2024 MATHEMATICA BOHEMICA 30 pp
Online first

GLOBAL WELL-POSEDNESS AND ENERGY DECAY FOR A ONE DIMENSIONAL POROUS-ELASTIC SYSTEM SUBJECT TO A NEUTRAL DELAY

Houssem Eddine Khochemane, Azzaba, Sara Labidi, Annaba, Sami Loucif, Tebessa, Abdelhak Djebabla, Annaba

Received July 10, 2023. Published online July 2, 2024.
Communicated by Michela Eleuteri

Abstract. We consider a one-dimensional porous-elastic system with porous-viscosity and a distributed delay of neutral type. First, we prove the global existence and uniqueness of the solution by using the Faedo-Galerkin approximations along with some energy estimates. Then, based on the energy method with some appropriate assumptions on the kernel of neutral delay term, we construct a suitable Lyapunov functional and we prove that, despite of the destructive nature of delays in general, the damping mechanism considered provokes an exponential decay of the solution for the case of equal speed of wave propagation. In the case of lack of exponential stability, we show that the solution decays polynomially.

Keywords: exponential decay; polynomial decay; porous-elastic system; neutral delay; multipliers method; Faedo-Galerkin approximations

MSC 2020: 35L70, 35B40, 93D20, 74D05, 93D15

1. Introduction

In 1972, Goodman and Cowin (see [17]) gave an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin (see [30]) presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils and to manufactured porous materials.

DOI: 10.21136/MB.2024.0104-23

In [31], Quintanilla gave the first investigation concerning the study of asymptotic behavior of the solutions for a one-dimensional porous-elastic system, where he proved that the damping through porous-viscosity is not strong enough to provoke an exponential decay. In [1], [2], Apalara showed that the same system considered in [31] is exponentially stable for the case of equal speeds of wave propagation. In [7], Casas and Quintanilla studied the one-dimensional porous-elastic system in the presence of the usual thermal effect with microtemperature damping and they used the semigroup approach to prove the exponential stability of the solutions irrespective of the speeds of wave propagations. In [6], Casas and Quintanilla proved that the combination of porous-viscosity and thermal effects provokes an exponential stability of the solutions. In [26], Magaña and Quintanilla showed that viscoelasticity damping and temperature produced slow decay in time and when the viscoelasticity is coupled with porous damping or with microtemperatures, the system decays in an exponential way.

Delay effect arises in many applications depending not only on the present state but also on some past occurrences and it has attracted lots of attentions from researchers in diverse fields of human endeavor such as mathematics, engineering, science, and economics. The presence of delay may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see [10], [11], [18], [28], [29], [35]). Also, the introducing of this complementary control may lead to ill-posedness as shown in many works such as [11], [32] and the references therein. In addition to the well-known discrete delays, there are several others and we are interested here in the neutral delay where the delay is occurring in the second (highest) derivative; for more details, see previous studies [12], [13], [19], [20], [24], [34] and the references therein.

Among the investigations that have been realized concerning the asymptotic behavior with neutral delay, we cite the work of Seghour et al. (see [33]), where they considered the thermoelastic laminated system subject to a neutral delay

$$\begin{cases} \varrho w_{tt} + G(\psi - w_x)_x + A w_t = 0, & x \in (0,1),\ t > 0, \\ I_\varrho(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi) + \mu\theta_x = 0, & x \in (0,1),\ t > 0, \\ 3I_\varrho\Big(s_t + \displaystyle\int_0^t h(t-r)s_t(r)\,\mathrm{d}r\Big)_t + 3G(\psi - w_x) & \\ \qquad\qquad\qquad + 4\gamma s - 3s_{xx} = 0, & x \in (0,1),\ t > 0, \\ \theta_t - \kappa\theta_{xx} + \mu(3s - \psi)_{tx} = 0, & x \in (0,1),\ t > 0, \end{cases}$$

with boundary conditions

$$\begin{cases} \psi(0,t) = s(0,t) = \theta_x(0,t) = w_x(0,t) = 0, & t \geqslant 0, \\ \theta(1,t) = w(1,t) = s_x(1,t) = \psi_x(1,t) = 0, & t \geqslant 0, \end{cases}$$

and initial data

$$\begin{cases} (w,\psi,s,\theta)(x,0) = (w_0,\psi_0,s_0,\theta_0), & x \in (0,1), \\ (w_t,\psi_t,s_t)(x,0) = (w_1,\psi_1,s_1), & x \in (0,1). \end{cases}$$

The authors showed, under some appropriate assumptions, that the dissipation produced by the heat equation with the frictional damping stabilize exponentially the system even in the presence of neutral delay for the case of equal wave speeds. In the opposite one, and with an additional assumption on the kernel, they proved a polynomial stability.

In this paper, we consider the following porous-elastic system with porous-viscosity subject to a distributed delay of neutral type

$$\begin{cases} \varrho u_{tt} - \mu u_{xx} - b\varphi_x = 0, & x \in (0,1),\ t > 0, \\ J\Big(\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_t - \delta\varphi_{xx} + bu_x & \\ \qquad\qquad +\xi\varphi + \mu_1\varphi_t = 0, & x \in (0,1),\ t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ u_x(0,t) = u_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0, & t > 0, \end{cases} \tag{1.1}$$

where the functions u and φ represent, respectively, the displacement of the solid elastic material and the volume fraction. The parameter ϱ designates the mass density and J equals to the product of the mass density by the equilibrated inertia. The coefficients μ, δ, ξ, μ_1 are positive constants representing the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2, \tag{1.2}$$

where b is a real number different from zero. The initial data u_0, u_1, φ_0, φ_1 belong to the suitable functional space and the integral represents the neutral delay term, where k is the relaxation function specified in the preliminaries. System (1.1) was constructed by considering the following basic evolution equations of the one-dimensional porous materials theory:

$$\varrho u_{tt} = T_x, \quad J\Big(\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_t = H_x + D, \tag{1.3}$$

where T, H and D represent, respectively, the stress tensor, the equilibrated stress vector and the equilibrated body force. Consequently, to get system (1.1) we take the constitutive equations T, H and D of the form

$$T = \mu u_x + b\varphi, \quad H = \delta\varphi_x, \quad D = -bu_x - \xi\varphi - \mu_1\varphi_t, \tag{1.4}$$

and by combining (1.4) in (1.3), we obtain (1.1).

The main goal of this paper is to prove a global well-posedness of problem (1.1) by using the Faedo-Galerkin method with some a priori estimates. Moreover, based on the multipliers method along side with some assumptions on the kernel of neutral delay, we construct a suitable Lyapunov functional and we show that the dissipation given by the porous-viscosity is strong enough to guarantee an exponential decay in spite of the existence of the neutral delay for the case of equal speeds of wave propagation, that is,

$$\chi = \frac{\mu}{\varrho} - \frac{\delta}{J} = 0. \tag{1.5}$$

In the opposite one, we establish a polynomial stability result. Furthermore, in our case and compared to the work of Seghour et al. (see [33]), we were able to dispense the thermal effect depending only on the damping mechanism to control the neutral delay term.

Introducing a neutral delay makes our problem different from those considered so far in the literature. Moreover, the study of the asymptotic behavior becomes different and more complicated than in the case of other types of delay that has appeared in the recent literature such as in [3], [4], [8], [9], [16], [21], [22]. In other words, a neutral-type delayed dynamical system is a more general class than delayed systems, in the sense where it is described by a model in which the highest derivative of the state at the present time is a function not only of the values of the passed state, but also of the highest derivative of the passed state and this strengthens the challenges. It is also worth mentioning that besides the fact that systems are very reactive to small delays, on the contrary, they can be stabilized by 'large' neutral delays. In fact, neutral delays are sometimes deliberately inserted into the systems to improve the performance of the structure and this has been proven by some works, such as [25] in which the authors showed that the dissipation given only by the neutral delay, without damping or other dissipation terms, provokes exponential stability of the solution.

This paper is organized as follows. In Section 2, we introduce some assumptions and transformations needed in the next sections to prove the main result. In Section 3, we prove the existence and uniqueness of the solution. In Section 4, we show the decay of the energy. In Section 5 and 6, we use the energy method to prove the exponential and polynomial stability result.

2. Preliminaries

In this section we present our assumptions on both kernels and introduce the energy functional and some other functionals. We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. Let us define the space $\mathcal{H}$ as

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

where $H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1)$ and

$$L_*^2(0,1) = \left\{ f \in L^2(0,1) \colon \int_0^1 f(x)\,\mathrm{d}x = 0 \right\}.$$

Moreover, we define the space

$$H_*^2(0,1) = \{\psi \in H^2(0,1) \colon \psi_x(0) = \psi_x(1) = 0\}.$$

To simplify the calculations, we are obliged to announce this lemma which is usable in the following sections.

Lemma 2.1 ([33]). *For any function $\psi \in C^1([0,\infty); L^2(0,1))$ and any $k \in C^1([0,\infty))$, we have the following identity:*

$$\begin{aligned}
&\int_0^1 \psi(t)\left(\int_0^t k(t-s)\psi_t(s)\,\mathrm{d}s\right)\mathrm{d}x \\
&\quad = -\frac{1}{2}(k' \mathbin{\Box} \psi)(t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\int_0^t k(t-s)\psi^2(s)\,\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \frac{k(t)}{2}\int_0^1 \psi^2\,\mathrm{d}x - k(t)\int_0^1 \psi(0)\psi(t)\,\mathrm{d}x,
\end{aligned}$$

where

$$(k' \mathbin{\Box} \psi) = \int_0^t k(t-s)\left(\int_0^1 (\psi(t)-\psi(s))^2\,\mathrm{d}x\right)\mathrm{d}s, \quad t \geqslant 0.$$

Also, we need the following hypothesis to reach our aim:

(H1) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leqslant 0 \quad \forall\, t \geqslant 0, \quad \overline{k} = \int_0^\infty k(s)\,\mathrm{d}s < 1.$$

(H2) $\exp(\varsigma t)k(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^\infty \mathrm{e}^{\varsigma s} k(s)\,\mathrm{d}s < \infty$ and $\lim\limits_{t\to\infty} \exp(\varsigma t)k(t) < \infty$, then

$$\int_0^\infty \mathrm{e}^{\varsigma s}|k'(s)|\,\mathrm{d}s = -\int_0^\infty \mathrm{e}^{\varsigma s}k'(s)\,\mathrm{d}s = -\mathrm{e}^{\varsigma s}k(s)\big|_0^\infty + \varsigma\int_0^\infty \mathrm{e}^{\varsigma s}k(s)\,\mathrm{d}s < \infty.$$

Theorem 2.1 ([5]). *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leqslant p, r \leqslant \infty$. For $T > 0$ we define*

$$E_{p,r} = \Big\{v \in L^p(0,T;B_0),\ \frac{\mathrm{d}v}{\mathrm{d}t} \in L^r(0,T;B_2)\Big\}.$$

(i) *If $p < \infty$, the embedding of $E_{p,r}$ in $L^p(0,T;B_1)$ is compact.*
(ii) *If $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0(0,T;B_1)$ is compact.*

Also, we need to use the following transformation in order to calculate the energy of the system and for other necessary estimations:

$$\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_t = k(t)\varphi_t(0) + \int_0^t k(t-s)\varphi_{tt}(s)\,\mathrm{d}s. \tag{2.1}$$

We shall consider the classical energy defined by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 (\varrho u_t^2 + \mu u_x^2 + J\varphi_t^2 + 2bu_x\varphi + \xi\varphi^2 + \delta\varphi_x^2)\,\mathrm{d}x \\ + \frac{J}{2}\int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x, \quad t \geqslant 0. \end{aligned} \tag{2.2}$$

Lemma 2.2. *The energy $E(t)$ given by* (2.2) *satisfies*

$$E'(t) \leqslant \frac{J}{2}(k' \,\square\, \varphi_t)(t) - \mu_1\int_0^1 \varphi_t^2\,\mathrm{d}x. \tag{2.3}$$

Proof. Multiplying $(1.1)_1, (1.1)_2$ by u_t, φ_t and integrating over $(0,1)$ and summing them up, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\varrho u_t^2 + \mu u_x^2 + J\varphi_t^2 + 2bu_x\varphi + \xi\varphi^2 + \delta\varphi_x^2)\,\mathrm{d}x \\ + J\int_0^1\Big(\varphi_t\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_t\Big)\,\mathrm{d}x = -\mu_1\int_0^1 \varphi_t^2\,\mathrm{d}x. \end{aligned} \tag{2.4}$$

By exploiting (2.1) and applying the result in Lemma 2.1, we obtain

(2.5)

$$J\int_0^1\Big(\varphi_t\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_t\Big)\,\mathrm{d}x = -\frac{J}{2}(k'\,\Box\,\varphi_t)(t) + J\frac{k(t)}{2}\int_0^1\varphi_t^2\,\mathrm{d}x + \frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x.$$

Inserting (2.5) in (2.4) and taking into account the positivity of $k(t)$, we have (2.3).

□

Remark 2.1. Note that

$$\mu u_x^2 + 2bu_x\varphi + \xi\varphi^2 = \mu\Big(u_x + \frac{b}{\mu}\varphi\Big)^2 + \Big(\xi - \frac{b^2}{\mu}\Big)(\varphi)^2$$

and because $\mu\xi > b^2$, we deduce that the energy $E(t)$ defined by (2.2) is nonnegative.

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. In other words, it is known that for the problem determined by (1.1) we can always take solutions where u is constant. For this reason, we impose that

$$\int_0^1 u_0\,\mathrm{d}x = \int_0^1 u_1\,\mathrm{d}x = 0. \tag{2.6}$$

It is worth noting that condition (2.6) is imposed to guarantee that the solution decays. Thus, if we want to avoid this behavior, we need to impose condition (2.6). In addition as in [3], to be able to use Poincaré's inequality for u, we perform the following transformation. From $(1.1)_1$, we observe that

$$\int_0^1 u_{tt}\,\mathrm{d}x = 0.$$

If we take $v(t) = \int_0^1 u\,\mathrm{d}x$, we observe that $v(0) = \int_0^1 u_0\,\mathrm{d}x = 0$ and $v'(0) = \int_0^1 u_1\,\mathrm{d}x = 0$. Moreover, v is a solution of the following initial value problem:

$$\begin{cases} v''(t) = 0 \quad \forall\, t \geqslant 0, \\ v(0) = \displaystyle\int_0^1 u_0(x)\,\mathrm{d}x = 0, \quad v'(0) = \int_0^1 u_1(x)\,\mathrm{d}x = 0. \end{cases}$$

The solution of the problem is given by

$$v(t) = \int_0^1 u(x,t)\,\mathrm{d}x = t\int_0^1 u_1(x)\,\mathrm{d}x + \int_0^1 u_0(x)\,\mathrm{d}x = 0.$$

Consequently,

$$\int_0^1 u(x,t)\,\mathrm{d}x = 0 \quad \forall\, t \geqslant 0.$$

3. Global well-posedness

In this section, we prove the global existence and the uniqueness of the solution of problem (1.1) by using the classical Faedo-Galerkin approximations along with some a priori estimates. The well-posedness of (1.1) is given by the following theorem.

Theorem 3.1. *Assume that* (H1)–(H2), (1.2) *hold, and the initial data are*

$$(u_0, u_1) \in H^1_*(0,1) \times L^2_*(0,1), \quad (\varphi_0, \varphi_1) \in H^1_0(0,1) \times L^2(0,1). \tag{3.1}$$

Then problem (1.1) *has a unique global weak solution*

$$\begin{aligned} u &\in C(\mathbb{R}_+, H^2_*(0,1) \cap H^1_*(0,1)) \cap C^1(\mathbb{R}_+, H^1_*(0,1)) \cap C^2(\mathbb{R}_+, L^2_*(0,1)),\\ \varphi &\in C(\mathbb{R}_+, H^2(0,1) \cap H^1_0(0,1)) \cap C^1(\mathbb{R}_+, H^1_0(0,1)) \cap C^2(\mathbb{R}_+, L^2(0,1)). \end{aligned} \tag{3.2}$$

In addition, the solution (u, φ) *depends continuously on the initial data.*

P r o o f. We divide the proof into three steps: We first construct Faedo-Galerkin approximations, then thanks to a priori estimates we look to prove that $t_n = T$ for $n \in \mathbb{N}^*$. Finally, we pass to the limit.

Step 1: Faedo-Galerkin approximations. We construct approximations of the solution (u, φ) by the Faedo-Galerkin method as follows (see [15] and [27]): For every $n \geqslant 1$, let $W_n = \operatorname{span}\{e_1, e_2, \dots, e_n\}$ be a Hilbert basis (orthonormal basis) of $H^2_*(0,1) \cap H^1_*(0,1)$ and $L^2_*(0,1)$. Also, we denote by $\Gamma_n = \operatorname{span}\{\sigma_1, \sigma_2, \dots, \sigma_n\}$ a Hilbertian basis of $H^2(0,1) \cap H^1_0(0,1)$ and $L^2(0,1)$. For given initial data

$$(u_0, u_1) \in H^1_*(0,1) \times L^2_*(0,1), \quad (\varphi_0, \varphi_1) \in H^1_0(0,1) \times L^2(0,1),$$

we seek functions $y^n_j, h^n_j \in C^2([0,T])$ such that the approximations

$$\begin{cases} u^n(x,t) = \displaystyle\sum_{j=1}^{j=n} y^n_j(t) e_j(x),\\ \varphi^n(x,t) = \displaystyle\sum_{j=1}^{j=n} h^n_j(t) \sigma_j(x), \end{cases} \tag{3.3}$$

check the approximate problem

$$\begin{cases} \varrho u^n_{tt}, \ -\mu u^n_{xx} - b\varphi^n_x = 0,\\ J\varphi^n_{tt} + J\Big(\displaystyle\int_0^t k(t-s)\varphi^n_t(s)\,\mathrm{d}s\Big)_t - \delta\varphi^n_{xx} + bu^n_x + \xi\varphi^n + \mu_1\varphi^n_t = 0 \end{cases} \tag{3.4}$$

with the initial data

$$
(3.5) \qquad \begin{cases} u^n(x,0) = u_0^n(x), \; u_t^n(x,0) = u_1^n(x), \\ \varphi^n(x,0) = \varphi_0^n(x), \; \varphi_t^n(x,0) = \varphi_1^n(x), \end{cases}
$$

which satisfies

$$
(3.6) \qquad \begin{cases} u_0^n = \displaystyle\sum_{j=1}^{n} \Big(\int_0^1 u_0 e_j \, \mathrm{d}x \Big) e_j \underset{n\to\infty}{\longrightarrow} u_0 \text{ strongly in } H_*^1(0,1), \\ u_1^n = \displaystyle\sum_{j=1}^{n} \Big(\int_0^1 u_1 e_j \, \mathrm{d}x \Big) e_j \underset{n\to\infty}{\longrightarrow} u_1 \text{ strongly in } L_*^2(0,1), \\ \varphi_0^n = \displaystyle\sum_{j=1}^{n} \Big(\int_0^1 \varphi_0 \sigma_j \, \mathrm{d}x \Big) \sigma_j \underset{n\to\infty}{\longrightarrow} \varphi_0 \text{ strongly in } H_0^1(0,1), \\ \varphi_1^n = \displaystyle\sum_{j=1}^{n} \Big(\int_0^1 \varphi_1 \sigma_j \, \mathrm{d}x \Big) \sigma_j \underset{n\to\infty}{\longrightarrow} \varphi_1 \text{ strongly in } L^2(0,1). \end{cases}
$$

Through (3.4), we get

$$
(3.7) \begin{cases} \varrho \langle u_{tt}^n, e_k \rangle_{L^2(0,1)} - \mu \langle u_{xx}^n, e_k \rangle_{L^2(0,1)} - b \langle \varphi_x^n, e_k \rangle_{L^2(0,1)} = 0, \\ J \langle \varphi_{tt}^n, \sigma_k \rangle_{L^2(0,1)} + J \Big\langle \Big(\displaystyle\int_0^t k(t-s) \varphi_t^n(s) \, \mathrm{d}s \Big)_t, \sigma_k \Big\rangle_{L^2(0,1)} \\ -\delta \langle \varphi_{xx}^n, \sigma_k \rangle_{L^2(0,1)} + b \langle u_x^n, \sigma_k \rangle_{L^2(0,1)} + \xi \langle \varphi^n, \sigma_k \rangle_{L^2(0,1)} + \mu_1 \langle \varphi_t^n, \sigma_k \rangle_{L^2(0,1)} = 0 \end{cases}
$$

with (u_0^n, u_1^n) and $(\varphi_0^n, \varphi_1^n)$, respectively, in W_n and Γ_n. According to the standard ordinary differential equations theory, the finite dimensional problem (3.7) has a solution $(y_j^n, h_j^n)_{j=1,\dots,n} \in C^2([0,t_n])^2$. Then the a priori estimates that follow imply that in fact, $t_n = T$ for all $T > 0$.

Step 2: Energy estimates.–A priori estimate I. For every $n \geqslant 1$, by integrating by parts in (3.7), we get

$$
(3.8) \qquad \begin{cases} \varrho \displaystyle\int_0^1 u_{tt}^n e_k \, \mathrm{d}x + \mu \int_0^1 u_x^n e_{kx} \, \mathrm{d}x - b \int_0^1 \varphi_x^n e_k \, \mathrm{d}x = 0, \\ J \displaystyle\int_0^1 \varphi_{tt}^n \sigma_k \, \mathrm{d}x + J \int_0^1 \sigma_k \Big(\int_0^t k(t-s) \varphi_t^n(s) \, \mathrm{d}s \Big)_t \, \mathrm{d}x \\ \qquad + \delta \displaystyle\int_0^1 \varphi_x^n \sigma_{kx} \, \mathrm{d}x + b \int_0^1 u_x^n \sigma_k \, \mathrm{d}x + \xi \int_0^1 \varphi^n \sigma_k \, \mathrm{d}x \\ \qquad + \mu_1 \displaystyle\int_0^1 \varphi_t^n \sigma_k \, \mathrm{d}x = 0 \quad \forall\, k = 1, \dots, n. \end{cases}
$$

By multiplying $(3.8)_1$ and $(3.8)_2$, respectively, by $(y_k^n)_t$ and $(h_k^n)_t$, and by using integration by parts, we obtain

$$
\begin{aligned}
(3.9)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\varrho(u_t^n)^2 + \mu(u_x^n)^2 + J(\varphi_t^n)^2 + 2bu_x^n\varphi^n + \xi(\varphi^n)^2 + \delta(\varphi_x^n)^2)\,\mathrm{d}x \\
&\quad + J\int_0^1 \varphi_t^n\Big(\int_0^t k(t-s)\varphi_t^n(s)\,\mathrm{d}s\Big)_t\,\mathrm{d}x + \mu_1\int_0^1 (\varphi_t^n)^2\,\mathrm{d}x = 0.
\end{aligned}
$$

We use the same technique in the proof of Lemma 2.2. We have

$$
\begin{aligned}
(3.10)\qquad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\varrho(u_t^n)^2 + \mu(u_x^n)^2 + J(\varphi_t^n)^2 + 2bu_x^n\varphi^n + \xi(\varphi^n)^2 + \delta(\varphi_x^n)^2)\,\mathrm{d}x \\
&\quad + \frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\int_0^t k(t-s)(\varphi_t^n)^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&= \frac{J}{2}(k' \,\square\, \varphi_t^n)(t) - \Big(J\frac{k(t)}{2} + \mu_1\Big)\int_0^1 (\varphi_t^n)^2\,\mathrm{d}x \leqslant 0.
\end{aligned}
$$

Now integrating (3.10), we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 (\varrho(u_t^n)^2 + \mu(u_x^n)^2 + J(\varphi_t^n)^2 + 2bu_x^n\varphi^n + \xi(\varphi^n)^2 + \delta(\varphi_x^n)^2)\,\mathrm{d}x \\
&\quad + \frac{J}{2}\int_0^1 \Big(\int_0^t k(t-s)(\varphi_t^n)^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_0^1 (\varrho(u_1^n)^2 + J(\varphi_1^n)^2 + (\delta(\varphi_x^n)^2 + \mu(u_x^n)^2 + 2bu_x^n\varphi^n + \xi(\varphi^n)^2)(x,0))\,\mathrm{d}x.
\end{aligned}
$$

Hence, the previous inequality takes the form

$$E^n(t) \leqslant E^n(0),$$

where

$$
\begin{aligned}
(3.11)\quad E^n(t) = &\frac{1}{2}\int_0^1 (\varrho(u_t^n)^2 + \mu(u_x^n)^2 + J(\varphi_t^n)^2 + 2bu_x^n\varphi^n + \xi(\varphi^n)^2 + \delta(\varphi_x^n)^2)\,\mathrm{d}x \\
&+ \frac{J}{2}\int_0^1 \Big(\int_0^t k(t-s)(\varphi_t^n)^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x.
\end{aligned}
$$

In view of the hypotheses on the function k and as in Remark 2.1, we deduce

$$0 \leqslant E^n(t) \leqslant E^n(0).$$

Now, since the sequences $(u_0^n)_{n\in\mathbb{N}}$, $(u_1^n)_{n\in\mathbb{N}}$, $(\varphi_0^n)_{n\in\mathbb{N}}$, $(\varphi_1^n)_{n\in\mathbb{N}}$ converge (see (3.6)), using (H1) and (H2), we can find a positive constant C independent of n such that

$$(3.12)\qquad E^n(t) \leqslant C.$$

Then $t_n = T$ for all $T > 0$.

A priori estimate II. Throught (3.3), also as $(y_j^n, h_j^n)_{j=1,\ldots,n} \in (C^2[0,T])^2$ and

$$(e_j)_{j\geqslant 1} \subset H_*^2(0,1) \cap H_*^1(0,1) \subset H^1(0,1) \hookrightarrow C(0,1),$$
$$(\sigma_j)_{j\geqslant 1} \subset H^2(0,1) \cap H_0^1(0,1) \subset H^1(0,1) \hookrightarrow C(0,1),$$

we have

$$\begin{cases} u^n \in C^2(0,T; H_*^2(0,1) \cap H_*^1(0,1)), \\ \varphi^n \in C^2(0,T; H^2(0,1) \cap H_0^1(0,1)). \end{cases} \tag{3.13}$$

Because $E = C^2(0,T; H_*^2(0,1) \cap H_*^1(0,1))$ is a Banach space equipped with the norm

$$\|u^n\|_E = \sup_{t\in[0,T]} \|u^n(\cdot,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|u_t^n(\cdot,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|u_{tt}^n(\cdot,t)\|_{H^2(0,1)} \quad \forall\, n \in \mathbb{N}^*,$$

also $F = C^2(0,T; H^2(0,1) \cap H_0^1(0,1))$ is a Banach space equipped with the norm

$$\|\varphi^n\|_F = \sup_{t\in[0,T]} \|\varphi^n(\cdot,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|\varphi_t^n(\cdot,t)\|_{H^2(0,1)} + \sup_{t\in[0,T]} \|\varphi_{tt}^n(\cdot,t)\|_{H^2(0,1)} \quad \forall\, n \in \mathbb{N}^*.$$

Taking into account (3.13), we get $\|u^n\|_E < \infty$, $\|\varphi^n\|_E < \infty$, and using the fact that

$$\|u_{xx}^n\|_{L^2(0,1)} \leqslant \|u^n\|_E < \infty \quad \forall\, n \in \mathbb{N}^* \text{ and } \forall\, t \in [0,T],$$
$$\|\varphi_{xx}^n\|_{L^2(0,1)} \leqslant \|\varphi^n\|_F < \infty \quad \forall\, n \in \mathbb{N}^* \text{ and } \forall\, t \in [0,T],$$

we get

$$\int_0^1 (u_{xx}^n)^2 + (\varphi_{xx}^n)^2 \,\mathrm{d}x < \infty \quad \forall\, t \in [0,T]. \tag{3.14}$$

Step 3: The limit process. From (3.12) and (3.14), we conclude that

(3.15)
$$(u^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty(0,T; H_*^2(0,1) \cap H_*^1(0,1)),$$
$$(u_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty(0,T; L_*^2(0,1)),$$
$$(\varphi^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty(0,T; H^2(0,1) \cap H_0^1(0,1)),$$
$$(\varphi_t^n)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty(0,T; L^2(0,1)).$$

Note that the boundedness in $L^\infty(0,T; H_*^2(0,1) \cap H_*^1(0,1))$ is not a consequence of (3.14) only, but we exploit it as follows:

Throughout (3.12), we conclude

$$\int_0^1 (u_x^n)^2 \,\mathrm{d}x < \infty \quad \forall\, n \geqslant 1,\ \forall\, t \in [0,T], \tag{3.16}$$

and by using Poincaré inequality with (3.14), we have

$$\sup_{t\in[0,T]} \left(\int_0^1 (u^n)^2 \,\mathrm{d}x + \int_0^1 (u_x^n)^2 \,\mathrm{d}x + \int_0^1 (u_{xx}^n)^2 \,\mathrm{d}x \right) < \infty \quad \forall\, n \geqslant 1.$$

Then

$$u^n \text{ is bounded in } L^\infty(0,T; H_*^2(0,1) \cap H_*^1(0,1)) \quad \forall\, n \geqslant 1.$$

By using Theorem 2.1, since
▷ the embedding of $H_*^1(0,1)$ in $L_*^2(0,1)$ is continuous,
▷ the embedding of $H_*^2(0,1) \cap H_*^1(0,1)$ in $H_*^1(0,1)$ is compact,
▷ the embedding of $H_0^1(0,1)$ in $L^2(0,1)$ is continuous,
▷ the embedding of $H^2(0,1) \cap H_0^1(0,1)$ in $H_0^1(0,1)$ is compact,
then the embedding of $E_{\infty,\infty}$ in $C(0,T; H_*^1(0,1))$ is compact, where

$$E_{\infty,\infty} = \left\{ u^n \colon\ u^n \in L^\infty(0,T; H_*^2(0,1) \cap H_*^1(0,1)),\ u_t^n = \frac{\mathrm{d}u^n}{\mathrm{d}t} \in L^\infty(0,T; L_*^2(0,1)) \right\},$$

and the embedding of $\widetilde{E}_{\infty,\infty}$ in $C([0,T], H_0^1(0,1))$ is compact with

$$\widetilde{E}_{\infty,\infty} = \left\{ \varphi^n \colon\ \varphi^n \in L^\infty(0,T; H^2(0,1) \cap H_0^1(0,1)),\ \varphi_t^n = \frac{\mathrm{d}\varphi^n}{\mathrm{d}t} \in L^\infty(0,T; L^2(0,1)) \right\}.$$

On the other hand, from (3.15) we have that $(u^n)_{n\in\mathbb{N}^*}$ and $(\varphi^n)_{n\in\mathbb{N}^*}$ are bounded in $E_{\infty,\infty}$ and $\widetilde{E}_{\infty,\infty}$, respectively. So, there exist two subsequences $(u^m)_{m\geqslant 1}$ of $(u^n)_{n\geqslant 1}$ and $(\varphi^m)_{m\geqslant 1}$ of $(\varphi^n)_{n\geqslant 1}$ such that

$$u^m \overset{m\to\infty}{\longrightarrow} u \text{ strongly in } C(0,T; H_*^1(0,1)), \tag{3.17}$$

$$\varphi^m \overset{m\to\infty}{\longrightarrow} \varphi \text{ strongly in } C(0,T; H_0^1(0,1)), \tag{3.18}$$

which implies that

$$\{u^m\}_{m\geqslant 1} \text{ simply converges to } u \quad \forall\, t \in [0,T]. \tag{3.19}$$

By using (3.13), we have $u_t^m \in C^1(0,T; H_*^2(0,1) \cap H_*^1(0,1))$ for all $m \geqslant 1$, and by (3.19) with the dominated convergence theorem, we obtain for any $t \in [0,T]$

and $k \in \mathbb{N}^*$

(3.20)

$$\begin{aligned}\lim_{m\to\infty} \|u_t^m(\cdot,t) - u_t^{m+k}(\cdot,t)\|^2_{L^2(0,1)} &= \lim_{m\to\infty} \int_0^1 |u_t^m(x,t) - u_t^{m+k}(x,t)|^2 \,\mathrm{d}x \\ &= \int_0^1 \lim_{m\to\infty} |u_t^m(x,t) - u_t^{m+k}(x,t)|^2 \,\mathrm{d}x = 0.\end{aligned}$$

Also, by the same way, we can write

(3.21)

$$\begin{aligned}\lim_{m\to\infty} \|u_{tx}^m(\cdot,t) - u_{tx}^{m+k}(\cdot,t)\|^2_{L^2(0,1)} &= \lim_{m\to\infty} \int_0^1 |u_{tx}^m(x,t) - u_{tx}^{m+k}(x,t)|^2 \,\mathrm{d}x \\ &= \int_0^1 \lim_{m\to\infty} |u_{tx}^m(x,t) - u_{tx}^{m+k}(x,t)|^2 \,\mathrm{d}x = 0.\end{aligned}$$

Combining (3.20)–(3.21), we get

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \|u_t^m(\cdot,t) - u_t^{m+k}(\cdot,t)\|^2_{H^1(0,1)} = 0,$$

which means that $(u_t^m)_{m\geqslant 1}$ is a Cauchy sequence in $X = C(0,T; H^1_*(0,1))$ equipped with the norm

$$\|u\|_X = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{H^1(0,1)}.$$

Since $X = (C(0,T; H^1_*(0,1)); \|\cdot\|_X)$ is a Banach space, then there exists a unique $g \in C(0,T; H^1_*(0,1))$ such that

$$u_t^m \overset{m\to\infty}{\longrightarrow} g \text{ strongly in } C(0,T; H^1_*(0,1)). \tag{3.22}$$

Now, it is left to prove that $g = u_t$. Since the operator A defined as

$$A\colon\ D(A) = C^1(0,T; H^1_*(0,1)) \subset C(0,T; H^1_*(0,1)) \to C(0,T; H^1_*(0,1)),\ u \to u_t$$

is closed, that is to say if $(u^m)_{m\geqslant 1} \subset D(A)$ converges strongly to $u \in C(0,T; H^1_*(0,1))$ and $u_t^m = Au^m$, $m \geqslant 1$ converges strongly to $g \in C(0,T; H^1_*(0,1))$, we get $u \in C^1(0,T; H^1_*(0,1))$ and $g = Au = u_t$. Using (3.17) and (3.22), we obtain

$$u_t^m \overset{m\to\infty}{\longrightarrow} g = u_t \text{ strongly in } X = C(0,T; H^1_*(0,1)). \tag{3.23}$$

Similarly, by using (3.13) and (3.18), we can easily prove that

$$\varphi_t^m \overset{m\to\infty}{\longrightarrow} \varphi_t \text{ strongly in } Y = C(0,T; H^1_0(0,1)). \tag{3.24}$$

Also, by using (3.13), we have $u_{tt}^m \in C(0,T;H_*^2(0,1)\cap H_*^1(0,1))$ for all $m \geqslant 1$, and by (3.19) with the dominated convergence theorem, we obtain for any $t \in [0,T]$ and $k \in \mathbb{N}^*$

(3.25)

$$\begin{aligned}\lim_{m\to\infty} \|u_{tt}^m(\cdot,t) - u_{tt}^{m+k}(\cdot,t)\|^2_{L^2(0,1)} &= \lim_{m\to\infty}\int_0^1 |u_{tt}^m(x,t) - u_{tt}^{m+k}(x,t)|^2\,\mathrm{d}x \\ &= \int_0^1 \lim_{m\to\infty} |u_{tt}^m(x,t) - u_{tt}^{m+k}(x,t)|^2\,\mathrm{d}x = 0;\end{aligned}$$

this last formula implies that

$$\lim_{m\to\infty}\sup_{t\in[0,T]} \|u_{tt}^m(\cdot,t) - u_{tt}^{m+k}(\cdot,t)\|_{L^2(0,1)} = 0,$$

which means that $(u_{tt}^m)_{m\geqslant 1}$ is a Cauchy sequence in $Z = C(0,T;L^2(0,1))$ equipped with the norm

$$\|u\|_Z = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^2(0,1)}.$$

Since $Z = (C(0,T,L^2(0,1)); \|\cdot\|_Z)$ is a Banach space, then there exists a unique $f \in C(0,T,L^2(0,1))$ such that

$$u_{tt}^m \overset{m\to\infty}{\longrightarrow} f \text{ strongly in } C(0,T;L^2(0,1)). \tag{3.26}$$

Now, it is left to prove that $f = u_{tt}$. By using (3.17) and (3.23), we get

$$u^m \overset{m\to\infty}{\longrightarrow} u \text{ strongly in } C^1(0,T;H_*^1(0,1)). \tag{3.27}$$

Since the operator B defined as

$$B\colon\ D(B) = C^2(0,T;L^2(0,1)) \subset C^1(0,T;L^2(0,1)) \to C(0,T;L^2(0,1)),\ u \to u_{tt}$$

is closed, by using (3.26) and (3.27) we obtain

$$f = Bu = u_{tt}, \tag{3.28}$$

which implies

$$u_{tt}^m \overset{m\to\infty}{\longrightarrow} f = u_{tt} \text{ strongly in } C(0,T;L^2(0,1)).$$

Similarly, by using (3.13), (3.18) and (3.24), we can easily prove that

$$\varphi_{tt}^m \overset{m\to\infty}{\longrightarrow} \varphi_{tt} \text{ strongly in } C(0,T;L^2(0,1)).$$

By passing to the limit in (3.5) and (3.8), problem (1.1) admits a global weak solution satisfying (3.2).

The proof now can be completed arguing as in [23], Théorème 3.1.

Continuous dependence and uniqueness.–Uniqueness: Let us assume that (Λ^1, Υ^1) and (Λ^2, Υ^2) are two global solutions of (1.1). Then $(\chi, \Xi) = (\Lambda^1 - \Lambda^2, \Upsilon^1 - \Upsilon^2)$ satisfies $(1.1)_1$ and $(1.1)_2$ with

$$\begin{cases} \chi(x,0) = \chi_t(x,0) = \Xi(x,0) = \Xi_t(x,0) = 0, & x \in (0,1), \\ \chi_x(0,t) = \chi_x(1,t) = \Xi(0,t) = \Xi(1,t) = 0, & t > 0. \end{cases} \tag{3.29}$$

From the linearity of the equations and the fact that the energy $E(t)$ is decreasing, so that, for (χ, Ξ), we have $0 \leqslant E(t) \leqslant E(0) = 0$ for any $t \geqslant 0$, where

$$\begin{aligned} E(t) = {}& \frac{1}{2} \int_0^1 (\varrho \chi_t^2 + \mu \chi_x^2 + J \Xi_t^2 + 2b\chi_x \Xi + \xi \Xi^2 + \delta \Xi_x^2) \, \mathrm{d}x \\ & + \frac{J}{2} \int_0^1 \Big(\int_0^t k(t-s) \Xi_t^2(s) \, \mathrm{d}s \Big) \, \mathrm{d}x \end{aligned}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \frac{J}{2} (k' \,\square\, \Xi_t)(t) - \Big(J \frac{k(t)}{2} + \mu_1 \Big) \int_0^1 \Xi_t^2 \, \mathrm{d}x \leqslant 0. \tag{3.30}$$

Hence, $(\chi, \Xi)(t) = (0,0)$, identically. So problem (1.1) has a unique global solution.

The continuous dependence on initial data. Let (Θ, Φ) be a global solution of (1.1). A simple integration, using Young's inequality and the positivity of energy, we get

(3.31)

$$\begin{aligned} E(t) \leqslant {}& E(0) + \frac{1}{2} \int_0^t \Big(\int_0^1 (\varrho \Theta_t^2 + \mu \Theta_x^2 + J \Phi_t^2 + 2b\Theta_x \Phi + \xi \Phi^2 + \delta \Phi_x^2) \, \mathrm{d}x. \\ & + \frac{J}{2} \int_0^1 \Big(\int_0^t k(t-s) \Phi_t^2(s) \, \mathrm{d}s \Big) \, \mathrm{d}x \Big) \, \mathrm{d}\tau \\ \leqslant {}& E(0) + \frac{1}{2} \int_0^t \Big(\int_0^1 (\varrho \Theta_t^2 + (\mu + |b|) \Theta_x^2 + J \Phi_t^2 + (\xi + |b|) \Phi^2 + \delta \Phi_x^2) \, \mathrm{d}x \\ & + \frac{J}{2} \int_0^1 \Big(\int_0^t k(t-s) \Phi_t^2(s) \, \mathrm{d}s \Big) \, \mathrm{d}x \Big) \, \mathrm{d}\tau \\ \leqslant {}& E(0) + \varsigma_1 \int_0^t \Big(\int_0^1 (\Theta_t^2 + \Theta_x^2 + \Phi_t^2 + \Phi^2 + \Phi_x^2) \, \mathrm{d}x + \int_0^t k(t-s) \Phi_t^2(s) \, \mathrm{d}s \Big) \, \mathrm{d}\tau, \end{aligned}$$

where

$$\varsigma_1 = \max\Big\{ \frac{\varrho}{2}, \frac{1}{2}(\mu + |b|), \frac{J}{2}, \frac{1}{2}(\xi + |b|), \frac{\delta}{2}, \frac{J}{4} \Big\}.$$

On the other hand, we have

$$E(t) > \varsigma_2 \int_0^1 \Big(\Theta_t^2 + \Theta_x^2 + \Xi_t^2 + \Xi^2 + \Xi_x^2 + \int_0^t k(t-s) \Xi_t^2(s) \, \mathrm{d}s \Big) \, \mathrm{d}x$$

with

$$\varsigma_2 = \min\left\{\frac{\varrho}{2}, \frac{J}{2}, \frac{\delta}{2}, \frac{1}{2}\Big(\mu - \frac{b^2}{\xi}\Big), \frac{1}{2}\Big(\xi - \frac{b^2}{\mu}\Big), \frac{J}{4}\right\}.$$

Applying Gronwall's inequality to (3.31), we obtain

$$\int_0^1 \left(\Theta_t^2 + \Theta_x^2 + \Phi_t^2 + \Phi^2 + \Phi_x^2 + \int_0^t k(t-s)\Phi_t^2(s)\,\mathrm{d}s\right)\mathrm{d}x \leqslant \mathrm{e}^{\varsigma_1 t} E(0).$$

This shows that the solution of problem (1.1) depends continuously on the initial data. This ends the proof of Theorem 3.1. □

4. Stability result

In this section, we use the energy method to study the asymptotic behavior of solutions of system (1.1).

4.1. Exponential stability. In this subsection, we establish an exponential decay result of solutions of problem (1.1) in the case when (1.5) holds. The same result is obtained in [21], where the authors considered the one dimensional porous-elastic system subject to a distributed delay and by some assumptions on the weight of delay they proved an exponential decay of the solution. Also, in [3] the author established an explicit and general decay rate of solution of the same system damped via a nonlinear damping term under some properties of convex functions. In our case the situation is completely different and this is due to the nature and form of the neutral delay. So, we need the following lemmas.

Lemma 4.1. *Let* (u, φ) *be the solution of system* (1.1). *Then the functional*

$$F_1(t) = J\int_0^1 \varphi\left(\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\right)\mathrm{d}x + \frac{b\varrho}{\mu}\int_0^1 \varphi\left(\int_0^x u_t(y)\,\mathrm{d}y\right)\mathrm{d}x + \frac{\mu_1}{2}\int_0^1 \varphi^2\,\mathrm{d}x$$

satisfies, for any $\varepsilon_0 > 0$,

$$F_1'(t) \leqslant -\delta\int_0^1 \varphi_x^2\,\mathrm{d}x - 2\xi_1\int_0^1 \varphi^2\,\mathrm{d}x + \Big(\frac{3J}{2} + \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\Big)\int_0^1 \varphi_t^2\,\mathrm{d}x + \varepsilon_0\int_0^1 u_t^2\,\mathrm{d}x + \frac{J\bar{k}}{2}\int_0^1\left(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\right)\mathrm{d}x. \tag{4.1}$$

P r o o f. By differentiating $F_1(t)$ and integrating by parts, we obtain

$$
\begin{aligned}
(4.2)\qquad F_1'(t) = {} & J\int_0^1 \varphi_t^2\,\mathrm{d}x + J\int_0^1 \varphi_t\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
& - \delta\int_0^1 \varphi_x^2\,\mathrm{d}x - b\int_0^1 u_x\varphi\,\mathrm{d}x - 2\xi_1\int_0^1 \varphi^2\,\mathrm{d}x + b\int_0^1 u_x\varphi\,\mathrm{d}x \\
& + \frac{b\varrho}{\mu}\int_0^1 \varphi_t\Big(\int_0^x u_t(y)\,\mathrm{d}y\Big)\,\mathrm{d}x.
\end{aligned}
$$

Using Young's and Cauchy-Schwarz inequalities, we obtain

(4.3)

$$
J\int_0^1 \varphi_t\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \leqslant \frac{J}{2}\int_0^1 \varphi_t^2\,\mathrm{d}x + \frac{J\overline{k}}{2}\int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x.
$$

Using Young's inequality, we get

$$
(4.4)\qquad \frac{b\varrho}{\mu}\int_0^1 \varphi_t\Big(\int_0^x u_t(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \leqslant \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\int_0^1 \varphi_t^2\,\mathrm{d}x + \varepsilon_0\int_0^1 u_t^2\,\mathrm{d}x.
$$

Inserting (4.3) and (4.4) into (4.2), we obtain (4.1). □

Lemma 4.2. *Let* (u,φ) *be the solution of system* (1.1). *Then the functional*

$$
F_2(t) = \frac{\delta\varrho b}{\mu J}\int_0^1 \varphi_x u_t\,\mathrm{d}x + b\int_0^1\Big(\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)u_x\,\mathrm{d}x
$$

satisfies, for any $\varepsilon_1 > 0$,

$$
\begin{aligned}
(4.5)\qquad F_2'(t) \leqslant {} & -\frac{b^2}{4J}\int_0^1 u_x^2\,\mathrm{d}x + C_{\varepsilon_1}\int_0^1 \varphi_x^2\,\mathrm{d}x + \varepsilon_1(2+k(0))\int_0^1 u_t^2\,\mathrm{d}x \\
& + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \varphi_{0x}^2\,\mathrm{d}x + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\Big(\int_0^t |k'(t-s)|\varphi_x^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
& + \frac{\mu_1^2}{J}\int_0^1 \varphi_t^2\,\mathrm{d}x + \frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_{tx}\,\mathrm{d}x,
\end{aligned}
$$

where

$$
C_{\varepsilon_1} = \frac{\delta b^2}{\mu J} + \frac{b^2k^2(0)}{4\varepsilon_1} + \frac{\xi^2}{2J}.
$$

P r o o f. By differentiating $F_2(t)$, and integrating by parts, we obtain

$$
\begin{aligned}
(4.6)\qquad F_2'(t) = {} & \frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_{tx}\,\mathrm{d}x - \frac{b^2}{J}\int_0^1 u_x^2\,\mathrm{d}x + \frac{\delta b^2}{\mu J}\int_0^1 \varphi_x^2\,\mathrm{d}x - \frac{b\xi}{J}\int_0^1 \varphi u_x\,\mathrm{d}x \\
& + b\int_0^1 u_{tx}\Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)\,\mathrm{d}x - \frac{b\mu_1}{J}\int_0^1 \varphi_t u_x\,\mathrm{d}x.
\end{aligned}
$$

Integrating by parts with respect to t the last term of (4.6), we have

$$b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\right)\mathrm{d}x$$
$$= b\int_0^1 u_{tx}\left(k(0)\varphi(t)-k(t)\varphi(0)+\int_0^t k'(t-s)\varphi(s)\,\mathrm{d}s\right)\mathrm{d}x$$
$$= -bk(0)\int_0^1 u_t\varphi_x\,\mathrm{d}x + bk(t)\int_0^1 u_t\varphi_x(0)\,\mathrm{d}x - b\int_0^1 u_t\left(\int_0^t k'(t-s)\varphi_x(s)\,\mathrm{d}s\right)\mathrm{d}x.$$

Then (4.6) becomes

$$F_2'(t) = \frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_{tx}\,\mathrm{d}x - \frac{b^2}{J}\int_0^1 u_x^2\,\mathrm{d}x + \frac{\delta b^2}{\mu J}\int_0^1 \varphi_x^2\,\mathrm{d}x - bk(0)\int_0^1 u_t\varphi_x\,\mathrm{d}x + bk(t)\int_0^1 u_t\varphi_x(0)\,\mathrm{d}x - \frac{b\xi}{J}\int_0^1 \varphi u_x\,\mathrm{d}x - b\int_0^1 u_t\left(\int_0^t k'(t-s)\varphi_x(s)\,\mathrm{d}s\right)\mathrm{d}x - \frac{b\mu_1}{J}\int_0^1 \varphi_t u_x\,\mathrm{d}x. \tag{4.7}$$

By using Young's inequality, we arrive at

$$-bk(0)\int_0^1 u_t\varphi_x\,\mathrm{d}x \leqslant \varepsilon_1\int_0^1 u_t^2\,\mathrm{d}x + \frac{b^2k^2(0)}{4\varepsilon_1}\int_0^1 \varphi_x^2\,\mathrm{d}x, \tag{4.8}$$

$$-\frac{b\mu_1}{J}\int_0^1 \varphi_t u_x\,\mathrm{d}x \leqslant \frac{b^2}{4J}\int_0^1 u_x^2\,\mathrm{d}x + \frac{\mu_1^2}{J}\int_0^1 \varphi_t^2\,\mathrm{d}x \tag{4.9}$$

and

$$+bk(t)\int_0^1 u_t\varphi_x(0)\,\mathrm{d}x \leqslant \varepsilon_1 k(t)\int_0^1 u_t^2\,\mathrm{d}x + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \varphi_{0x}^2\,\mathrm{d}x \leqslant \varepsilon_1 k(0)\int_0^1 u_t^2\,\mathrm{d}x + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \varphi_{0x}^2\,\mathrm{d}x. \tag{4.10}$$

Young's and Cauchy-Schwarz inequalities lead to

$$-b\int_0^1 u_t\left(\int_0^t k'(t-s)\varphi_x(s)\,\mathrm{d}s\right)\mathrm{d}x \leqslant \varepsilon_1\int_0^1 u_t^2\,\mathrm{d}x + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t |k'(t-s)|\varphi_x^2(s)\,\mathrm{d}s\right)\mathrm{d}x. \tag{4.11}$$

By using Young's and Poincaré inequalities, we have

$$-\frac{b\xi}{J}\int_0^1 \varphi u_x\,\mathrm{d}x \leqslant \frac{b^2}{4J}\int_0^1 u_x^2\,\mathrm{d}x + \frac{\xi^2}{J}\int_0^1 \varphi_x^2\,\mathrm{d}x. \tag{4.12}$$

By substituting (4.8)–(4.12) in (4.7) and taking into account that $\chi = 0$, we get (4.5). □

Lemma 4.3. *Let* (u,φ) *be the solution of system* (1.1). *Then the functional*

$$F_3(t) = -\int_0^1 u u_t \,\mathrm{d}x$$

satisfies

$$F_3'(t) \leqslant -\varrho \int_0^1 u_t^2 \,\mathrm{d}x + \frac{b^2}{2\mu}\int_0^1 \varphi_x^2 \,\mathrm{d}x + \frac{3\mu}{2}\int_0^1 u_x^2 \,\mathrm{d}x. \tag{4.13}$$

P r o o f. Differentiating $F_3(t)$ and integrating by parts, we obtain

$$F_3'(t) = -\varrho \int_0^1 u_t^2 \,\mathrm{d}x + \mu \int_0^1 u_x^2 \,\mathrm{d}x - b\int_0^1 u\varphi_x \,\mathrm{d}x,$$

Young's and Poincaré inequalities give (4.13). □

Lemma 4.4 ([33])**.** *Let* (u,φ) *be the solution of system* (1.1). *Then the functionals*

$$F_4(t) = \mathrm{e}^{-\varsigma t}\int_0^1 \left(\int_0^t \mathrm{e}^{\varsigma s}\widetilde{H}_1(t-s)\varphi_t^2(s)\,\mathrm{d}s\right)\mathrm{d}x,$$
$$F_5(t) = \mathrm{e}^{-\tau t}\int_0^1 \left(\int_0^t \mathrm{e}^{\tau s}\widetilde{H}_2(t-s)\varphi_x^2(s)\,\mathrm{d}s\right)\mathrm{d}x$$

satisfy, for all $t \geqslant 0$,

$$F_4'(t) = -\varsigma F_4(t) + \widetilde{H}_1(0)\int_0^1 \varphi_t^2\,\mathrm{d}x - \int_0^1\left(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\right)\mathrm{d}x, \tag{4.14}$$

$$F_5'(t) = -\tau F_5(t) + \widetilde{H}_2(0)\int_0^1 \varphi_x^2\,\mathrm{d}x - \int_0^1\left(\int_0^t |k'(t-s)|\varphi_x^2(s)\,\mathrm{d}s\right)\mathrm{d}x, \tag{4.15}$$

where $\widetilde{H}_1(t) = \int_t^\infty \mathrm{e}^{\varsigma s}|k(s)|\,\mathrm{d}s$ *and* $\widetilde{H}_2(t) = \int_t^\infty \mathrm{e}^{\tau s}|k'(s)|\,\mathrm{d}s$.

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + F_3(t) + N_3F_4(t) + N_4F_5(t), \tag{4.16}$$

where N, N_1, N_2, N_3 and N_4 are positive constants.

Lemma 4.5. *Let* (u,φ) *be the solution of* (1.1). *Then there exist two positive constants* κ_1 *and* κ_2 *such that the Lyapunov functional* (4.16) *satisfies*

$$\kappa_1(E(t) + F_4(t) + F_5(t)) \leqslant \mathcal{L}(t) \leqslant \kappa_2(E(t) + F_4(t) + F_5(t)) \quad \forall\, t \geqslant 0, \tag{4.17}$$

and

$$\mathcal{L}'(t) \leqslant -\beta_1(E(t) + F_4(t) + F_5(t)) + C_2k(t) + N_2\frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_{tx}\,\mathrm{d}x, \quad \beta_1 > 0. \tag{4.18}$$

P r o o f. From (4.16) we have

$$
\begin{aligned}
&|\mathcal{L}(t) - NE(t) - N_3F_4(t) - N_4F_5(t)| \\
&\qquad \leqslant N_1 J \int_0^1 |\varphi| \Big|\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big|\,\mathrm{d}x + N_1 \frac{\mu_1}{2} \int_0^1 \varphi^2\,\mathrm{d}x \\
&\qquad + N_1 \frac{|b|\varrho}{\mu} \int_0^1 |\varphi| \Big(\int_0^x |u_t(y)|\,\mathrm{d}y\Big)\,\mathrm{d}x + N_2 \frac{\delta\varrho|b|}{\mu J} \int_0^1 |\varphi_x||u_t|\,\mathrm{d}x \\
&\qquad + N_2|b| \int_0^1 |u_x| \Big|\varphi_t + \int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big|\,\mathrm{d}x + \varrho \int_0^1 |u||u_t|\,\mathrm{d}x.
\end{aligned}
$$

By using Young's, Cauchy-Schwarz and Poincaré inequalities, we obtain

$$
|\mathcal{L}(t) - NE(t) - N_3F_4(t) - N_4F_5(t)| \leqslant \lambda_1 E(t).
$$

Therefore,

$$
(N - \lambda_1)E(t) + N_3F_4(t) + N_4F_5(t) \leqslant \mathcal{L}(t) \leqslant (N + \lambda_1)E(t) + N_3F_4(t) + N_4F_5(t);
$$

by choosing N (depending on N_1, N_2, N_3, N_4) sufficiently large we obtain (4.17) with

$$
\kappa_1 = \min\{N - \lambda_1, N_3, N_4\}, \quad \kappa_2 = \max\{N + \lambda_1, N_3, N_4\}.
$$

Now, by differentiating $\mathcal{L}(t)$, exploiting (2.3), (4.1), (4.5), (4.13), (4.14), (4.15) and setting $\varepsilon_0 = \varrho/(4N_1)$, $\varepsilon_1 = \varrho/(4N_2(2 + k(0)))$, we get

$$
\begin{aligned}
\mathcal{L}'(t) \leqslant &- \Big(N\mu_1 - N_1\Big(\frac{3J}{2} + \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\Big) - N_3\widetilde{H}_1(0) - N_2\frac{\mu_1^2}{J}\Big) \int_0^1 \varphi_t^2\,\mathrm{d}x \\
&+ \frac{NJ}{2}(k' \,\Box\, \varphi_t)(t) - \frac{\varrho}{2} \int_0^1 u_t^2\,\mathrm{d}x - 2N_1\xi_1 \int_0^1 \varphi^2\,\mathrm{d}x \\
&- \Big(\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\widetilde{H}_2(0)\Big) \int_0^1 \varphi_x^2\,\mathrm{d}x \\
&- \Big(\frac{b^2}{4J}N_2 - \frac{3\mu}{2}\Big) \int_0^1 u_x^2\,\mathrm{d}x - \varsigma N_3F_4(t) - \tau N_4F_5(t) \\
&- \Big(N_3 - \frac{J\overline{k}}{2}N_1\Big) \int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&- \Big(N_4 - \frac{N_2^2 b^2 k(0)(2 + k(0))}{\varrho}\Big) \int_0^1 \Big(\int_0^t |k'(t-s)|\varphi_x^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
&+ \frac{b^2 N_2^2 k(t)(2 + k(0))}{\varrho} \int_0^1 \varphi_{0x}^2\,\mathrm{d}x + N_2\frac{\varrho b}{\mu}\chi \int_0^1 \varphi_t u_{tx}\,\mathrm{d}x.
\end{aligned}
$$

We select our parameters appropriately as follows. First, we choose N_2 large enough such that

$$\frac{b^2}{4J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large enough such that

$$N_4 - \frac{N_2^2 b^2 k(0)(2+k(0))}{\varrho} > 0.$$

We select N_1 large enough such that

$$\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4 \widetilde{H}_2(0) > 0.$$

We choose N_3 large such that

$$N_3 - \frac{J\overline{k}}{2}N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.17) remains valid) such that

$$N\mu_1 - N_1\Big(\frac{3J}{2} + \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\Big) - N_3\widetilde{H}_1(0) - N_2\frac{\mu_1^2}{J} > 0.$$

All these choices lead to

(4.19)
$$\begin{aligned}\mathcal{L}'(t) \leqslant & -\alpha_1\int_0^1 (\varphi_t^2 + \varphi_x^2 + u_t^2 + u_x^2 + \varphi^2)\,\mathrm{d}x - \int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\ & + \alpha_2 k(t)\int_0^1 \varphi_{0x}^2\,\mathrm{d}x - \varsigma N_3 F_4(t) - N_4\tau F_5(t) + N_2\frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_{tx}\,\mathrm{d}x,\end{aligned}$$

where $\alpha_1, \alpha_2 > 0$.

On the other hand, from (2.2) and by using Young's inequality, we obtain

$$\begin{aligned}E(t) &\leqslant \frac{1}{2}\int_0^1 (\varrho u_t^2 + J\varphi_t^2 + (\mu+|b|)u_x^2 + \delta\varphi_x^2 + (\xi+|b|)\varphi^2)\,\mathrm{d}x \\ &\quad + \frac{J}{2}\int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\ &\leqslant \varrho_1\Big(\int_0^1 (u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2)\,\mathrm{d}x + \int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\Big),\end{aligned}$$

where $\varrho_1 > 0$, and which implies that

$$\text{(4.20)}\quad -\int_0^1 (u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2)\,\mathrm{d}x - \int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \leqslant -\varrho_2 E(t),$$

where $\varrho_2 > 0$. The combination of (4.19) and (4.20) gives (4.18) with $C_2 = \alpha_2 \times \int_0^1 \varphi_{0x}^2\,\mathrm{d}x$. □

We are now ready to state and prove the following exponential stability result.

Theorem 4.1. *Let* (u, φ) *be the solution of* (1.1) *and assume that* (1.2), (H1)–(H2) *hold and* $\chi = 0$. *Then there exist two positive constants* τ_1 *and* τ_2 *such that*

$$E(t) \leqslant \tau_2 \mathrm{e}^{-\tau_1 t} \quad \forall\, t \geqslant 0. \tag{4.21}$$

Proof. By using (4.18) and the right side of (4.17), we get

$$\mathcal{L}'(t) \leqslant -C_1 \mathcal{L}(t) + C_2 k(t), \tag{4.22}$$

where $C_1 = \beta_1/\kappa_2 > 0$. Multiplying (4.22) by $\exp(C_1 t)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{L}(t)\exp(C_1 t)) \leqslant C_2 \exp(C_1 t) k(t). \tag{4.23}$$

Integrating over $(0, T)$ inequation (4.23) and choosing C_1 smaller than ς, we have

$$\mathcal{L}(T)\exp(C_1 T) \leqslant \mathcal{L}(0) + C_2 \int_0^T \exp(\varsigma t) k(t)\,\mathrm{d}t \leqslant \mathcal{L}(0) + C_2 \int_0^\infty \exp(\varsigma t) k(t)\,\mathrm{d}t.$$

Thanks to the hypothesis (H2), we can write

$$\mathcal{L}(T) \leqslant C_3 \exp(-C_1 T), \quad C_3 > 0,$$

which yields the serial result (4.21), using the fact that $F_4(t)$, $F_5(t)$ are positive and the other side of the equivalence relation (4.17) again. The proof is complete. □

4.2. Polynomial stability. Here, similarly to [33], we prove a polynomial decay result of solutions of problem (1.1) when (1.5) does not hold by assuming that the function k verifies the same hypotheses (H1)–(H2) and the following additional assumption:

(H3) $-\omega k(t) \leqslant k'(t) \leqslant 0$, where ω is a positive constant.

In order to establish this result, we need to introduce the second-order energy $E_2(t)$ by using the multipliers technique as in the case of $E(t)$. For that, by differentiating $(1.1)_1$ and $(1.1)_2$ with respect to time, we arrive at

$$\begin{cases} \varrho u_{ttt} = \mu u_{xxt} + b\varphi_{xt}, & x \in (0,1),\ t > 0, \\ J\varphi_{ttt} + J\Big(\displaystyle\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_{tt} & \\ \qquad = \delta\varphi_{xxt} - bu_{xt} - \xi\varphi_t + \mu_1\varphi_{tt}, & x \in (0,1),\ t > 0, \end{cases} \tag{4.24}$$

with boundary conditions

$$u_{xt}(0,t) = u_{xt}(1,t) = \varphi_t(0,t) = \varphi_t(1,t) = 0, \quad t \geqslant 0,$$

and initial data

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x), & x \in (0,1), \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \varphi_{tt}(x,0) = \varphi_2(x), & x \in (0,1). \end{cases}$$

Note that

$$\begin{aligned} \Big(\int_0^t k(t-s)\varphi_t(s)\,\mathrm{d}s\Big)_{tt} &= \Big(\int_0^t k(s)\varphi_t(t-s)\,\mathrm{d}s\Big)_{tt} \\ &= \Big(\int_0^t k(s)\varphi_{tt}(t-s)\,\mathrm{d}s + k(t)\varphi_t(0)\Big)_t \\ &= \int_0^t k(t-s)\varphi_{ttt}(s)\,\mathrm{d}s + k(t)\varphi_{tt}(0) + k'(t)\varphi_t(0). \end{aligned}$$

Then system (4.24) can be rewritten as

$$(4.25)\quad \begin{cases} \varrho u_{ttt} = \mu u_{xxt} + b\varphi_{xt}, & x \in (0,1),\ t > 0 \\ J\varphi_{ttt} + J\displaystyle\int_0^t k(t-s)\varphi_{ttt}(s)\,\mathrm{d}s + Jk(t)\varphi_2 + Jk'(t)\varphi_1 & \\ \qquad = \delta\varphi_{xxt} - bu_{xt} - \xi\varphi_t - \mu_1\varphi_{tt}, & x \in (0,1),\ t > 0, \end{cases}$$

where $\varphi_2 = \varphi_{tt}(0)$ and $\varphi_1 = \varphi_t(0)$ depend on x.

Theorem 4.2. *The second-order energy $E_2(t)$ associated to system (1.1) defined by*

$$(4.26)\quad \begin{aligned} E_2(t) = {} & \frac{1}{2}\int_0^1 (\varrho u_{tt}^2 + J\varphi_{tt}^2 + \xi\varphi^2 + \delta\varphi_{xt}^2 + \mu u_{xt}^2 + 2b\varphi_t u_{tx})\,\mathrm{d}x \\ & + \frac{J}{2}\int_0^1 \Big(\int_0^t k(t-s)\varphi_{tt}^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \end{aligned}$$

satisfies

$$(4.27)\quad E_2'(t) \leqslant -Jk'(t)\int_0^1 \varphi_1\varphi_{tt}\,\mathrm{d}x - \mu_1\int_0^1 \varphi_{tt}^2\,\mathrm{d}x + \frac{J}{2}(k' \,\Box\, \varphi_{tt})(t),$$

and

$$(4.28)\quad E_2(t) \leqslant l \quad \forall\, t \geqslant 0.$$

P r o o f. By multiplying $(4.25)_1$ by u_{tt}, $(4.25)_2$ by φ_{tt}, integrating over $(0,1)$ and summing up, we obtain

$$
\begin{aligned}
(4.29)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (\varrho u_{tt}^2 + J\varphi_{tt}^2 + \xi\varphi^2 + \delta\varphi_{xt}^2 + \mu u_{xt}^2 + 2b\varphi_t u_{tx})\,\mathrm{d}x \\
& + Jk(t)\int_0^1 \varphi_{tt}\varphi_2\,\mathrm{d}x + Jk'(t)\int_0^1 \varphi_{tt}\varphi_1\,\mathrm{d}x \\
& + J\int_0^1 \varphi_{tt}\Big(\int_0^t k(t-s)\varphi_{ttt}(s)\,\mathrm{d}s\Big)\,\mathrm{d}x = -\mu_1\int_0^1 \varphi_{tt}^2\,\mathrm{d}x.
\end{aligned}
$$

By using again the result in Lemma 2.1 to estimate the last term of (4.29), we get

$$
\begin{aligned}
(4.30)\qquad & J\int_0^1 \varphi_{tt}\Big(\int_0^t k(t-s)\varphi_{ttt}(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\
& = \frac{J}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big(\int_0^t k(t-s)\varphi_{tt}^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x - Jk(t)\int_0^1 \varphi_2\varphi_{tt}\,\mathrm{d}x \\
& + \frac{Jk(t)}{2}\int_0^1 \varphi_{tt}^2\,\mathrm{d}x - \frac{J}{2}(k'\,\square\,\varphi_{tt})(t).
\end{aligned}
$$

By using the positivity of $k(t)$ and the combination of (4.29) with (4.30), we have (4.26) and (4.27).

Now, by using the hypothesis (H3) and Young's inequality, we can write

$$
(4.31)\qquad -Jk'(t)\int_0^1 \varphi_1\varphi_{tt}\,\mathrm{d}x \leqslant J\delta_1\omega k(t)\int_0^1 \varphi_{tt}^2\,\mathrm{d}x + \frac{J\omega k(t)}{4\delta_1}\int_0^1 \varphi_1^2\,\mathrm{d}x,
$$

letting $\delta_1 = 1/(2\omega)$ and because $k'(t) \leqslant 0$, then (4.27) becomes

$$
E_2'(t) \leqslant \frac{J\omega^2 k(t)}{2}\int_0^1 \varphi_1^2\,\mathrm{d}x = \zeta k(t),
$$

where

$$
\zeta = \frac{J\omega^2}{2}\int_0^1 \varphi_1^2\,\mathrm{d}x > 0.
$$

A simple integration over $(0,T)$ and by the hypothesis (H1), we obtain (4.28). □

We introduce the functional

$$
\widetilde{F}_2(t) = -\frac{\varrho b}{\mu}\chi\int_0^1 \varphi_t u_x\,\mathrm{d}x
$$

that satisfies

$$
\widetilde{F}_2'(t) = -\frac{\varrho b}{\mu}\chi\int_0^1 u_{tx}\varphi_t\,\mathrm{d}x - \frac{\varrho b}{\mu}\chi\int_0^1 \varphi_{tt}u_x\,\mathrm{d}x.
$$

By using Young's inequality, we get

$$-\frac{\varrho b}{\mu}\chi\int_0^1 \varphi_{tt}u_x\,\mathrm{d}x \leqslant \frac{b^2}{8J}\int_0^1 u_x^2\,\mathrm{d}x + C_0\int_0^1 \varphi_{tt}^2\,\mathrm{d}x.$$

Then

$$\widetilde{F}_2'(t) \leqslant \frac{b^2}{8J}\int_0^1 u_x^2\,\mathrm{d}x + C_0\int_0^1 \varphi_{tt}^2\,\mathrm{d}x - \frac{\varrho b}{\mu}\chi\int_0^1 u_{tx}\varphi_t\,\mathrm{d}x.$$

We define the following Lyapunov functional as

(4.32)

$$\mathcal{L}_1(t) = N(E(t)+E_2(t)) + N_1F_1(t) + N_2(F_2(t)+\widetilde{F}_2(t)) + F_3(t) + N_3F_4(t) + N_4F_5(t).$$

The Lyapunov functional $\mathcal{L}_1$ defined by (4.32) is not equivalent to the energy functional E, but it is equivalent to $E + E_2 + F_4(t) + F_5(t)$. Indeed, by using (4.32) Young's, Poincaré's and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}
&|\mathcal{L}_1(t) - N(E(t)+E_2(t)) - N_3F_4(t) - N_4F_5(t)| \\
&\qquad \leqslant \lambda_1 E(t) + \lambda_2 E_2(t) \leqslant \beta(E(t)+E_2(t)), \quad \beta = \max(\lambda_1,\lambda_2), \\
&(N-\beta)(E(t)+E_2(t)) + N_3F_4(t) + N_4F_5(t) \\
&\qquad \leqslant \mathcal{L}_1(t) \leqslant (N+\beta)(E(t)+E_2(t)) + N_3F_4(t) + N_4F_5(t).
\end{aligned}$$

Now by choosing N sufficiently large, we obtain

$$\varrho_1(E(t)+E_2(t)+F_4(t)+F_5(t)) \leqslant \mathcal{L}_1(t) \leqslant \varrho_2(E(t)+E_2(t)+F_4(t)+F_5(t)),$$

where

$$\varrho_1 = \min\{N-\beta, N_3, N_4\}, \quad \varrho_2 = \max\{N+\beta, N_3, N_4\}.$$

Therefore,

$$\mathcal{L}_1(t) \sim E + E_2 + F_4 + F_5.$$

Now, we are ready to state and prove the polynomial stability result.

Theorem 4.3. *Let* (u,φ) *be the solution of* (1.1) *and assume that* (1.2), (H1)–(H3) *hold and* $\chi \neq 0$. *Then there exits a positive constant* C_3 *such that*

$$E(t) \leqslant \frac{C_3}{t}, \quad t > 0.$$

Proof. First, note that when $\chi \neq 0$, we have

$$F_2'(t) + \widetilde{F}_2'(t) \leqslant -\frac{b^2}{8J}\int_0^1 u_x^2\,\mathrm{d}x + \Big(\frac{\delta b^2}{\mu J} + \frac{b^2k^2(0)}{4\varepsilon_1} + \frac{\xi^2}{2J}\Big)\int_0^1 \varphi_x^2\,\mathrm{d}x + \varepsilon_1(2+k(0))\int_0^1 u_t^2\,\mathrm{d}x + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \varphi_{0x}^2\,\mathrm{d}x + \frac{\mu_1^2}{J}\int_0^1 \varphi_t^2\,\mathrm{d}x + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\Big(\int_0^t k'(t-s)\varphi_x^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x + C_0\int_0^1 \varphi_{tt}^2\,\mathrm{d}x. \tag{4.33}$$

By differentiating $\mathcal{L}_1$ and using (2.3), (4.1), (4.33), (4.13), (4.14) and (4.15), we get

$$\begin{aligned}\mathcal{L}_1'(t) \leqslant & -\delta_1\int_0^1 \varphi_t^2\,\mathrm{d}x + \delta_2(k'\,\Box\,\varphi_t)(t) - \delta_3\int_0^1 u_t^2\,\mathrm{d}x - \delta_4\int_0^1 \varphi^2\,\mathrm{d}x - \delta_5\int_0^1 \varphi_x^2\,\mathrm{d}x\\ & - \delta_6\int_0^1 u_x^2\,\mathrm{d}x - \delta_7F_4(t) - \delta_8F_5(t) - \delta_9\int_0^1\Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x\\ & - \delta_{10}\int_0^1\Big(\int_0^t |k'(t-s)|\varphi_x^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x + (\delta_{11}+\delta_{14})k(t)\\ & - \delta_{12}\int_0^1 \varphi_{tt}^2\,\mathrm{d}x + \delta_{13}(k'\,\Box\,\varphi_{tt})(t),\end{aligned}$$

where

$$\begin{gathered}\delta_1 = \Big(N\mu_1 - N_1\Big(\frac{3J}{2} + \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\Big) - N_3\widetilde{H}_1(0) - N_2\frac{\mu_1^2}{J}\Big), \quad \delta_2 = \frac{NJ}{2},\\ \delta_3 = \frac{\varrho}{2}, \quad \delta_4 = 2N_1\xi_1, \quad \delta_5 = \Big(\delta N_1 - N_2C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\widetilde{H}_2(0)\Big),\\ \delta_6 = \Big(\frac{b^2}{8J}N_2 - \frac{3\mu}{2}\Big), \quad \delta_7 = \varsigma N_3, \quad \delta_8 = \tau N_4, \quad \delta_9 = \Big(N_3 - \frac{J\overline{k}}{2}N_1\Big),\\ \delta_{10} = \Big(N_4 - \frac{N_2^2b^2k(0)(2+k(0))}{\varrho}\Big), \quad \delta_{11} = \frac{b^2N_2^2(2+k(0))}{\varrho}\int_0^1 \varphi_{0x}^2\,\mathrm{d}x,\\ \delta_{12} = (N\mu_1 - N_2C_0), \quad \delta_{13} = \frac{JN}{2}, \quad \delta_{14} = N\zeta.\end{gathered}$$

We select our parameters as follows. First, we choose N_2 large enough such that

$$\delta_6 = \frac{b^2}{8J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large such that

$$\delta_{10} = N_4 - \frac{N_2^2b^2k(0)(2+k(0))}{\varrho} > 0.$$

We select N_1 large enough such that

$$\delta_5 = \delta N_1 - N_2C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\widetilde{H}_2(0) > 0.$$

We choose N_3 large such that

$$\delta_9 = N_3 - \frac{J\bar{k}}{2} N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.17) remains valid) such that

$$\delta_1 = N\mu_1 - N_1\Big(\frac{3J}{2} + \frac{b^2\varrho^2}{4\mu^2\varepsilon_0}\Big) - N_3\widetilde{H}_1(0) - N_2\frac{\mu_1^2}{J} > 0, \quad \text{and} \quad \delta_{12} = N\mu_1 - N_2C_0 > 0.$$

Because $k'(t) \leqslant 0$ and $\delta_2, \delta_7, \delta_8, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{13} > 0$, then

$$\begin{aligned} \mathcal{L}_1'(t) \leqslant & -\delta_1 \int_0^1 \varphi_t^2 \,\mathrm{d}x - \delta_3 \int_0^1 u_t^2 \,\mathrm{d}x - \delta_4 \int_0^1 \varphi^2 \,\mathrm{d}x - \delta_5 \int_0^1 \varphi_x^2 \,\mathrm{d}x \\ & - \delta_6 \int_0^1 u_x^2 \,\mathrm{d}x - \delta_9 \int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s \Big) \,\mathrm{d}x + (\delta_{11} + \delta_{14})k(t) \\ \leqslant & -\upsilon_1 \int_0^1 \Big(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2 + \int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s \Big) \,\mathrm{d}x + \upsilon_2 k(t), \end{aligned}$$

where $\upsilon_1 = \min\{\delta_1, \delta_3, \delta_4, \delta_5, \delta_6, \delta_9\}$, $\upsilon_2 = \delta_{11} + \delta_{14}$.

On the other hand, from the energy formula and by using Young's inequality, we obtain

$$\begin{aligned} E(t) \leqslant & \frac{1}{2} \int_0^1 (\varrho u_t^2 + J\varphi_t^2 + (\mu + |b|)u_x^2 + \delta\varphi_x^2 + (\xi + |b|)\varphi^2) \,\mathrm{d}x \\ & + \frac{J}{2} \int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s \Big) \,\mathrm{d}x \\ \leqslant & \varrho_1 \Big(\int_0^1 (u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2) \,\mathrm{d}x + \int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s \Big) \,\mathrm{d}x \Big), \end{aligned}$$

where $\varrho_1 > 0$, and which implies that

$$-\int_0^1 \Big(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2 + \int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s \Big) \,\mathrm{d}x \leqslant -\varrho_2 E(t),$$

where $\varrho_2 > 0$. Then

$$\mathcal{L}_1'(t) \leqslant -\omega_0 E(t) + \omega_1 k(t)$$

with $\omega_0 = \upsilon_1\varrho_2$, $\omega_1 = \upsilon_2$. By integrating over $(0, T)$, we get

$$\omega_0 E(T)T \leqslant -\mathcal{L}_1(T) + \mathcal{L}_1(0) + \omega_1 \int_0^T k(t)\,\mathrm{d}t \leqslant \mathcal{L}_1(0) + \omega_1 \int_0^\infty k(t)\,\mathrm{d}t = l.$$

So

$$E(T) \leqslant \frac{C_3}{T} \quad \text{with } C_3 = \frac{l}{\omega_0}.$$

The proof is complete. □

Remark 4.1. We note that the results obtained hold even for $\mu\xi = b^2$. In this case, we have to redefine the energy as in [14] as

$$E(t) = \frac{1}{2}\int_0^1 \Big(\varrho u_t^2 + J\varphi_t^2 + \delta\varphi_x^2 + \mu\Big(u_x + \frac{b}{\mu}\varphi\Big)^2 + \Big(\xi - \frac{b^2}{\mu}\Big)(\varphi)^2\Big)\,\mathrm{d}x + \frac{J}{2}\int_0^1 \Big(\int_0^t k(t-s)\varphi_t^2(s)\,\mathrm{d}s\Big)\,\mathrm{d}x,$$

and adjust our calculations accordingly.

5. Conclusion

In this paper, we studied the asymptotic behavior of the solution of porous-elastic system in the presence of neutral delay. Introducing a single damping mechanism given by this type of delay makes our problem different from those considered so far in the literature and under some assumptions imposed on the kernel of delay, we have been able to prove an explicit energy decay rate that depends on the wave speeds of propagation.

References

[1] *T. A. Apalara*: Exponential decay in one-dimensional porous dissipation elasticity. Q. J. Mech. Appl. Math. *70* (2017), 363–372. zbl MR doi

[2] *T. A. Apalara*: Corrigendum to: "Exponential decay in one-dimensional porous dissipation elasticity". Q. J. Mech. Appl. Math. *70* (2017), 553–555. zbl MR doi

[3] *T. A. Apalara*: A general decay for a weakly nonlinearly damped porous system. J. Dyn. Control Syst. *25* (2019), 311–322. zbl MR doi

[4] *T. A. Apalara*: General decay of solutions in one-dimensional porous-elastic system with memory. J. Math. Anal. Appl. *469* (2019), 457–471. zbl MR doi

[5] *F. Boyer, P. Fabrie*: Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models. Applied Mathematical Sciences 183. Springer, New York, 2013. zbl MR doi

[6] *P. S. Casas, R. Quintanilla*: Exponential decay in one-dimensional porous-thermo-elasticity. Mech. Res. Commun. *32* (2005), 652–658. zbl MR doi

[7] *P. S. Casas, R. Quintanilla*: Exponential stability in thermoelasticity with microtemperatures. Int. J. Eng. Sci. *43* (2005), 33–47. zbl MR doi

[8] *A. Choucha, S. M. Boulaaras, D. Ouchenane, B. B. Cherif, M. Abdalla*: Exponential stability of swelling porous elastic with a viscoelastic damping and distributed delay term. J. Funct. Spaces *2021* (2021), Article ID 5581634, 8 pages. zbl MR doi

[9] *A. Choucha, D. Ouchenane, K. Zennir*: General decay of solutions in one-dimensional porous-elastic with memory and distributed delay term. Tamkang J. Math. *52* (2021), 479–495. zbl MR doi

[10] *R. Datko*: Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM J. Control Optim. *26* (1988), 697–713. zbl MR doi

[11] *R. Datko, J. Lagnese, M. P. Polis*: An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM J. Control Optim. *24* (1986), 152–156. zbl MR doi
[12] *R. D. Driver*: A mixed neutral system. Nonlinear Anal., Theory Methods Appl. *8* (1984), 155–158. zbl MR doi
[13] *R. D. Driver*: A neutral system with state-dependent delays. Trends in Theory and Practice of Nonlinear Differential Equations. Lecture Notes in Pure and Applied Mathematics 90. Marcel Dekker, Basel, 1984, pp. 157–161. zbl MR
[14] *A. Fareh, S. A. Messaoudi*: Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy. Appl. Math. Comput. *293* (2017), 493–507. zbl MR doi
[15] *B. Feng*: Global well-posedness and stability for a viscoelastic plate equation with a time delay. Math. Probl. Eng. *2015* (2015), Article ID 585021, 10 pages. zbl MR doi
[16] *F. Foughali, S. Zitouni, L. Bouzettouta, H. E. Khochemane*: Well-posedness and general decay for a porous-elastic system with microtemperatures effects and time-varying delay term. Z. Angew. Math. Phys. *73* (2022), Artile ID 183, 31 pages. zbl MR doi
[17] *M. A. Goodman, S. C. Cowin*: A continuum theory for granular materials. Arch. Ration. Mech. Anal. *44* (1972), 249–266. zbl MR doi
[18] *A. Guesmia*: Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay. IMA J. Math. Control Inf. *30* (2013), 507–526. zbl MR doi
[19] *M. E. Hernández Morales, H. R. Henríquez*: Existence results for second order partial neutral functional differential equations. Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal. *15* (2008), 645–670. zbl MR
[20] *M. E. Hernández Morales, H. R. Henríquez, M. A. McKibben*: Existence of solutions for second order partial neutral functional differential equations. Integral Equations Oper. Theory *62* (2008), 191–217. zbl MR doi
[21] *H. E. Khochemane, L. Bouzettouta, A. Guerouah*: Exponential decay and well-posedness for a one-dimensional porous-elastic system with distributed delay. Appl. Anal. *100* (2021), 2950–2964. zbl MR doi
[22] *H. E. Khochemane, S. Zitouni, L. Bouzettouta*: Stability result for a nonlinear damping porous-elastic system with delay term. Nonlinear Stud. *27* (2020), 487–503. zbl MR
[23] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Études mathématiques. Dunod, Paris, 1969. (In French.) zbl MR
[24] *G. Liu, J. Yan*: Global asymptotic stability of nonlinear neutral differential equation. Commun. Nonlinear Sci. Numer. Simul. *19* (2014), 1035–1041. zbl MR doi
[25] *S. Loucif, R. Guefaifia, S. Zitouni, H. E. Khochemane*: Global well-posedness and exponential decay of fully dynamic and electrostatic or quasi-static piezoelectric beams subject to a neutral delay. Z. Angew. Math. Phys. *74* (2023), Article ID 83, 22 pages. zbl MR doi
[26] *A. Magaña, R. Quintanilla*: On the time decay of solutions in one-dimensional theories of porous materials. Int. J. Solids Struct. *43* (2006), 3414–3427. zbl MR doi
[27] *J. E. Muñoz Rivera, E. C. Lapa, R. Barreto*: Decay rates for viscoelastic plates with memory. J. Elasticity *44* (1996), 61–87. zbl MR doi
[28] *S. Nicaise, C. Pignotti*: Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM J. Control Optim. *45* (2006), 1561–1585. zbl MR doi
[29] *S. Nicaise, C. Pignotti*: Stabilization of the wave equation with boundary or internal distributed delay. Differ. Integral Equ. *21* (2008), 935–958. zbl MR doi
[30] *J. W. Nunziato, S. C. Cowin*: A nonlinear theory of elastic materials with voids. Arch. Ration. Mech. Anal. *72* (1979), 175–201. zbl MR doi
[31] *R. Quintanilla*: Slow decay for one-dimensional porous dissipation elasticity. Appl. Math. Lett. *16* (2003), 487–491. zbl MR doi

[32] *R. Racke*: Instability of coupled systems with delay. Commun. Pure. Appl. Anal. *11* (2012), 1753–1773. zbl MR doi
[33] *L. Seghour, N.-e. Tatar, A. Berkani*: Stability of a thermoelastic laminated system subject to a neutral delay. Math. Methods Appl. Sci. *43* (2020), 281–304. zbl MR doi
[34] *J. Wang*: Existence and stability of solutions for neutral differential equations with delay. International Conference on Multimedia Technology. IEEE, Piscataway, 2011, pp. 2462–2465. doi
[35] *G. Q. Xu, S. P. Yung, L. K. Li*: Stabilization of wave systems with input delay in the boundary control. ESAIM, Control Optim. Calc. Var. *12* (2006), 770–785. zbl MR doi

Authors' addresses: *Houssem Eddine Khochemane* (corresponding author), École Normale Supérieure d'Enseignement Technologique de Skikda, Rue du 1er Novembre, 21000 Azzaba, Algeria, e-mail: `khochmanehoussem@hotmail.com`; *Sara Labidi*, University of Badji Mokhtar-Annaba, B. P. 12, 23000 Annaba, Algeria; Laboratory of Numerical Analysis, Optimization and Statistics (LANOS), 23000 Annaba, Algeria, e-mail: `sarralabidi2222@gmail.com`; *Sami Loucif*, Laboratory of Mathematics, Informatics and Systems (LAMIS), University of Larbi Tebessi, Route de Constantine, 12022 Tebessa, Algeria, e-mail: `loucifsami2022@gmail.com`; *Abdelhak Djebabla*, University of Badji Mokhtar-Annaba, B. P. 12, 23000 Annaba, Algeria; Laboratory of Mathematics, Dynamics and Modelization, 23000 Annaba, Algeria, e-mail: `adjebabla@yahoo.com`.