp
Online first

FINITE LOGARITHMIC ORDER MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENCE/DIFFERENTIAL-DIFFERENCE EQUATIONS

Abdelkader Dahmani, Benharrat Belaïdi, Mostaganem

Received July 15, 2023. Published online June 5, 2024.
Communicated by Grigore Sălăgean

Abstract. Firstly we study the growth of meromorphic solutions of linear difference equation of the form

$$A_k(z)f(z+c_k)+\ldots+A_1(z)f(z+c_1)+A_0(z)f(z)=F(z),$$

where $A_k(z),\ldots,A_0(z)$ and $F(z)$ are meromorphic functions of finite logarithmic order, c_i $(i=1,\ldots,k,\ k\in\mathbb{N})$ are distinct nonzero complex constants. Secondly, we deal with the growth of solutions of differential-difference equation of the form

$$\sum_{i=0}^{n}\sum_{j=0}^{m}A_{ij}(z)f^{(j)}(z+c_i)=F(z),$$

where $A_{ij}(z)$ $(i=0,1,\ldots,n,\ j=0,1,\ldots,m,n,\ m\in\mathbb{N})$ and $F(z)$ are meromorphic functions of finite logarithmic order, c_i $(i=0,\ldots,n)$ are distinct complex constants. We extend some previous results obtained by Zhou and Zheng and Biswas to the logarithmic lower order.

Keywords: linear difference equation; linear differential-difference equation; meromorphic function; logarithmic order; logarithmic lower order

MSC 2020: 30D35, 39B32, 39A10

1. Introduction and main results

Throughout this paper, we assume the readers are familiar with the fundamental results and standard notations of the Nevanlinna distribution theory of meromorphic functions which can be found in [11], [12], [18]. Further, we denote, respectively, by $\varrho(f)$, $\lambda(1/f)$, $\tau(f)$ the order, the convergence exponent of the pole-sequence and the

This paper is supported by University of Mostaganem (UMAB) (PRFU Project Code C00L03UN270120220007).

DOI: 10.21136/MB.2024.0107-23

type of a meromorphic function f. Many results have been obtained by many different mathematicians on studying the growth of solutions of the different types of the linear difference and q-difference equations and the linear differential equations where their coefficients are entire or meromorphic functions, see, for example, [4], [7], [14], [15], [17], [19], [20]. Recently some of these results were obtained by using the concept of the logarithmic order due to Chern (see [8]), as a better technique for the case when these coefficients are entire or meromorphic functions of zero order in the complex plane see, for example, [1]–[3], [5], [6], [10], [16]. This inspired us to investigate the logarithmic order of solutions to these equations given in [20], where we give some results on the logarithmic lower order. At first let us recall some related definitions.

Definition 1 ([12], [14]). Let f be a meromorphic function. The counting function of f is defined by

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}\,\mathrm{d}t+n(0,f)\log r,$$

where $n(t,\infty,f)=n(t,f)$ is the number of poles of $f(z)$ lying in $|z|\leqslant t$ counted according to their multiplicity. The proximity function of f is defined by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(r\mathrm{e}^{\mathrm{i}\varphi})|\,\mathrm{d}\varphi,$$

where $\log^+ x=\max\{0,\log x\}$ for $x\geqslant 0$. The characteristic function of f is defined by

$$T(r,f)=m(r,f)+N(r,f),\qquad r>0.$$

Definition 2 ([6], [8]). The logarithmic order of a meromorphic function f is defined by

$$\varrho_{\log}(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log\log r}.$$

When $1\leqslant \varrho_{\log}(f)=\varrho<\infty$, the logarithmic type of f is defined by

$$\tau_{\log}(f)=\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^{\varrho}}.$$

Definition 3 ([3]). The logarithmic lower order of a meromorphic function f is defined by

$$\mu_{\log}(f)=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log\log r}.$$

When $1\leqslant \mu_{\log}(f)=\mu<\infty$, the logarithmic lower type of f is defined by

$$\underline{\tau}_{\log}(f)=\liminf_{r\to\infty}\frac{T(r,f)}{(\log r)^{\mu}}.$$

Definition 4 ([1], [6])**.** Let f be a meromorphic function. Then the logarithmic exponent of convergence of zeros of $f(z)$ is defined by

$$\lambda_{\log}(f) = \lambda_{\log}(f,0) = \limsup_{r\to\infty} \frac{\log n(r,1/f)}{\log\log r} = \limsup_{r\to\infty} \frac{\log N(r,1/f)}{\log\log r} - 1,$$

where $n(r,1/f)$ denotes the number of zeros of f in the disk $|z| \leqslant r$.

Definition 5 ([12], [18])**.** Let $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f is given by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,1/(f-a))}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,1/(f-a))}{T(r,f)}, \quad a \neq \infty,$$
$$\delta(\infty,f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,f)}{T(r,f)}.$$

In [20], Zhou and Zheng considered the linear difference equation

$$A_k(z)f(z+c_k) + \ldots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z), \tag{1}$$

where $A_0(z), \ldots, A_k(z)$ and $F(z)$ are meromorphic functions of finite order, c_i $(i = 1, \ldots, k,\ k \in \mathbb{N})$ are distinct nonzero complex constants, and proved the following result.

Theorem A ([20])**.** *Let $A_j(z)$ $(j = 0,1,\ldots,k)$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_l(z)$ satisfies*

$$\lambda\Big(\frac{1}{A_l}\Big) < \varrho(A_l) < \infty, \quad \max\{\varrho(A_j)\colon\ j = 0,1\ldots k,\ j \neq l\} \leqslant \varrho(A_l),$$
$$\sum_{\varrho(A_j)=\varrho(A_l),j\neq l} \tau(A_j) < \tau(A_l) < \infty.$$

(1) *If $\varrho(F) < \varrho(A_l)$, or $\varrho(F) = \varrho(A_l)$ and $\sum_{\varrho(A_j)=\varrho(A_l),j\neq l} \tau(A_j) + \tau(F) < \tau(A_l)$, or $\varrho(F) = \varrho(A_l)$ and $\sum_{\varrho(A_j)=\varrho(A_l)} \tau(A_j) < \tau(F)$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of* (1) *satisfies $\varrho(f) \geqslant \varrho(A_l)$.*
(2) *If $\varrho(F) > \varrho(A_l)$, then every meromorphic solution $f(z)$ of* (1) *satisfies $\varrho(f) \geqslant \varrho(F)$.*

Further, they considered the more general complex differential-difference equation

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i) = F(z), \tag{2}$$

where $A_{ij}(z)$ $(i = 0,1,\ldots,n,\ j = 0,1,\ldots,m, n,\ m \in \mathbb{N})$ and $F(z)$ are meromorphic functions of finite order, c_i $(i = 0,\ldots,n)$ are distinct complex constants, and obtained the following theorems for the homogeneous and non-homogeneous equations of (2).

Theorem B ([20]). *Let $A_{ij}(z)$ $(i = 0,1,\dots,n,\ j = 0,1,\dots,m, n,\ m \in \mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_{l0}(z)$ satisfies*

$$\max\{\varrho(A_{ij})\colon\ (i,j) \neq (l,0)\} < \varrho(A_{l0}), \quad \delta(\infty, A_{l0}) > 0.$$

(1) *If $\varrho(F) < \varrho(A_{l0})$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (2) satisfies $\varrho(f) \geqslant \varrho(A_{l0})$. Further, if $F(z) \equiv 0$, then $\varrho(f) \geqslant \varrho(A_{l0}) + 1$.*
(2) *If $\varrho(F) > \varrho(A_{l0})$, then every meromorphic solution $f(z)$ of (2) satisfies $\varrho(f) \geqslant \varrho(F)$.*

Theorem C ([20]). *Let $A_{ij}(z)$ $(i = 0,1,\dots,n,\ j = 0,1,\dots,m, n,\ m \in \mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_{l0}(z)$ satisfies*

$$\lambda\Big(\frac{1}{A_{l0}}\Big) < \varrho(A_{l0}) < \infty, \quad \max\{\varrho(A_{ij})\colon\ (i,j) \neq (l,0)\} \leqslant \varrho(A_{l0}),$$
$$\sum_{\varrho(A_{ij})=\varrho(A_{l0}),(i,j)\neq(l,0)} \tau(A_{ij}) < \tau(A_{l0}) < \infty.$$

(1) *If $\varrho(F) < \varrho(A_{l0})$, or $\varrho(F) = \varrho(A_{l0})$ and $\sum\limits_{\varrho(A_{ij})=\varrho(A_{l0}),(i,j)\neq(l,0)} \tau(A_{ij}) + \tau(F) < \tau(A_{l0})$, or $\varrho(F) = \varrho(A_{l0})$ and $\sum\limits_{\varrho(A_{ij})=\varrho(A_{l0}),(i,j)=(l,0)} \tau(A_{ij}) < \tau(F)$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (2) satisfies $\varrho(f) \geqslant \varrho(A_{l0})$. Further, if $F(z) \equiv 0$, then $\varrho(f) \geqslant \varrho(A_{l0}) + 1$.*
(2) *If $\varrho(F) > \varrho(A_{l0})$, then every meromorphic solution $f(z)$ of (2) satisfies $\varrho(f) \geqslant \varrho(F)$.*

There are many interesting results on the logarithmic order obtained as an answer to the question how to express the growth of solutions of (1) and (2), for the case when their coefficients are meromorphic functions of order zero, we state here some of these results. In previous paper [1], Belaïdi investigated the meromorphic solutions of the special homogeneous case of (1)

$$A_k(z)f(z+k) + \ldots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{3}$$

where $A_k(z), \dots, A_0(z)$ are meromorphic functions of finite logarithmic order, and obtained the following result.

Theorem D ([1]). *Let $A_j(z)$ $(j = 0, 1, \ldots, k)$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_l(z)$ satisfies*

$$\lambda_{\log}\left(\frac{1}{A_l}\right) < \varrho_{\log}(A_l) < \infty, \quad \max\{\varrho_{\log}(A_j)\colon\ j = 0, 1, \ldots, k,\ j \neq l\} \leqslant \varrho_{\log}(A_l),$$

$$\sum_{\varrho_{\log}(A_j)=\varrho_{\log}(A_l), j\neq l} \tau_{\log}(A_j) < \tau_{\log}(A_l) < \infty.$$

If f is a meromorphic solution of (3)*, then $\varrho_{\log}(f) \geqslant \varrho_{\log}(A_l) + 1$.*

He also in [3] considered the homogeneous case of (2)

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z + c_i) = 0, \tag{4}$$

where $A_{ij}(z)$ $(i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m, n,\ m \in \mathbb{N})$ are meromorphic functions of finite logarithmic order, c_i $(i = 0, \ldots, n)$ are distinct complex constants, and obtained the following theorem.

Theorem E ([3]). *Let $A_{ij}(z)$ $(i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m, n,\ m \in \mathbb{N})$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_{l0}(z)$ satisfies*

$$\max\{\varrho_{\log}(A_{ij})\colon\ (i, j) \neq (l, 0)\} < \varrho_{\log}(A_{l0}), \quad \delta(\infty, A_{l0}) > 0.$$

Then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (4) *satisfies $\varrho_{\log}(f) \geqslant \varrho_{\log}(A_{l0}) + 1$.*

In recent paper [5], Biswas considered the logarithmic order of meromorphic solutions of the non-homogeneous equation (2), and obtained the following theorems.

Theorem F ([5]). *Let $A_{ij}(z)$ $(i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m, n,\ m \in \mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0 \leqslant l \leqslant k)$ such that $A_{l0}(z)$ satisfies*

$$\max\{\varrho_{\log}(A_{ij})\colon\ (i, j) \neq (l, 0)\} < \varrho_{\log}(A_{l0}), \quad \delta(\infty, A_{l0}) > 0.$$

(1) *If $\varrho_{\log}(F) < \varrho_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of* (2) *satisfies $\varrho_{\log}(f) \geqslant \varrho_{\log}(A_{l0})$.*
(2) *If $\varrho_{\log}(F) > \varrho_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of* (2) *satisfies $\varrho_{\log}(f) \geqslant \varrho_{\log}(F)$.*

Theorem G ([5]). *Let $A_{ij}(z)$ $(i=0,1,\dots,n,\ j=0,1,\dots,m,n,\ m\in\mathbb{N})$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0\leqslant l\leqslant k)$ such that $A_{l0}(z)$ satisfies*

$$\lambda_{\log}\Big(\frac{1}{A_{l0}}\Big)<\varrho_{\log}(A_{l0})<\infty,\quad \max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}\leqslant\varrho_{\log}(A_{l0}),$$
$$\sum_{\varrho_{\log}(A_{ij})=\varrho_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\tau_{\log}(A_{l0})<\infty.$$

(1) *If $\varrho_{\log}(F)<\varrho_{\log}(A_{l0})$, or $\varrho_{\log}(F)=\varrho_{\log}(A_{l0})$ and*

$$\sum_{\varrho_{\log}(A_{ij})=\varrho_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})+\tau_{\log}(F)<\tau_{\log}(A_{l0}),$$

or $\varrho_{\log}(F)=\varrho_{\log}(A_{l0})$ and

$$\sum_{\varrho_{\log}(A_{ij})=\varrho_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})+\tau_{\log}(A_{l0})<\tau_{\log}(F),$$

then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (2) satisfies $\varrho_{\log}(f)\geqslant\varrho_{\log}(A_{l0})$.

(2) *If $\varrho_{\log}(F)>\varrho_{\log}(A_{l0})$, then every meromorphic solution $f(z)$ of (2) satisfies $\varrho_{\log}(f)\geqslant\varrho_{\log}(F)$.*

Remark 1. We note that $\lambda_{\log}(1/A_{l0})$ in Theorems D and G should be replaced by $\lambda_{\log}(1/A_{l0})+1$.

The main aim of this paper is to continue investigating the logarithmic order of meromorphic solutions of equations (1) and (2) to extend and improve the above theorems. Firstly, for the linear difference equation (1), when one coefficient dominates by its logarithmic lower order, we obtain the following result.

Theorem 1. *Let $A_j(z)$ $(j=0,1,\dots,k)$ and $F(z)$ be meromorphic functions. Suppose there exists an integer l $(0\leqslant l\leqslant k)$ such that $A_l(z)$ satisfies $\delta(\infty,A_l)>0$ and $\max\{\varrho_{\log}(A_j)\colon\ j=0,1,\dots,k,\ j\neq l\}<\mu_{\log}(A_l)\leqslant\varrho_{\log}(A_l)<\infty$.*

(1) *If $\mu_{\log}(F)<\mu_{\log}(A_l)$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (1) satisfies $\varrho_{\log}(f)\geqslant\mu_{\log}(A_l)$. Further, if $F(z)\equiv 0$, then $\mu_{\log}(f)\geqslant\mu_{\log}(A_l)+1$.*
(2) *If $\mu_{\log}(F)>\mu_{\log}(A_l)$, then every meromorphic solution $f(z)$ of (1) satisfies $\varrho_{\log}(f)\geqslant\mu_{\log}(F)$.*

R e m a r k 2. We can replace the condition $\max\{\varrho_{\log}(A_j)\colon\ j{=}0,1,\ldots,k,\ j{\neq}l\}{<}\mu_{\log}(A_l){\leqslant}\varrho_{\log}(A_l)$ in Theorem 1 by

$$\limsup_{r\to\infty}\frac{\sum_{j=0,j\neq l}^{k} m(r,A_j)}{m(r,A_l)}<1$$

for the homogeneous case $F(z)\equiv 0$.

Secondly, for the linear differential-difference equation (2), where we generalize our previous results, we obtain the following theorems.

Theorem 2. *Let* $A_{ij}(z)$ $(i=0,1,\ldots,n,\ j=0,1,\ldots,m,n,\ m\in\mathbb{N})$ *and* $F(z)$ *be meromorphic functions. Suppose there exists an integer* l $(0\leqslant l\leqslant k)$ *such that* $A_{l0}(z)$ *satisfies* $\delta(\infty,A_{l0})>0$ *and* $\max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})\leqslant\varrho_{\log}(A_{l0})<\infty$.

(1) *If* $\mu_{\log}(F)<\mu_{\log}(A_{l0})$, *then every meromorphic solution* $f(z)$ $(\not\equiv 0)$ *of* (2) *satisfies* $\varrho_{\log}(f)\geqslant\mu_{\log}(A_{l0})$. *Further, if* $F(z)\equiv 0$, *then* $\mu_{\log}(f)\geqslant\mu_{\log}(A_{l0})+1$.
(2) *If* $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, *then every meromorphic solution* $f(z)$ *of* (2) *satisfies* $\varrho_{\log}(f)\geqslant\mu_{\log}(F)$.

R e m a r k 3. We can also replace the condition $\max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})\leqslant\varrho_{\log}(A_{l0})$ in Theorem 2 by

$$\limsup_{r\to\infty}\frac{\sum_{(i,j)\neq(l,0)} m(r,A_{ij})}{m(r,A_{l0})}<1$$

for the homogeneous case $F(z)\equiv 0$.

Theorem 3. *Let* $A_{ij}(z)$ $(i=0,1,\ldots,n,\ j=0,1,\ldots,m,n,\ m\in\mathbb{N})$ *and* $F(z)$ *be meromorphic functions. Suppose there exists an integer* l $(0\leqslant l\leqslant k)$ *such that* $A_{l0}(z)$ *satisfies*

$$\lambda_{\log}\Big(\frac{1}{A_{l0}}\Big)+1<\mu_{\log}(A_{l0})<\infty,\quad \max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}\leqslant\mu_{\log}(A_{l0}),$$
$$\tau=\sum_{\varrho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0})<\infty.$$

(1) *If* $\varrho_{\log}(F)<\mu_{\log}(A_{l0})$, *or* $\varrho_{\log}(F)=\mu_{\log}(A_{l0})$ *and* $\tau+\tau_{\log}(F)<\underline{\tau}_{\log}(A_{l0})$, *or* $\mu_{\log}(F)=\mu_{\log}(A_{l0})$ *and* $\tau+\underline{\tau}_{\log}(A_{l0})<\underline{\tau}_{\log}(F)$, *then every meromorphic solution* $f(z)$ $(\not\equiv 0)$ *of* (2) *satisfies* $\varrho_{\log}(f)\geqslant\mu_{\log}(A_{l0})$. *Further, if* $F(z)\equiv 0$, *then* $\mu_{\log}(f)\geqslant\mu_{\log}(A_{l0})+1$.
(2) *If* $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, *then every meromorphic solution* $f(z)$ *of* (2) *satisfies* $\varrho_{\log}(f)\geqslant\mu_{\log}(F)$.

R e m a r k 4. The condition $\lambda_{\log}(1/A_{l0}) + 1 < \mu_{\log}(A_{l0})$ in Theorem 3 can be replaced by $\delta(\infty, A_{l0}) > 0$ with $\delta\underline{\tau}_{\log}(A_{l0})$ instead of $\underline{\tau}_{\log}(A_{l0})$, the only difference between the two conditions that by the condition $\delta(\infty, A_{l0}) > 0$ the case when $\mu_{\log}(A_{l0}) = 1$ is also included.

2. Some lemmas

For the proof of our results we need the following lemmas.

Lemma 1 ([1]). *Let c_1, c_2 be two arbitrary complex numbers such that $c_1 \neq c_2$ and let f be a finite logarithmic order meromorphic function. Let ϱ be the logarithmic order of f. Then for each $\varepsilon > 0$ we have*

$$m\Big(r, \frac{f(z+c_1)}{f(z+c_2)}\Big) = O((\log r)^{\varrho-1+\varepsilon}).$$

Lemma 2 ([11]). *Let f be a meromorphic function, c be a nonzero complex constant. Then we have that for $r \to \infty$*

$$(1+o(1))T(r-|c|, f) \leqslant T(r, f(z+c)) \leqslant (1+o(1))T(r+|c|, f).$$

It follows that $\varrho_{\log}(f(z+c)) = \varrho_{\log}(f)$ and $\mu_{\log}(f(z+c)) = \mu_{\log}(f)$.

Lemma 3 ([2], [3]). *Let f be a meromorphic function with finite logarithmic lower order $1 \leqslant \mu_{\log}(f) < \infty$. Then there exists a subset E_1 of $[1,\infty)$ that has infinite logarithmic measure such that for all $r \in E_1$ we have*

$$T(r, f) < (\log r)^{\mu_{\log}(f)+\varepsilon}.$$

Lemma 4 ([9]). *Let α, R, R' be real numbers such that $0 < \alpha < 1$, $R > 0$, and let η be a nonzero complex number. Then there is a positive constant C_α depending only on α such that for a given meromorphic function f we have, when $|z| = r$, $\max\{1, r+|\eta|\} < R < R'$, the estimate*

$$\begin{aligned} m\Big(r, \frac{f(z+\eta)}{f(z)}\Big) &+ m\Big(r, \frac{f(z)}{f(z+\eta)}\Big) \\ &\leqslant \frac{2|\eta|R}{(R-r-|\eta|)^2}\Big(m(R,f) + m\Big(R, \frac{1}{f}\Big)\Big) \\ &\quad + \frac{2R'}{R'-R}\Big(\frac{|\eta|}{R-r-|\eta|} + \frac{C_\alpha|\eta|^\alpha}{(1-\alpha)r^\alpha}\Big)\Big(N(R',f) + N\Big(R', \frac{1}{f}\Big)\Big). \end{aligned}$$

Lemma 5. *Let η_1, η_2 be two arbitrary complex numbers such that $\eta_1 \neq \eta_2$, and let f be finite logarithmic lower order meromorphic function. Let μ be the logarithmic lower order of f. Then for each $\varepsilon > 0$, there exists a subset $E_2 \subset [1,\infty)$ of infinite logarithmic measure such that for all $r \in E_2$ we have*

$$m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) = O((\log r)^{\mu-1+\varepsilon}).$$

Proof. We have

$$\begin{aligned} m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) &\leqslant m\Big(r, \frac{f(z+\eta_1)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+\eta_2)}\Big) \\ &\leqslant m\Big(r, \frac{f(z+\eta_1)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+\eta_1)}\Big) \\ &\quad + m\Big(r, \frac{f(z+\eta_2)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+\eta_2)}\Big). \end{aligned} \tag{5}$$

Since f has finite logarithmic lower order $\mu_{\log}(f) = \mu < \infty$, so by Lemma 3, for any given ε $(0 < \varepsilon < 2)$, there exists a subset $E_2 \subset [1,\infty)$ of infinite logarithmic measures such that for all $r \in E_2$ we have

$$T(r,f) \leqslant (\log r)^{\mu+\varepsilon/2}. \tag{6}$$

By Lemma 4, we obtain from (5)

(7)

$$\begin{aligned} m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) \leqslant\ & \frac{2|\eta_1|R}{(R-r-|\eta_1|)^2}\Big(m(R,f) + m\Big(R, \frac{1}{f}\Big)\Big) \\ & + \frac{2R'}{R'-R}\Big(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\alpha|\eta_1|^\alpha}{(1-\alpha)r^\alpha}\Big)\Big(N(R',f) + N\Big(R', \frac{1}{f}\Big)\Big) \\ & + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\Big(m(R,f) + m\Big(R, \frac{1}{f}\Big)\Big) \\ & + \frac{2R'}{R'-R}\Big(\frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\alpha|\eta_2|^\alpha}{(1-\alpha)r^\alpha}\Big)\Big(N(R',f) + N\Big(R', \frac{1}{f}\Big)\Big) \\ =\ & \Big(\frac{2|\eta_1|R}{(R-r-|\eta_1|)^2} + \frac{2|\eta_2|R}{(R-r-|\eta_2|)^2}\Big) \\ & \times \Big(m(R,f) + m\Big(R, \frac{1}{f}\Big)\Big) + \frac{2R'}{R'-R} \\ & \times \Big(\frac{|\eta_1|}{R-r-|\eta_1|} + \frac{C_\alpha|\eta_1|^\alpha}{(1-\alpha)r^\alpha} + \frac{|\eta_2|}{R-r-|\eta_2|} + \frac{C_\alpha|\eta_2|^\alpha}{(1-\alpha)r^\alpha}\Big) \\ & \times \Big(N(R',f) + N\Big(R', \frac{1}{f}\Big)\Big). \end{aligned}$$

We choose $\alpha = 1 - \frac{1}{2}\varepsilon$, $R = 2r$, $R' = 3r$ and $r > \max\{|\eta_1|, |\eta_2|, \frac{1}{2}\}$ in (7), we obtain

$$
\begin{aligned}
(8)\quad m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) &\leqslant \Big(\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2}\Big)\Big(m(2r,f) + m\Big(2r, \frac{1}{f}\Big)\Big) \\
&\quad + 6\Big(\frac{|\eta_1|}{r-|\eta_1|} + \frac{2C_\alpha|\eta_1|^{1-\varepsilon/2}}{\varepsilon r^{1-\varepsilon/2}} + \frac{|\eta_2|}{r-|\eta_2|} + \frac{2C_\alpha|\eta_2|^{1-\varepsilon/2}}{\varepsilon r^{1-\varepsilon/2}}\Big) \\
&\quad \times \Big(N(3r,f) + N\Big(3r, \frac{1}{f}\Big)\Big) \\
&\leqslant 4\Big(\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2} + 6\Big(\frac{|\eta_1|}{r-|\eta_1|} + \frac{|\eta_2|}{r-|\eta_2|} \\
&\quad + \frac{2C_\alpha(|\eta_1|^{1-\varepsilon/2} + |\eta_2|^{1-\varepsilon/2})}{\varepsilon r^{1-\varepsilon/2}}\Big)\Big)T(3r,f).
\end{aligned}
$$

Using estimate (6), we get

$$
\begin{aligned}
m\Big(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\Big) &\leqslant 4K\Big(\frac{4|\eta_1|r}{(r-|\eta_1|)^2} + \frac{4|\eta_2|r}{(r-|\eta_2|)^2} + 6\Big(\frac{|\eta_1|}{r-|\eta_1|} + \frac{|\eta_2|}{r-|\eta_2|} \\
&\quad + \frac{2C_\alpha(|\eta_1|^{1-\varepsilon/2} + |\eta_2|^{1-\varepsilon/2})}{\varepsilon r^{1-\varepsilon/2}}\Big)\Big)(\log 3r)^{\mu+\varepsilon/2} \\
&\leqslant M(\log r)^{\mu+\varepsilon-1},
\end{aligned}
$$

where $K > 0$, $M > 0$ are some constants. The proof is completed. $\square$

Lemma 6. *Let f be a meromorphic function with finite logarithmic lower order $1 \leqslant \mu_{\log}(f) < \infty$. Then there exists a subset E_3 of $[1,\infty)$ that has infinite logarithmic measure such that for all $r \in E_3$ we have*

$$\underline{\tau}_{\log}(f) = \lim_{r\to\infty} \frac{T(r,f)}{(\log r)^{\mu_{\log}(f)}}.$$

Consequently, for any given $\varepsilon > 0$ and for all $r \in E_3$ we have

$$T(r,f) < (\underline{\tau}_{\log}(f) + \varepsilon)(\log r)^{\mu_{\log}(f)}.$$

P r o o f. To prove Lemma 6 we use a similar proof as in ([2], Lemma 10) for the case when f is an entire function. $\square$

Lemma 7 ([12])**.** *Let f be a meromorphic function and $k \geqslant 1$ be an integer. Then we have*

$$T(r, f^{(k)}) \leqslant (k+1)T(r,f) + S(r,f),$$

where $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set $E_4 \subset (0,\infty)$ of r of finite linear measure. If f is of finite order, then $S(r,f) = o(T(r,f))$ as $r \to \infty$.

Lemma 8 ([13]). *Let k and j be integers such that $k > j \geqslant 0$. Let f be a meromorphic function in the plane $\mathbb{C}$ such that $f^{(j)}$ does not vanish identically. Then there exists an $r_0 > 1$ such that*

$$m\Big(r, \frac{f^{(k)}}{f^{(j)}}\Big) \leqslant (k-j)\log^+ \frac{\varrho(T(\varrho, f))}{r(\varrho - r)} + \log\frac{k!}{j!} + 5.3078(k-j)$$

for all $r_0 < r < \varrho < \infty$. If f is of finite order s, then

$$\limsup_{r\to\infty} \frac{m(r, f^{(k)}/f^{(j)})}{\log r} \leqslant \max\{0, (k-j)(s-1)\}.$$

3. Proof of the theorems

In our proofs, we always suppose that f is of finite logarithmic order ($\varrho_{\log}(f) < \infty$), otherwise the results are trivial.

Proof of Theorem 1. Let $f(z)$ ($\not\equiv 0$) be a meromorphic solution of (1). We divide (1) by $f(z+c_l)$ to get

$$-A_l(z) = \sum_{j=1, j\neq l}^{k} A_j(z)\frac{f(z+c_j)}{f(z+c_l)} + A_0(z)\frac{f(z)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}, \tag{9}$$

it follows that

$$\begin{aligned} m(r, A_l(z)) \leqslant{} & \sum_{j=0, j\neq l}^{k} m(r, A_j(z)) + \sum_{j=1, j\neq l}^{k} m\Big(r, \frac{f(z+c_j)}{f(z+c_l)}\Big) + m\Big(r, \frac{f(z)}{f(z+c_l)}\Big) \\ & + m(r, F(z)) + m\Big(r, \frac{1}{f(z+c_l)}\Big) + O(1). \end{aligned} \tag{10}$$

By (10), Lemma 1 and Lemma 2, for any given $\varepsilon > 0$ we have

$$\begin{aligned} m(r, A_l(z)) &\leqslant \sum_{j=0, j\neq l}^{k} T(r, A_j(z)) + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) \\ &\quad + T(r, F(z)) + T(r, f(z+c_l)) + O(1) \\ &\leqslant \sum_{j=0, j\neq l}^{k} T(r, A_j(z)) + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) \\ &\quad + T(r, F(z)) + (1+o(1))T(r+|c_l|, f(z)) \\ &\leqslant \sum_{j=0, j\neq l}^{k} T(r, A_j(z)) + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) \\ &\quad + T(r, F(z)) + 2T(2r, f(z)). \end{aligned} \tag{11}$$

Setting

$$\liminf_{r\to\infty} \frac{m(r,A_l)}{T(r,A_l)} = \delta(\infty, A_l) = \delta > 0 \tag{12}$$

and $\max\{\varrho_{\log}(A_j)\colon\ j=0,1,\ldots,k,\ j\neq l\} = \varrho < \mu_{\log}(A_l)$, by (12) and the definition of $\mu_{\log}(A_l)$, for any given ε $(0<\varepsilon<\frac{1}{2}(\mu_{\log}(A_l)-\varrho))$ and sufficiently large r we have

$$m(r,A_l) \geqslant \frac{\delta}{2} T(r,A_l) \geqslant \frac{\delta}{2}(\log r)^{\mu_{\log}(A_l)-\varepsilon/2} \geqslant (\log r)^{\mu_{\log}(A_l)-\varepsilon}. \tag{13}$$

By the definition of $\varrho_{\log}(A_j)$, $j=0,1,\ldots,k$, $j\neq l$, for the above ε and sufficiently large r we obtain

$$T(r,A_j) \leqslant (\log r)^{\varrho+\varepsilon}, \quad j=0,1,\ldots,k,\ j\neq l. \tag{14}$$

(1) If $\mu_{\log}(F) < \mu_{\log}(A_l)$, then by Lemma 3, there exists a subset $E_1 \subset [1,\infty)$ with infinite logarithmic measure such that for any given ε $(0<\varepsilon<\frac{1}{2}(\mu_{\log}(A_l)-\mu_{\log}(F)))$ and for all $r\in E_1$ we have

$$T(r,F) \leqslant (\log r)^{\mu_{\log}(F)+\varepsilon}. \tag{15}$$

By substituting (13)–(15) into (11), for any given ε satisfying

$$0<\varepsilon<\min\Big\{\frac{\mu_{\log}(A_l)-\varrho}{2}, \frac{\mu_{\log}(A_l)-\mu_{\log}(F)}{2}\Big\}$$

and for all $r\in E_1$ we obtain

$$\begin{aligned}(\log r)^{\mu_{\log}(A_l)-\varepsilon} &\leqslant k(\log r)^{\varrho+\varepsilon} + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) \\ &\quad + (\log r)^{\mu_{\log}(F)+\varepsilon} + O((\log r)^{\varrho_{\log}(f)+\varepsilon}),\end{aligned} \tag{16}$$

which implies that

$$(1-o(1))(\log r)^{\mu_{\log}(A_l)-\varepsilon} \leqslant O((\log r)^{\varrho_{\log}(f)+\varepsilon}). \tag{17}$$

By (17), we get $\mu_{\log}(A_l)-2\varepsilon \leqslant \varrho_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we deduce that $\mu_{\log}(A_l) \leqslant \varrho_{\log}(f)$.

Further, for the homogeneous case $F(z)\equiv 0$, by (10) and Lemma 5, there exists a subset $E_2 \subset [1,\infty)$ with infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $r\in E_2$ we get

$$m(r,A_l(z)) \leqslant \sum_{j=0,j\neq l}^{k} T(r,A_j(z)) + O((\log r)^{\mu_{\log}(f)-1+\varepsilon}). \tag{18}$$

Substituting (13) and (14) into (18), for any given ε $(0<\varepsilon<\frac{1}{2}(\mu_{\log}(A_l)-\varrho))$ and for all $r\in E_2$ we obtain

$$(\log r)^{\mu_{\log}(A_l)-\varepsilon}\leqslant k(\log r)^{\varrho+\varepsilon}+O((\log r)^{\mu_{\log}(f)-1+\varepsilon}). \tag{19}$$

Then

$$(1-o(1))(\log r)^{\mu_{\log}(A_l)-\varepsilon}\leqslant O((\log r)^{\mu_{\log}(f)-1+\varepsilon}). \tag{20}$$

It follows that $\mu_{\log}(A_l)+1-2\varepsilon\leqslant\mu_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we obtain $\mu_{\log}(A_l)+1\leqslant\mu_{\log}(f)$.

(2) Let f be a meromorphic solution of (1). If $\mu_{\log}(F)>\mu_{\log}(A_l)$, then for any given ε $(0<\varepsilon<\frac{1}{2}(\mu_{\log}(F)-\mu_{\log}(A_l)))$ and sufficiently large r we have

$$T(r,F)\geqslant(\log r)^{\mu_{\log}(F)-\varepsilon}. \tag{21}$$

By Lemma 3, there exists a subset $E_1\subset[1,\infty)$ with infinite logarithmic measure such that for the above ε and for all $r\in E_1$ we obtain

$$T(r,A_l)\leqslant(\log r)^{\mu_{\log}(A_l)+\varepsilon}. \tag{22}$$

By (1) and Lemma 2, we have

$$\begin{aligned} T(r,F(z))&\leqslant\sum_{j=0,j\neq l}^{k}T(r,A_j(z))+T(r,A_l(z))\\ &\quad+\sum_{j=1}^{k}T(r,f(z+c_j))+T(r,f(z))+O(1)\\ &\leqslant\sum_{j=0,j\neq l}^{k}T(r,A_j(z))+T(r,A_l(z))+(2k+1)T(2r,f(z))+O(1). \end{aligned} \tag{23}$$

Substituting (14), (21) and (22) into (23), for the above ε and for all $r\in E_1$ we get

$$(\log r)^{\mu_{\log}(F)-\varepsilon}\leqslant k(\log r)^{\varrho+\varepsilon}+(\log r)^{\mu_{\log}(A_l)+\varepsilon}+(2k+1)T(2r,f(z))+O(1). \tag{24}$$

So

$$(1-o(1))(\log r)^{\mu_{\log}(F)-\varepsilon}\leqslant O((\log r)^{\varrho_{\log}(f)+\varepsilon}). \tag{25}$$

It follows that $\mu_{\log}(F)-2\varepsilon\leqslant\varrho_{\log}(f)$. Since $\varepsilon>0$ is arbitrary, we get $\mu_{\log}(F)\leqslant\varrho_{\log}(f)$. $\square$

P r o o f of Theorem 2. Let $f(z)$ ($\not\equiv 0$) be a meromorphic solution of (2). We divide (2) by $f(z+c_l)$ to get

(26)
$$-A_{l0}(z)=\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}A_{ij}(z)\frac{f^{(j)}(z+c_i)f(z+c_i)}{f(z+c_i)f(z+c_l)}+\sum_{j=1}^{m}A_{ij}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}.$$

By (26), it follows

$$\begin{aligned}(27)\qquad m(r,A_{l0}(z))\leqslant&\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}m(r,A_{ij}(z))+\sum_{j=1}^{m}m(r,A_{lj}(z))\\&+\sum_{i=0}^{n}\sum_{j=1}^{m}m\Big(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\Big)+\sum_{i=0,i\neq l}^{n}m\Big(r,\frac{f(z+c_i)}{f(z+c_l)}\Big)\\&+m\Big(r,\frac{F(z)}{f(z+c_l)}\Big)+O(1).\end{aligned}$$

From Lemma 8, for sufficiently large r we obtain

$$(28)\qquad m\Big(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\Big)\leqslant 2j\log^{+}T(2r,f),\quad (i=0,1,\dots,n,\ j=1,\dots,m).$$

By (27), (28), Lemma 1 and Lemma 2, for any given $\varepsilon>0$ we have

$$\begin{aligned}(29)\quad m(r,A_{l0}(z))\leqslant&\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij}(z))+\sum_{j=1}^{m}T(r,A_{lj}(z))+O(\log^{+}T(2r,f))\\&+O((\log r)^{\varrho_{\log}(f)-1+\varepsilon})+T(r,F(z))\\&+(1+o(1))T(r+|c_l|,f(z))+O(1)\\\leqslant&\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij}(z))+\sum_{j=1}^{m}T(r,A_{lj}(z))+O(\log(\log r))\\&+O((\log r)^{\varrho_{\log}(f)-1+\varepsilon})+T(r,F(z))+2T(2r,f(z))\\\leqslant&\sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij}(z))+\sum_{j=1}^{m}T(r,A_{lj}(z))+O(\log(\log r))\\&+O((\log r)^{\varrho_{\log}(f)-1+\varepsilon})+T(r,F(z))+O((\log r)^{\varrho_{\log}(f)+\varepsilon}).\end{aligned}$$

We suppose that $\delta(\infty,A_{l0})=\delta>0$ and $\max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}=\varrho<\mu_{\log}(A_{l0})$.

(1) If $\mu_{\log}(F)<\mu_{\log}(A_{l0})$, then by using a similar reasoning method as in (11)–(17) from the proof of Theorem 1, we obtain $\mu_{\log}(A_{l0})\leqslant\varrho_{\log}(f)$.

Further, if $F(z)\equiv 0$, then by (27), (28) and Lemma 5, there exists a subset $E_2\subset[1,\infty)$ with infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $r \in E_2$ we have

(30)

$$m(r, A_{l0}) \leqslant \sum_{i=0, i\neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + O(\log(\log r)) + O((\log r)^{\mu_{\log}(f)-1+\varepsilon}).$$

Similarly as in (18)–(20) from the proof of Theorem 1, we get $\mu_{\log}(A_{l0}) + 1 \leqslant \mu_{\log}(f)$.

(2) Let f be a meromorphic solution of (2). If $\mu_{\log}(F) > \mu_{\log}(A_{l0})$, then by (2), Lemma 2 and Lemma 7, we have

(31)

$$\begin{aligned} T(r, F(z)) &\leqslant \sum_{(i,j)\neq(l,0)} T(r, A_{ij}(z)) + T(r, A_{l0}(z)) + \sum_{i=0}^{n} \sum_{j=0}^{m} T(r, f^{(j)}(z + c_i)) + O(1) \\ &\leqslant \sum_{(i,j)\neq(l,0)} T(r, A_{ij}(z)) + T(r, A_{l0}(z)) + \sum_{i=0}^{n} \sum_{j=0}^{m} ((j + 1)T(r, f(z + c_i)) \\ &\quad + S(r, f)) + O(1) \\ &\leqslant \sum_{(i,j)\neq(l,0)} T(r, A_{ij}(z)) + T(r, A_{l0}(z)) + O(T(2r, f(z))) + o(T(r, f)). \end{aligned}$$

Then by using a similar reasoning method as in (23)–(25) from the proof of Theorem 1, we get $\mu_{\log}(F) \leqslant \varrho_{\log}(f)$. $\square$

P r o o f of Theorem 3. Let $f(z)$ ($\not\equiv 0$) be a meromorphic solution of (2). By (29), for any given $\varepsilon > 0$ we have

$$\begin{aligned} T(r, A_{l0}(z)) &= m(r, A_{l0}(z)) + N(r, A_{l0}(z)) \\ &\leqslant \sum_{i=0, i\neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}(z)) + \sum_{j=1}^{m} T(r, A_{lj}(z)) + O(\log(\log r)) \\ &\quad + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) + T(r, F(z)) \\ &\quad + O((\log r)^{\varrho_{\log}(f)+\varepsilon}) + N(r, A_{l0}(z)). \end{aligned} \tag{32}$$

(1) If $\varrho_{\log}(F) < \mu_{\log}(A_{l0})$, then for any given ε $(0 < \varepsilon < \frac{1}{2}(\mu_{\log}(A_{l0}) - \varrho_{\log}(F)))$ and sufficiently large r we have

$$T(r, F) \leqslant (\log r)^{\varrho_{\log}(F)+\varepsilon}. \tag{33}$$

Setting $k = m + n(m + 1)$, we suppose that $\varrho = \max\{\varrho_{\log}(A_{ij})\colon (i, j) \neq (l, 0)\} < \mu_{\log}(A_{l0})$. Then by the definitions of $\mu_{\log}(A_{l0})$ and $\varrho_{\log}(A_{ij})$, for any given ε $(0 < \varepsilon < \frac{1}{2}(\mu_{\log}(A_{l0}) - \varrho))$ and sufficiently large r we get

$$T(r, A_{l0}) \geqslant (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \tag{34}$$

and

$$T(r, A_{ij}) \leqslant (\log r)^{\varrho_{\log}(A_{ij})+\varepsilon} \leqslant (\log r)^{\varrho+\varepsilon}, \quad (i,j) \neq (l,0). \tag{35}$$

By the definition of $\lambda_{\log}(1/A_{l0})$, for any given ε $(0 < \varepsilon < \frac{1}{2}(\mu_{\log}(A_{l0}) - \lambda_{\log}(1/A_{l0}) - 1))$ and sufficiently large r we have

$$N(r, A_{l0}) \leqslant (\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}. \tag{36}$$

By substituting the assumptions (33)–(36) into (32), for any given ε satisfying

$$0 < \varepsilon < \min\Big\{\frac{\mu_{\log}(A_{l0}) - \varrho}{2}, \frac{\mu_{\log}(A_{l0}) - \lambda_{\log}(1/A_{l0}) - 1}{2}, \frac{\mu_{\log}(A_{l0}) - \varrho_{\log}(F)}{2}\Big\}$$

and sufficiently large r we obtain

$$\begin{aligned}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leqslant & \sum_{i=0, i\neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}(z)) + \sum_{j=1}^{m} T(r, A_{lj}(z)) + O(\log(\log r)) \\ & + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) + (\log r)^{\varrho_{\log}(F)+\varepsilon} \\ & + O((\log r)^{\varrho_{\log}(f)+\varepsilon}) + (\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}.\end{aligned} \tag{37}$$

Then

$$(1 - o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \leqslant O((\log r)^{\varrho_{\log}(f)+\varepsilon}), \tag{38}$$

which implies that $\varrho_{\log}(f) \geqslant \mu_{\log}(A_{l0}) - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\varrho_{\log}(f) \geqslant \mu_{\log}(A_{l0})$. Now we suppose that $\max\{\varrho_{\log}(A_{ij}) \colon (i,j) \neq (l,0)\} = \mu_{\log}(A_{l0})$ and $\tau = \sum_{\varrho_{\log}(A_{ij}) = \mu_{\log}(A_{l0}), (i,j)\neq(l,0)} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0})$. Then there exist two sets $\Gamma_1 \subseteq \{(i,j) \colon i = 0,1,\ldots,n,\ j = 0,1,\ldots,m,\ (i,j) \neq (l,0)\}$ and $\Gamma_2 = \{(i,j) \colon i = 0,1,\ldots,n,\ j = 0,1,\ldots,m,\ (i,j) \neq (l,0)\} \setminus \Gamma_1$ such that for $(i,j) \in \Gamma_1$ we have $\varrho_{\log}(A_{ij}) = \mu_{\log}(A_{l0})$ with $\tau = \sum_{(i,j)\in\Gamma_1} \tau_{\log}(A_{ij}) < \underline{\tau}_{\log}(A_{l0})$ and for $(i,j) \in \Gamma_2$ we have $\varrho_{\log}(A_{ij}) < \mu_{\log}(A_{l0})$. Hence, for any given ε $(0 < \varepsilon < (\underline{\tau}_{\log}(A_{l0}) - \tau)/(k+1))$ and sufficiently large r we get

$$T(r, A_{ij}) \leqslant (\tau_{\log}(A_{ij}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}, \quad (i,j) \in \Gamma_1 \tag{39}$$

and

$$T(r, A_{ij}) \leqslant (\log r)^{\mu_{\log}(A_{l0})-\varepsilon}, \quad (i,j) \in \Gamma_2. \tag{40}$$

By the definition of $\underline{\tau}_{\log}(A_{l0})$, for the above ε and sufficiently large r we have

$$T(r, A_{l0}) \geqslant (\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{41}$$

By substituting assumptions (33), (36), (39)–(41) into (32), for any given ε satisfying

$$0 < \varepsilon < \min\Big\{\frac{\underline{\tau}_{\log}(A_{l0}) - \tau}{k+1}, \frac{\mu_{\log}(A_{l0}) - \lambda_{\log}(1/A_{l0}) - 1}{2}, \frac{\mu_{\log}(A_{l0}) - \varrho_{\log}(F)}{2}\Big\}$$

and for sufficiently large r we obtain

$$\begin{aligned} &(\underline{\tau}_{\log}(A_{l0}) - \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \\ &\quad\leqslant \sum_{(i,j)\in\Gamma_1} (\tau_{\log}(A_{ij}) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + \sum_{(i,j)\in\Gamma_2} (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} \\ &\qquad + O(\log(\log r)) + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) + (\log r)^{\varrho_{\log}(F)+\varepsilon} \\ &\qquad + O((\log r)^{\varrho_{\log}(f)+\varepsilon}) + (\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon} \\ &\quad\leqslant (\tau + k\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} + O((\log r)^{\mu_{\log}(A_{l0})-\varepsilon}) + O(\log(\log r)) \\ &\qquad + O((\log r)^{\varrho_{\log}(f)-1+\varepsilon}) + (\log r)^{\varrho_{\log}(F)+\varepsilon} \\ &\qquad + O((\log r)^{\varrho_{\log}(f)+\varepsilon}) + (\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}. \end{aligned} \tag{42}$$

Thus,

$$(1 - o(1))(\underline{\tau}_{\log}(A_{l0}) - \tau - (k+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})} \leqslant O((\log r)^{\varrho_{\log}(f)+\varepsilon}). \tag{43}$$

It follows by (43) that $\varrho_{\log}(f) \geqslant \mu_{\log}(A_{l0}) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\varrho_{\log}(f) \geqslant \mu_{\log}(A_{l0})$.

If $\varrho_{\log}(F) = \mu_{\log}(A_{l0})$ and $\tau + \tau_{\log}(F) < \underline{\tau}_{\log}(A_{l0})$, then for any given $\varepsilon > 0$ and for sufficiently large r we have

$$T(r, F) \leqslant (\tau_{\log}(F) + \varepsilon)(\log r)^{\mu_{\log}(A_{l0})}. \tag{44}$$

By substituting assumptions (36), (39)–(41) and (44) into (32), for any given ε satisfying

$$0 < \varepsilon < \min\Big\{\frac{\underline{\tau}_{\log}(A_{l0}) - \tau - \tau_{\log}(F)}{k+2}, \frac{\mu_{\log}(A_{l0}) - \lambda_{\log}(1/A_{l0}) - 1}{2}\Big\}$$

and for sufficiently large r we get

$$(45)\qquad \begin{aligned}&(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\ &\leqslant \sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\ &\quad+O(\log(\log r))+O((\log r)^{\varrho_{\log}(f)-1+\varepsilon})+(\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\ &\quad+O((\log r)^{\varrho_{\log}(f)+\varepsilon})+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}\\ &\leqslant(\tau+k\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O((\log r)^{\mu_{\log}(A_{l0})-\varepsilon})+O(\log(\log r))\\ &\quad+O((\log r)^{\varrho_{\log}(f)-1+\varepsilon})+(\tau_{\log}(F)+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\ &\quad+O((\log r)^{\varrho_{\log}(f)+\varepsilon})+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}.\end{aligned}$$

It follows that

$$(46)\quad (1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau-\tau_{\log}(F)-(k+2)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leqslant O((\log r)^{\varrho_{\log}(f)+\varepsilon}).$$

This implies that $\varrho_{\log}(f)\geqslant\mu_{\log}(A_{l0})-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\varrho_{\log}(f)\geqslant\mu_{\log}(A_{l0})$.

If $\mu_{\log}(F)=\mu_{\log}(A_{l0})$ and $\tau+\underline{\tau}_{\log}(A_{l0})<\underline{\tau}_{\log}(F)$, then for any sufficiently small $\varepsilon>0$ and for sufficiently large r we have

$$(47)\qquad T(r,F)>(\underline{\tau}_{\log}(F)-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}.$$

By Lemma 6, there exists a subset $E_3\subset[1,\infty)$ of infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $r\in E_3$ we have

$$(48)\qquad T(r,A_{l0})\leqslant(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}.$$

Substituting assumptions (39), (40), (47)–(48), into (31), for every sufficiently small ε satisfying $0<\varepsilon<(\underline{\tau}_{\log}(F)-\tau-\underline{\tau}_{\log}(A_{l0}))/(k+2)$ and for all $r\in E_3$ we obtain

$$(49)\qquad \begin{aligned}&(\underline{\tau}_{\log}(F)-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\ &\leqslant\sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\ &\quad+(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O(T(2r,f(z)))+o(T(r,f))\\ &\leqslant(\tau+\underline{\tau}_{\log}(A_{l0})+(k+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O((\log r)^{\mu_{\log}(A_{l0})-\varepsilon})\\ &\quad+O(T(2r,f(z)))+o(T(r,f)).\end{aligned}$$

So

$$(50)\qquad \begin{aligned}&(1-o(1))(\underline{\tau}_{\log}(F)-\tau-\underline{\tau}_{\log}(A_{l0})-(k+2)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\ &\qquad\leqslant O(T(2r,f(z)))+o(T(r,f)),\end{aligned}$$

which implies that $\varrho_{\log}(f)\geqslant\mu_{\log}(A_{l0})$.

Further, for the homogeneous case $F(z)\equiv 0$, by (30), for any given $\varepsilon>0$ and for all $r\in E_2$ we have

$$
\begin{aligned}
(51)\qquad T(r,A_{l0}(z)) &= m(r,A_{l0}(z))+N(r,A_{l0}(z))\\
&\leqslant \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}T(r,A_{ij})+\sum_{j=1}^{m}T(r,A_{lj})+O(\log(\log r))\\
&\quad+O((\log r)^{\mu_{\log}(f)-1+\varepsilon})+N(r,A_{l0}(z)).
\end{aligned}
$$

If $\varrho=\max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})$, then by substituting assumptions (34)–(36) into (51), for any given ε satisfying

$$
0<\varepsilon<\min\Big\{\frac{\mu_{\log}(A_{l0})-\varrho}{2},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(1/A_{l0})-1}{2}\Big\}
$$

and for all $r\in E_2$ we obtain

$$
\begin{aligned}
(52)\qquad (\log r)^{\mu_{\log}(A_{l0})-\varepsilon} &\leqslant \sum_{i=0,i\neq l}^{n}\sum_{j=0}^{m}(\log r)^{\varrho+\varepsilon}+\sum_{j=1}^{m}(\log r)^{\varrho+\varepsilon}+O(\log(\log r))\\
&\quad+O((\log r)^{\mu_{\log}(f)-1+\varepsilon})+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}\\
&\leqslant k(\log r)^{\varrho+\varepsilon}+O(\log(\log r))+O((\log r)^{\mu_{\log}(f)-1+\varepsilon})\\
&\quad+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}.
\end{aligned}
$$

Then

$$
(53)\qquad (1-o(1))(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\leqslant O((\log r)^{\mu_{\log}(f)-1+\varepsilon}),
$$

which implies that $\mu_{\log}(f)\geqslant\mu_{\log}(A_{l0})+1-2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we deduce that $\mu_{\log}(f)\geqslant\mu_{\log}(A_{l0})+1$. If $\max\{\varrho_{\log}(A_{ij})\colon\ (i,j)\neq(l,0)\}=\mu_{\log}(A_{l0})$ and $\tau=\sum_{\varrho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0})$, then by substituting assumptions (36), (39)–(41) into (51), for any given ε satisfying

$$
0<\varepsilon<\min\Big\{\frac{\underline{\tau}_{\log}(A_{l0})-\tau}{k+1},\frac{\mu_{\log}(A_{l0})-\lambda_{\log}(1/A_{l0})-1}{2}\Big\}
$$

and for all $r\in E_2$ we get

$$
\begin{aligned}
(54)\qquad &(\underline{\tau}_{\log}(A_{l0})-\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\\
&\leqslant \sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\
&\quad+O(\log(\log r))+O((\log r)^{\mu_{\log}(f)-1+\varepsilon})+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}\\
&\leqslant(\tau+k\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O((\log r)^{\mu_{\log}(A_{l0})-\varepsilon})\\
&\quad+O(\log(\log r))+O((\log r)^{\mu_{\log}(f)-1+\varepsilon})+(\log r)^{\lambda_{\log}(1/A_{l0})+1+\varepsilon}.
\end{aligned}
$$

It follows that

$$(55)\qquad (1-o(1))(\underline{\tau}_{\log}(A_{l0})-\tau-(k+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}\leqslant O((\log r)^{\mu_{\log}(f)-1+\varepsilon}),$$

that is, $\mu_{\log}(f)\geqslant \mu_{\log}(A_{l0})+1-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\mu_{\log}(f)\geqslant \mu_{\log}(A_{l0})+1$.

(2) Let f be a meromorphic solution of (2). If $\mu_{\log}(F)>\mu_{\log}(A_{l0})$, then for any given ε $(0<\varepsilon<\frac{1}{2}(\mu_{\log}(F)-\mu_{\log}(A_{l0})))$ and sufficiently large r we have

$$(56)\qquad T(r,F)\geqslant (\log r)^{\mu_{\log}(F)-\varepsilon}.$$

By Lemma 3, there exists a subset $E_1\subset[1,\infty)$ of infinite logarithmic measure such that for any given $\varepsilon>0$ and for all $r\in E_1$ we have

$$(57)\qquad T(r,A_{l0})\leqslant (\log r)^{\mu_{\log}(A_{l0})+\varepsilon}.$$

If $\varrho=\max\{\varrho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}<\mu_{\log}(A_{l0})$, then by substituting assumptions (35), (56) and (57) into (31), for any given ε satisfying $0<\varepsilon<\frac{1}{2}(\mu_{\log}(F)-\mu_{\log}(A_{l0}))$ and for all $r\in E_1$ we get

(58)
$$\begin{aligned}(\log r)^{\mu_{\log}(F)-\varepsilon}&\leqslant \sum_{(i,j)\neq(l,0)}(\log r)^{\varrho+\varepsilon}+(\log r)^{\mu_{\log}(A_{l0})+\varepsilon}+O(T(2r,f(z)))+o(T(r,f))\\&=k(\log r)^{\varrho+\varepsilon}+(\log r)^{\mu_{\log}(A_{l0})+\varepsilon}+O(T(2r,f(z)))+o(T(r,f)).\end{aligned}$$

Then

$$(59)\qquad (1-o(1))(\log r)^{\mu_{\log}(F)-\varepsilon}\leqslant O(T(2r,f(z)))+o(T(r,f)).$$

It follows by (59) that $\varrho_{\log}(f)\geqslant\mu_{\log}(F)-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we deduce that $\varrho_{\log}(f)\geqslant\mu_{\log}(F)$.

If $\max\{\varrho_{\log}(A_{ij})\colon (i,j)\neq(l,0)\}=\mu_{\log}(A_{l0})$ and

$$\tau=\sum_{\varrho_{\log}(A_{ij})=\mu_{\log}(A_{l0}),(i,j)\neq(l,0)}\tau_{\log}(A_{ij})<\underline{\tau}_{\log}(A_{l0}),$$

then by substituting assumptions (39), (40), (48) and (56) into (31), for any given ε satisfying $0<\varepsilon<\frac{1}{2}(\mu_{\log}(F)-\mu_{\log}(A_{l0}))$ and for all $r\in E_3$ we have

(60)
$$\begin{aligned}(\log r)^{\mu_{\log}(F)-\varepsilon}\leqslant{}&\sum_{(i,j)\in\Gamma_1}(\tau_{\log}(A_{ij})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+\sum_{(i,j)\in\Gamma_2}(\log r)^{\mu_{\log}(A_{l0})-\varepsilon}\\&+(\underline{\tau}_{\log}(A_{l0})+\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O(T(2r,f(z)))+o(T(r,f))\\\leqslant{}&(\tau+\underline{\tau}_{\log}(A_{l0})+(k+1)\varepsilon)(\log r)^{\mu_{\log}(A_{l0})}+O((\log r)^{\mu_{\log}(A_{l0})-\varepsilon})\\&+O(T(2r,f(z)))+o(T(r,f)).\end{aligned}$$

It follows that

$$(61)\qquad (1-o(1))(\log r)^{\mu_{\log}(F)-\varepsilon} \leqslant O(T(2r, f(z))) + o(T(r,f)).$$

By (61), we conclude that $\varrho_{\log}(f) \geqslant \mu_{\log}(F)$. $\square$

A c k n o w l e d g e m e n t s. The author would like to thank the anonymous referee for his/her helpful remarks and suggestions to improve this article.

References

[1] *B. Belaïdi*: Growth of meromorphic solutions of finite logarithmic order of linear difference equations. Fasc. Math. *54* (2015), 5–20. zbl MR doi

[2] *B. Belaïdi*: Some properties of meromorphic solutions of logarithmic order to higher order linear difference equations. Bul. Acad. Ştiinţe Repub. Mold., Mat. *2017* (2017), 15–28. zbl MR

[3] *B. Belaïdi*: Study of solutions of logarithmic order to higher order linear differential-difference equations with coefficients having the same logarithmic order. Univ. Iagell. Acta Math. *54* (2017), 15–32. zbl MR doi

[4] *B. Belaïdi*: Differential polynomials generated by solutions of second order non-homogeneous linear differential equations. Rad Hrvat. Akad. Znan. Umjet., Mat. Znan. *26* (2022), 139–153. zbl MR doi

[5] *N. Biswas*: Growth of solutions of linear differential-difference equations with coefficients having the same logarithmic order. Korean J. Math. *29* (2021), 473–481. zbl MR doi

[6] *T.-B. Cao, K. Liu, J. Wang*: On the growth of solutions of complex differential equations with entire coefficients of finite logarithmic order. Math. Rep., Buchar. *15* (2013), 249–269. zbl MR

[7] *B. Q. Chen, Z. X. Chen, S. Li*: Properties on solutions of some q-difference equations. Acta Math. Sin., Engl. Ser. *26* (2010), 1877–1886. zbl MR doi

[8] *P. T.-Y. Chern*: On meromorphic functions with finite logarithmic order. Trans. Am. Math. Soc. *358* (2006), 473–489. zbl MR doi

[9] *Y.-M. Chiang, S.-J. Feng*: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. *16* (2008), 105–129. zbl MR doi

[10] *A. Ferraoun, B. Belaïdi*: Growth and oscillation of solutions to higher order linear differential equations with coefficients of finite logarithmic order. Sci. Stud. Res., Ser. Math. Inform. *26* (2016), 115–144. zbl MR

[11] *A. A. Goldberg, I. V. Ostrovskii*: Value Distribution of Meromorphic Functions. Translations of Mathematical Monographs 236. AMS, Providence, 2008. zbl MR doi

[12] *W. K. Hayman*: Meromorphic Functions. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964. zbl MR

[13] *J. Heittokangas, R. Korhonen, J. Rättyä*: Generalized logarithmic derivative estimates of Gol'dberg-Grinshtein type. Bull. Lond. Math. Soc. *36* (2004), 105–114. zbl MR doi

[14] *I. Laine*: Nevanlinna Theory and Complex Differential Equations. de Gruyter Studies in Mathematics 15. Walter de Gruyter, Berlin, 1993. zbl MR doi

[15] *H. Liu, Z. Mao*: On the meromorphic solutions of some linear difference equations. Adv. Difference Equ. *2013* (2013), Article ID 133, 12 pages. zbl MR doi

[16] *Z.-T. Wen*: Finite logarithmic order solutions of linear q-difference equations. Bull. Korean Math. Soc. *51* (2014), 83–98. zbl MR doi

[17] *S. Wu, X. Zheng*: Growth of meromorphic solutions of complex linear differential-difference equations with coefficients having the same order. J. Math. Res. Appl. *34* (2014), 683–695. zbl MR doi
[18] *C.-C. Yang, H.-X. Yi*: Uniqueness Theory of Meromorphic Functions. Mathematics and its Applications 557. Kluwer Academic, Dordrecht, 2003. zbl MR doi
[19] *X.-M. Zheng, Z.-X. Chen*: Some properties of meromorphic solutions of q-difference equations. J. Math. Anal. Appl. *361* (2010), 472–480. zbl MR doi
[20] *Y.-P. Zhou, X.-M. Zheng*: Growth of meromorphic solutions to homogeneous and non-homogeneous linear (differential)-difference equations with meromorphic coefficients. Electron. J. Differ. Equ. *2017* (2017), Article ID 34, 15 pages. zbl MR

Authors' address: *Abdelkader Dahmani*, *Benharrat Belaïdi* (corresponding author), Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria, e-mail: `abdelkader.dahmani.etu@univ-mosta.dz`, `benharrat.belaidi@univ-mosta.dz`.