2023 MATHEMATICA BOHEMICA 25 pp
Online first

ON A KIRCHHOFF-CARRIER EQUATION WITH NONLINEAR TERMS CONTAINING A FINITE NUMBER OF UNKNOWN VALUES

Nguyen Vu Dzung, Ho Chi Minh City,
Le Thi Phuong Ngoc, Nha Trang City,
Nguyen Huu Nhan, Nguyen Thanh Long, Ho Chi Minh City

Received October 4, 2021. Published online April 25, 2023.
Communicated by Reinhard Farwig

Abstract. We consider problem (P) of Kirchhoff-Carrier type with nonlinear terms containing a finite number of unknown values $u(\eta_1, t), \ldots, u(\eta_q, t)$ with $0 \leqslant \eta_1 < \eta_2 < \ldots < \eta_q < 1$. By applying the linearization method together with the Faedo-Galerkin method and the weak compact method, we first prove the existence and uniqueness of a local weak solution of problem (P). Next, we consider a specific case (P_q) of (P) in which the nonlinear term contains the sum $S_q[u^2](t) = q^{-1} \sum_{i=1}^{q} u^2((i-1)/q, t)$. Under suitable conditions, we prove that the solution of (P_q) converges to the solution of the corresponding problem (P_∞) as $q \to \infty$ (in a certain sense), here (P_∞) is defined by (P_q) in which $S_q[u^2](t)$ is replaced by $\int_0^1 u^2(y,t)\,\mathrm{d}y$. The proof is done by using the compactness lemma of Aubin-Lions and the method of continuity with a priori estimates. We end the paper with remarks related to similar problems.

Keywords: Kirchhoff-Carrier equation; Robin-Dirichlet problem; nonlocal term; Faedo-Galerkin method; linearization method

MSC 2020: 35A01, 35A02, 35B45, 35M11, 35L05

1. Introduction

We investigate the Robin-Dirichlet problem for a nonlinear wave equation

$$
(1.1) \qquad \begin{cases} u_{tt} - \lambda u_{txx} - \mu(t, u(0,t), u(\eta_1,t), \ldots, u(\eta_q,t), \|u(t)\|^2, \|u_x(t)\|^2) u_{xx} \\ \qquad + \displaystyle\int_0^t g(t-s) u_{xx}(x,s)\,\mathrm{d}s = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \end{cases}
$$

DOI: 10.21136/MB.2023.0153-21

where μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions and $\lambda > 0$, $\zeta \geqslant 0$, $\eta_1, \eta_2, \ldots, \eta_q$ are given constants with $0 \leqslant \eta_1 < \eta_2 < \ldots < \eta_q < 1$ and $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,\mathrm{d}x$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,\mathrm{d}x$.

In the one-dimensional case, the first equation $(1.1)_1$ of problem (1.1) is regarded as a model of nonlinear wave equations of the Kirchhoff-Carrier type with strong damping and memory terms. It is well known that the mathematical model of Kirchhoff and Carrier comes from a description of small vibrations of an elastic stretched string. In [9], Kirchhoff first investigated the nonlinear vibration of an elastic string

$$\varrho h u_{tt} = \Big(P_0 + \frac{Eh}{2L}\int_0^L \Big|\frac{\partial u}{\partial y}(y,t)\Big|^2\,\mathrm{d}y\Big)u_{xx}, \tag{1.2}$$

where $u = u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ϱ is the mass density, h is the cross-section area, L is the length, E is the Young modulus, P_0 is the initial axial tension. Carrier in [3] established a model of the type

$$u_{tt} - \Big(P_0 + P_1\int_0^L u^2(y,t)\,\mathrm{d}y\Big)u_{xx} = 0, \tag{1.3}$$

where P_0, P_1 are given constants, which models vibrations of an elastic string when changes in tension are not small.

There is a great number of works in this aspect, for during the last decades, initial-boundary value problems of the Kirchhoff-Carrier model have been studied extensively providing many interesting results. Among the works of the Kirchhoff-Carrier type we can cite, for example, Cavalcanti et al. (see [4]–[5]), Larkin (see [10]), Long et al. (see [13]) , Medeiros (see [14]), Park and Bae (see [21]), Santos (see [23]) and the references given therein. A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros et al. (see [15], [16]). By using different methods together with various techniques in functional analysis, several results concerning the existence/global existence and the properties of solutions such as blow-up, decay, stability have been established. Especially, in case of the presence of viscoelastic terms, the reciprocal effects between viscoelastic terms and the source term can cause the decayed property or the blow-up phenomena of solutions in some cases, see for example, [7], [8], [11], [17]–[19], [22], [25]–[27].

In [20], Nhan et al. considered the Robin problem for a wave equation with the source containing nonlocal terms, i.e.,

$$\begin{cases} u_{tt} - u_{xx} = f(x,t,u(x,t),u(\eta_1,t),\ldots,u(\eta_q,t),u_t(x,t)), \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - h_0 u(0,t) = u_x(1,t) + h_1 u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \end{cases} \tag{1.4}$$

where f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions and h_0, $h_1 \geqslant 0$, $\eta_1, \eta_2, \ldots, \eta_q$ are given constants with $h_0 + h_1 > 0$, $0 \leqslant \eta_1 < \eta_2 < \ldots < \eta_q \leqslant 1$. Here, the authors proved the existence and uniqueness of a weak solution and established an asymptotic expansion of high order in a small parameter of a weak solution. However, to the best of our knowledge, there are relatively few results about such a problem with the source containing multi-point nonlocal terms.

Motivated by the above-mentioned inspiring works, we investigated problem (1.1) and we first proved the existence and uniqueness of a local weak solution of this problem. We further note that, when f has the general form

$$f \equiv f(x, t, u, u_t, u_x, u(0,t), u(\eta_1, t), \ldots, u(\eta_q, t), \|u(t)\|^2, \|u_x(t)\|^2),$$

the existence and uniqueness of a local weak solution are also valid. This result can of course be immediately applied to specific cases of problem (1.1), namely, problem (P_q) with $\mu \equiv \mu\Big(t, q^{-1} \sum\limits_{i=0}^{q-1} u^2(i/q, t), \|u_x(t)\|^2\Big)$, or problem (P_∞) with $\mu \equiv \mu(t, \|u(t)\|^2, \|u_x(t)\|^2)$, or some similar specific cases (see remarks in Section 5 below). Therefore, based on the solvability of them, we then can consider the behavior of solutions. Let us discuss the situation where the sum $q^{-1} \sum\limits_{i=1}^{q} u^2((i-1)/q, t)$ can be considered as a special combination of the discrete family $\{u(\eta_i, t)\}_{i=1}^q$ in $(1.1)_1$. It is clear to see that, if the functions $y \mapsto u^2(y, t)$ are continuous on $[0, 1]$ with $t \in [0, T]$ fixed, then

$$\frac{1}{q} \sum_{i=1}^{q} u^2\Big(\frac{i-1}{q}, t\Big) \to \int_0^1 u^2(y, t)\,\mathrm{d}y = \|u(t)\|^2 \quad \text{as } q \to \infty,$$

therefore, problem (P_q) can have a close relationship (in a certain sense) with problem (P_∞) for the equation

$$u_{tt} - \lambda u_{txx} - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2) u_{xx} + \int_0^t g(t-s) u_{xx}(x, s)\,\mathrm{d}s = f(x, t),$$

$0 < x < 1$, $0 < t < T$, associated with the Robin-Dirichlet condition and initial condition $(1.1)_{2,3}$. We will prove this relationship in this paper to obtain an approximate solution of the Kirchhoff-Carrier problem (P_∞). Obviously, similar relationships also can be discussed and proposed.

This paper consists of five sections. In Section 2, we present some preliminaries. In Section 3, by applying the linearization method together with the Faedo-Galerkin method and the weak compact method, we prove the existence and uniqueness of a local weak solution. In Section 4, we consider the couple of problems (P_q), (P_∞)

and prove that the solution of (P_q) converges to the solution of (P_∞) as $q \to \infty$ (in the same sense as in Theorem 4.4 below). The proof of this section is obtained by the compactness lemma of Aubin-Lions and the method of continuity with a priori estimates. Finally, in Section 5, we remark that the methods used can be applied again to similar problems to obtain the same results (see Remarks 5.1, 5.2 below).

2. Preliminaries

Put $\Omega = (0,1)$. We omit the definitions of the usual function spaces and denote them $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$, the Banach space of real measurable functions $u\colon (0,T) \to X$ such that $\|u\|_{L^p(0,T;X)} < \infty$ with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X & \text{if } p = \infty. \end{cases}$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, put $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

Let $T^* > 0$ with $\mu \in C^k([0,T^*] \times \mathbb{R}^{q+3})$, $\mu = \mu(t, z_1, \ldots, z_{q+3})$, we take $D_1 \mu = \frac{\partial \mu}{\partial t}$, $D_{i+1}\mu = \frac{\partial \mu}{\partial z_i}$ with $i = 1, \ldots, q+3$, and $D^\alpha \mu = D_1^{\alpha_1} \ldots D_{q+4}^{\alpha_{q+4}} \mu$, $\alpha = (\alpha_1, \ldots, \alpha_{q+4}) \in \mathbb{Z}_+^{q+4}$, $|\alpha| = \alpha_1 + \ldots + \alpha_{q+4} \leqslant k$, $D^{(0,\ldots,0)}\mu = \mu$.

On H^1, we use the norm

$$\|v\|_{H^1} = \sqrt{\|v\|^2 + \|v_x\|^2}.$$

We put

$$V = \{v \in H^1 \colon v(1) = 0\}, \tag{2.1}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) \, \mathrm{d}x + \zeta u(0) v(0), \quad u, v \in V. \tag{2.2}$$

Obviously,V is a closed subspace of H^1 and on V, the three norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v_x\|$ and $v \mapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent norms. It is well known that the imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact satisfying the inequality

$$\|v\|_{C^0(\overline{\Omega})} \leqslant \sqrt{2} \|v\|_{H^1} \quad \forall\, v \in H^1.$$

Moreover, we have the following lemmas, the proofs of which are straightforward hence we omit the details.

Lemma 2.1. *Let $\zeta \geqslant 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leqslant \|v_x\| \leqslant \|v\|_a, \\ \dfrac{1}{\sqrt{2}}\|v\|_{H^1} \leqslant \|v_x\| \leqslant \|v\|_a \leqslant \sqrt{1+\zeta}\|v_x\| \leqslant \sqrt{1+\zeta}\|v\|_{H^1} \end{cases} \qquad \forall\, v \in V.$$

Lemma 2.2. *Let $\zeta \geqslant 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.2) is continuous on $V \times V$ and coercive on V.*

Lemma 2.3. *Let $\zeta \geqslant 0$. Then there exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalues λ_j such that*

$$\begin{cases} 0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_j \leqslant \ldots, & \lim\limits_{j\to\infty} \lambda_j = \infty, \\ a(w_j, v) = \lambda_j \langle w_j, v\rangle & \forall\, v \in V,\ j = 1, 2, \ldots \end{cases}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is a Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$. On the other hand, we have w_j satisfying the boundary value problem

$$\begin{cases} -\Delta w_j = \lambda_j w_j, & \text{in } (0,1), \\ w_{jx}(0) - \zeta w_j(0) = w_j(1) = 0 & w_j \in C^\infty(\overline{\Omega}). \end{cases}$$

The proof of Lemma 2.3 can be found in [24], Theorem 7.7, page 87 with $H = L^2$ and V, $a(\cdot,\cdot)$ are defined by (2.1), (2.2), respectively.

Definition 2.4. A function $u = u(x,t)$ is a weak solution of problem (1.1) if $u \in \widetilde{V}_T = \{v \in L^\infty(0,T; H^2 \cap V),\ v' \in L^\infty(0,T; H^2 \cap V),\ v'' \in L^\infty(0,T; L^2) \cap L^2(0,T; V)\}$, and u satisfies the variational equation

$$\begin{aligned} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \mu[u](t) a(u(t), w) \\ = \int_0^t g(t-s) a(u(s), w)\,\mathrm{d}s + \langle f(t), w\rangle \end{aligned} \tag{2.3}$$

for all $w \in V$, a.e. $t \in (0,T)$, together with the initial conditions

$$u(0) = \widetilde{u}_0, \quad u'(0) = \widetilde{u}_1, \tag{2.4}$$

where

$$\mu[u](t) = \mu(t, u(0,t), u(\eta_1,t), \ldots, u(\eta_q,t), \|u(t)\|^2, \|u_x(t)\|^2). \tag{2.5}$$

3. Existence and uniqueness

In this section, we establish the local existence and uniqueness of problem (1.1). First, we make the following assumptions:

(H$_1$) $\widetilde{u}_0, \widetilde{u}_1 \in V \cap H^2$, $\widetilde{u}_{0x}(0) - \zeta\widetilde{u}_0(0) = 0$;

(H$_2$) $g \in H^1(0, T^*)$;

(H$_3$) $\mu \in C^1([0, T^*] \times \mathbb{R}^{q+1} \times \mathbb{R}^2_+)$ such that $\mu(t, z_1, \ldots, z_{q+3}) \geqslant \mu_* > 0$ for all $t \in [0, T^*]$, for all $(z_1, \ldots, z_{q+1}) \in \mathbb{R}^{q+1}$, for all $(z_{q+2}, z_{q+3}) \in \mathbb{R}^2_+$;

(H$_4$) $f \in L^\infty(0, T^*; L^2)$, $f' \in L^2(0, T^*; L^2) = L^2(Q_{T^*})$, where $Q_{T^*} = (0, 1) \times (0, T^*)$.

For each $M > 0$ given, we set the constants $\widetilde{K}_M(\mu)$, $\overline{K}_f$ as

$$\widetilde{K}_M(\mu) = \|\mu\|_{C^1(\widetilde{A}_M)} = \|\mu\|_{C^0(\widetilde{A}_M)} + \sum_{i=1}^{q+4} \|D_i\mu\|_{C^0(\widetilde{A}_M)},$$

$$\overline{K}_f = \|f\|_{L^\infty(0,T^*;L^2)} + \|f'\|_{L^2(Q_{T^*})}, \quad \|\mu\|_{C^0(\widetilde{A}_M)} = \sup_{(t,z_1,\ldots,z_{q+3}) \in \widetilde{A}_M} |\mu(t, z_1, \ldots, z_{q+3})|,$$

where

$$\widetilde{A}_M = [0, T^*] \times [-M, M]^{q+1} \times [0, M^2]^2.$$

For every $T \in (0, T^*]$, if we take

$$V_T = \{v \in L^\infty(0, T; H^2 \cap V)\colon\ v' \in L^\infty(0, T; H^2 \cap V),\ v'' \in L^2(0, T; V)\},$$

then V_T is a Banach space with respect to the norm (see Lions [12])

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;H^2\cap V)}, \|v'\|_{L^\infty(0,T;H^2\cap V)}, \|v''\|_{L^2(0,T;V)}\}.$$

For every $M > 0$, we put

$$W(M, T) = \{v \in V_T\colon\ \|v\|_{V_T} \leqslant M\},$$
$$W_1(M, T) = \{v \in W(M, T)\colon\ v'' \in L^\infty(0, T; L^2)\}.$$

Note that

$$W_1(T) = \{v \in C^0([0, T]; V) \cap C^1([0, T]; L^2)\colon\ v' \in L^2(0, T; V)\}$$

is also a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{C^0([0,T];V)} + \|v'\|_{C^0([0,T];L^2)} + \|v'\|_{L^2(0,T;V)}.$$

Now, we establish the recurrent sequence $\{u_m\}$ defined by choosing the first iteration $u_0 \equiv \widetilde{u}_0$ and suppose that

$$u_{m-1} \in W_1(M, T). \tag{3.1}$$

We associate problem (1.1) with finding $u_m \in W(M,T)$, $m \geqslant 1$, such that u_m satisfies the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \lambda a(u_m'(t), w) + \mu_m(t) a(u_m(t), w) \\ \qquad = \displaystyle\int_0^t g(t-s) a(u_m(s), w)\,\mathrm{d}s + \langle f(t), w\rangle \quad \forall\, w \in V, \\ u_m(0) = \widetilde{u}_0,\ u_m'(0) = \widetilde{u}_1, \end{cases} \tag{3.2}$$

where

(3.3)

$$\begin{aligned} \mu_m(t) &= \mu[u_{m-1}](t) \\ &= \mu(t, u_{m-1}(0,t), u_{m-1}(\eta_1, t), \ldots, u_{m-1}(\eta_q, t), \|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2). \end{aligned}$$

Then, the existence of the sequence $\{u_m\}$ is verified by the following theorem.

Theorem 3.1. *Let* (H_1)–(H_4) *hold. Then there exist positive constants* M *and* T *such that, for* $u_0 \equiv \widetilde{u}_0$, *there exists a recurrent sequence* $\{u_m\} \subset W(M,T)$ *defined by* (3.2)–(3.3).

P r o o f. The proof consists of three steps.

Step 1. The Faedo-Galerkin approximation (introduced by Lions in [12]). Starting from the basis $\{w_j\}$ defined for L^2 as in Lemma 2.3, we obtain the approximation of the data in the form

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j,$$

where the coefficients $c_{mj}^{(k)}$, $j = 1, \ldots, k$, satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \lambda a(\dot{u}_m^{(k)}(t), w_j) + \mu_m(t) a(u_m^{(k)}(t), w_j) \\ \qquad = \displaystyle\int_0^t g(t-s) a(u_m^{(k)}(s), w_j)\,\mathrm{d}s + \langle f(t), w_j\rangle, \quad 1 \leqslant j \leqslant k, \\ u_m^{(k)}(0) = \widetilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \widetilde{u}_{1k}, \end{cases} \tag{3.4}$$

where

$$\begin{cases} \widetilde{u}_{0k} = \sum\limits_{j=1}^{k} \alpha_j^{(k)} w_j \to \widetilde{u}_0 \quad \text{strongly in } H^2 \cap V, \\ \widetilde{u}_{1k} = \sum\limits_{j=1}^{k} \beta_j^{(k)} w_j \to \widetilde{u}_1 \quad \text{strongly in } H^2 \cap V. \end{cases} \tag{3.5}$$

Using contraction mapping principle, it is not difficult to show that the system (3.4) has a unique solution $u_m^{(k)}(t)$ in $[0, T]$.

Step 2. A priori estimates. Set

$$S_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_m^{(k)}(t)\|_a^2 + \lambda\|\Delta\dot{u}_m^{(k)}(t)\|^2 + \mu_m(t)(\|u_m^{(k)}(t)\|_a^2 + \|\Delta u_m^{(k)}(t)\|^2) + 2\int_0^t (\lambda(\|\dot{u}_m^{(k)}(s)\|_a^2 + \|\Delta\dot{u}_m^{(k)}(s)\|^2) + \|\ddot{u}_m^{(k)}(s)\|_a^2)\,\mathrm{d}s,$$

then we deduce from (3.4) that

$$\begin{aligned}(3.6)\quad \bar{\mu}_*\overline{S}_m^{(k)}(t) &\leqslant S_m^{(k)}(t) \\ &= S_m^{(k)}(0) + 2\langle\Delta\widetilde{u}_{0k}, \Delta\widetilde{u}_{1k}\rangle + 2\langle f(0), \Delta\widetilde{u}_{1k}\rangle + g(0)\|\Delta\widetilde{u}_{0k}\|^2 \\ &\quad + \int_0^t (\mu_m'(s) - 2g(0))(\|u_m^{(k)}(s)\|_a^2 + \|\Delta u_m^{(k)}(s)\|^2)\,\mathrm{d}s \\ &\quad + 2\int_0^t g(t-s) \\ &\quad \times (a(u_m^{(k)}(s), u_m^{(k)}(t)) + \langle\Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\rangle)\,\mathrm{d}s \\ &\quad - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) \\ &\quad \times (a(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \langle\Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) + \Delta\dot{u}_m^{(k)}(\tau)\rangle)\,\mathrm{d}s \\ &\quad + 2\int_0^t \langle f(s), \dot{u}_m^{(k)}(s) - \Delta\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s + 2\int_0^t \langle f'(s), \Delta\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \\ &\quad + 2\int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s - g(0)\|\Delta u_m^{(k)}(t)\|^2 \\ &\quad - 2\langle\Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle - 2\langle f(t), \Delta\dot{u}_m^{(k)}(t)\rangle,\end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$(3.7)\quad \overline{S}_m^{(k)}(t) = \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{u}_m^{(k)}(t)\|_a^2 + \|\Delta\dot{u}_m^{(k)}(t)\|^2 + \|u_m^{(k)}(t)\|_a^2 + \|\Delta u_m^{(k)}(t)\|^2 + \int_0^t (\|\dot{u}_m^{(k)}(s)\|_a^2 + \|\Delta\dot{u}_m^{(k)}(s)\|^2 + \|\ddot{u}_m^{(k)}(s)\|_a^2)\,\mathrm{d}s.$$

Now, for convenience, we rewrite (3.6) in the form

$$\bar{\mu}_*\overline{S}_m^{(k)}(t) \leqslant S_m^{(k)}(0) + 2\langle\Delta\widetilde{u}_{0k}, \Delta\widetilde{u}_{1k}\rangle + 2\langle f(0), \Delta\widetilde{u}_{1k}\rangle + g(0)\|\Delta\widetilde{u}_{0k}\|^2 + \sum_{j=1}^{9} I_j$$

with the terms $I_1, \ldots, I_9$ defined and estimated as follows.

By (3.3), we first note here that

$$\begin{aligned}\mu_m'(t) &= D_1\mu[u_{m-1}](t) + D_2\mu[u_{m-1}](t)u_{m-1}'(0,t) + \sum_{i=1}^{q} D_{i+2}\mu[u_{m-1}](t)u_{m-1}'(\eta_i, t) \\ &\quad + 2D_{q+3}\mu[u_{m-1}](t)\langle u_{m-1}(t), u_{m-1}'(t)\rangle \\ &\quad + 2D_{q+4}\mu[u_{m-1}](t)\langle\nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle,\end{aligned}$$

so

$$\begin{aligned}|\mu_m'(t)| &\leqslant \widetilde{K}_M(\mu)(1+(q+1)\|\nabla u_{m-1}'(t)\| + 2\|u_{m-1}(t)\|\|u_{m-1}'(t)\| \\ &\quad + 2\|\nabla u_{m-1}(t)\|\|\nabla u_{m-1}'(t)\|) \\ &\leqslant \widetilde{K}_M(\mu)(1+(q+1)M+4M^2) \equiv \widehat{\eta}_M.\end{aligned}$$

Therefore

$$\begin{aligned}I_1 &= \int_0^t (\mu_m'(s) - 2g(0))(\|u_m^{(k)}(s)\|_a^2 + \|\Delta u_m^{(k)}(s)\|^2)\,\mathrm{d}s \\ &\leqslant (\widehat{\eta}_M + 2|g(0)|)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s.\end{aligned} \tag{3.8}$$

Applying the Cauchy inequality

$$2ab \leqslant \beta a^2 + \frac{1}{\beta}b^2 \quad \forall\, a,b\in\mathbb{R},\ \beta = \frac{\bar{\mu}_*}{6} \equiv \beta_*,$$

we make the following estimation

$$\begin{aligned}I_2 &= 2\int_0^t g(t-s)(a(u_m^{(k)}(s), u_m^{(k)}(t)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t)\rangle)\,\mathrm{d}s \\ &\leqslant 2\int_0^t |g(t-s)|(\|u_m^{(k)}(s)\|_a\|u_m^{(k)}(t)\|_a + \|\Delta u_m^{(k)}(s)\|\|\Delta u_m^{(k)}(t)\| \\ &\quad + \|\Delta u_m^{(k)}(s)\|\|\Delta \dot{u}_m^{(k)}(t)\|)\,\mathrm{d}s \\ &\leqslant 2\int_0^t |g(t-s)|(\|u_m^{(k)}(s)\|_a^2 + \|\Delta u_m^{(k)}(s)\|^2 + \|\Delta u_m^{(k)}(s)\|^2)^{1/2} \\ &\quad \times (\|u_m^{(k)}(t)\|_a^2 + \|\Delta u_m^{(k)}(t)\|^2 + \|\Delta \dot{u}_m^{(k)}(t)\|^2)^{1/2}\,\mathrm{d}s \\ &\leqslant 4\int_0^t |g(t-s)|\sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(t)}\,\mathrm{d}s \\ &\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{4}{\beta_*}\|g\|_{L^2(0,T^*)}^2\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s; \\ I_3 &= -2\int_0^t \mathrm{d}\tau\int_0^\tau g'(\tau-s) \\ &\quad \times (a(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) + \Delta \dot{u}_m^{(k)}(\tau)\rangle)\,\mathrm{d}s \\ &\leqslant 4\int_0^t \mathrm{d}\tau\int_0^\tau |g'(\tau-s)|\sqrt{\overline{S}_m^{(k)}(s)}\sqrt{\overline{S}_m^{(k)}(\tau)}\,\mathrm{d}s \\ &\leqslant 4\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s;\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
I_4 &= 2\int_0^t \langle f(s), \dot{u}_m^{(k)}(s) - \Delta\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant 2\sqrt{2}\|f\|_{L^\infty(0,T^*;L^2)}\int_0^t \sqrt{\overline{S}_m^{(k)}(s)}\,\mathrm{d}s \\
&\leqslant 2T\|f\|^2_{L^\infty(0,T^*;L^2)} + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \leqslant 2T\overline{K}_f^2 + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s; \\
I_5 &= 2\int_0^t \langle f'(s), \Delta\dot{u}_m^{(k)}(s)\rangle\,\mathrm{d}s \leqslant \|f'\|^2_{L^2(Q_{T^*})} + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s \\
&\leqslant \overline{K}_f^2 + \int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s; \\
I_6 &= 2\int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s \leqslant 2\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s; \\
I_7 &= -g(0)\|\Delta u_m^{(k)}(t)\|^2 \leqslant 2|g(0)|\Big(\|\Delta\widetilde{u}_{0k}\|^2 + T^*\int_0^t \|\Delta\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s\Big) \\
&\leqslant 2|g(0)|\|\Delta\widetilde{u}_{0k}\|^2 + 2T^*|g(0)|\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s; \\
I_8 &= -2\langle\Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle \leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{1}{\beta_*}\|\Delta u_m^{(k)}(t)\|^2 \\
&\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}\Big(\|\Delta\widetilde{u}_{0k}\|^2 + T^*\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s\Big); \\
I_9 &= -2\langle f(t), \Delta\dot{u}_m^{(k)}(t)\rangle \leqslant \beta\overline{S}_m^{(k)}(t) + \frac{1}{\beta}\|f(t)\|^2 \\
&\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}(\|f(0)\|^2 + T\|f'\|^2_{L^2(Q_{T^*})}) \\
&\leqslant \beta_*\overline{S}_m^{(k)}(t) + \frac{2}{\beta_*}(\|f(0)\|^2 + T\overline{K}_f^2).
\end{aligned}
$$

It follows from (3.6), (3.8) and (3.9) that

$$
\overline{S}_m^{(k)}(t) \leqslant \overline{S}_{0m}^{(k)} + TD_1(f) + D_2(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \tag{3.10}
$$

where

$$
\begin{aligned}
\overline{S}_{0m}^{(k)} &= \frac{1}{3\beta_*}\Big(S_m^{(k)}(0) + \Big(3|g(0)| + \frac{2}{\beta_*}\Big)\|\Delta\widetilde{u}_{0k}\|^2\Big) \\
&\quad + \frac{2}{3\beta_*}(\langle\Delta\widetilde{u}_{0k}, \Delta\widetilde{u}_{1k}\rangle + \langle f(0), \Delta\widetilde{u}_{1k}\rangle) + \frac{1}{3\beta_*}\Big(\frac{2}{\beta_*}\|f(0)\|^2 + \overline{K}_f^2\Big), \\
D_1(f) &= \frac{2}{3\beta_*}\Big(1 + \frac{1}{\beta_*}\Big)\overline{K}_f^2, \\
D_2(M) &= \frac{1}{3\beta_*}\Big(4 + \frac{2}{\beta_*}T^* + \widehat{\eta}_M + 2(1+T^*)|g(0)|\Big) \\
&\quad + \frac{4}{3\beta_*}\Big(\frac{1}{\beta_*}\|g\|^2_{L^2(0,T^*)} + \sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big).
\end{aligned} \tag{3.11}
$$

By (3.5), it follows from $(3.11)_1$ that

$$\overline{S}_{0m}^{(k)} \leqslant \frac{1}{2}M^2 \quad \forall\, m, k \in \mathbb{N}, \tag{3.12}$$

where M is a constant depending only on μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$, λ, ζ, q. We choose $T \in (0, T^*]$ such that

$$\Big(\frac{1}{2}M^2 + TD_1(f)\Big)\exp(TD_2(M)) \leqslant M^2 \tag{3.13}$$

and

$$k_T = 3M\widetilde{K}_M(\mu)(1+q+4M)\sqrt{\frac{2T}{\bar{\mu}_*}}\exp(T\widetilde{D}_2(M)) < 1, \tag{3.14}$$

where

$$\widetilde{D}_2(M) = \frac{1}{\bar{\mu}_*}\Big(1 + \widehat{\eta}_M + 2\Big(|g(0)| + \frac{1}{\bar{\mu}_*}\|g\|^2_{L^2(0,T^*)} + \sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big)\Big).$$

By using Gronwall's lemma, we deduce from (3.10), (3.12) and (3.13) that

$$\overline{S}_m^{(k)}(t) \leqslant M^2\exp(-TD_2(M))\exp(tD_2(M)) \leqslant M^2 \quad \forall\, t \in [0,T] \text{ and } \forall\, m \text{ and } k \in \mathbb{N}.$$

Therefore, we have

$$u_m^{(k)} \in W(M,T) \quad \forall\, m \text{ and } k \in \mathbb{N}. \tag{3.15}$$

Step 3. Limit procedure. From (3.15), there exists a subsequence of the sequence $\{u_m^{(k)}\}$, with the elements also denoted by $u_m^{(k)}$, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*},\\ \dot{u}_m^{(k)} \to u_m' & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*},\\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } L^2(0,T;V) \text{ weakly},\\ u_m \in W(M,T). \end{cases} \tag{3.16}$$

Passing to limit in (3.4)–(3.5), we have u_m satisfying (3.2)–(3.3) in $L^2(0,T)$ weakly.

Furthermore, $(3.2)_1$ and $(3.16)_4$ imply that

$$u_m'' = \lambda\Delta u_m' + \mu_m(t)\Delta u_m - \int_0^t g(t-s)\Delta u_m(s)\,\mathrm{d}s + f \in L^\infty(0,T;L^2),$$

so we obtain $u_m \in W_1(M,T)$, Theorem 3.1 is proved. □

Theorem 3.2. *Let* (H_1)–(H_4) *hold. Then, there exist positive constants* M, T *such that the recurrent sequence defined by* (3.2)–(3.3) *converges strongly to a function* u *in* $W_1(T)$. *Moreover,* $u \in W_1(M,T)$ *and* u *is a local unique weak solution of problem* (1.1) *satisfying the estimate*

$$\|u_m - u\|_{W_1(T)} \leqslant C_T k_T^m \quad \forall\, m \in \mathbb{N}, \tag{3.17}$$

where $k_T \in [0,1)$ *and* C_T *are constants depending only on* T, μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$, λ, ζ, q.

Proof. (a) Existence of the solution. By Theorem 3.1, there exist positive constants M, T such that, for $u_0 \equiv \widetilde{u}_0$, there exists the recurrent sequence $\{u_m\} \subset W(M,T)$ defined by (3.2)–(3.3). We prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$.

Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \lambda a(w_m'(t), w) + \mu_{m+1}(t) a(w_m(t), w) \\ \qquad = \displaystyle\int_0^t g(t-s) a(w_m(s), w)\,\mathrm{d}s + (\mu_{m+1}(t) - \mu_m(t))\langle \Delta u_m(t), w\rangle \quad \forall\, w \in V, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{3.18}$$

Taking $w = w_m'$ in $(3.18)_1$ and integrating with respect to t, we get

$$\begin{aligned} \bar{\mu}_* \overline{X}_m(t) \leqslant{}& \int_0^t (\mu_{m+1}'(s) - 2g(0))\|w_m(s)\|_a^2\,\mathrm{d}s + 2\int_0^t g(t-s) a(w_m(s), w_m(t))\,\mathrm{d}s \\ &- 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) a(w_m(s), w_m(\tau))\,\mathrm{d}s \\ &+ 2\int_0^t (\mu_{m+1}(s) - \mu_m(s))\langle \Delta u_m(s), w_m'(s)\rangle\,\mathrm{d}s, \end{aligned} \tag{3.19}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and

$$\overline{X}_m(t) = \|w_m'(t)\|^2 + \|w_m(t)\|_a^2 + \int_0^t \|w_m'(s)\|_a^2\,\mathrm{d}s.$$

With similar estimations as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} &\int_0^t (\mu_{m+1}'(s) - 2g(0))\|w_m(s)\|_a^2\,\mathrm{d}s \leqslant (\widehat{\eta}_M + 2|g(0)|)\int_0^t \overline{X}_m(s)\,\mathrm{d}s, \\ &2\int_0^t g(t-s) a(w_m(s), w_m(t))\,\mathrm{d}s \leqslant \frac{\bar{\mu}_*}{2}\overline{X}_m(t) + \frac{2}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2 \int_0^t \overline{X}_m(s)\,\mathrm{d}s, \\ &-2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) a(w_m(s), w_m(\tau))\,\mathrm{d}s \leqslant 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)} \int_0^t \overline{X}_m(s)\,\mathrm{d}s, \\ &2\int_0^t (\mu_{m+1}(s) - \mu_m(s))\langle \Delta u_m(s), w_m'(s)\rangle\,\mathrm{d}s \\ &\qquad\qquad \leqslant T M^2 \widetilde{K}_M^2(\mu)(1 + q + 4M)^2 \|w_{m-1}\|_{W_1(T)}^2 + \int_0^t \overline{X}_m(s)\,\mathrm{d}s. \end{aligned} \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\overline{X}_m(t) \leqslant T\widetilde{D}_1(M)\|w_{m-1}\|^2_{W_1(T)} + 2\widetilde{D}_2(M)\int_0^t \overline{X}_m(s)\,\mathrm{d}s, \tag{3.21}$$

where

$$\widetilde{D}_1(M) = \frac{2}{\bar{\mu}_*}M^2\widetilde{K}^2_M(\mu)(1+q+4M)^2,$$
$$\widetilde{D}_2(M) = \frac{1}{\bar{\mu}_*}\Big(1+\widehat{\eta}_M + 2\Big(|g(0)| + \frac{1}{\bar{\mu}_*}\|g\|^2_{L^2(0,T^*)} + \sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big)\Big).$$

Using Gronwall's lemma, (3.21) gives

$$\|w_m\|_{W_1(T)} \leqslant k_T\|w_{m-1}\|_{W_1(T)} \quad \forall\, m\in\mathbb{N},$$

where $k_T \in (0,1)$ is defined as in (3.14). It leads to

$$\|u_m - u_{m+p}\|_{W_1(T)} \leqslant \|u_0 - u_1\|_{W_1(T)}(1-k_T)^{-1}k_T^m \quad \forall m,p\in\mathbb{N}. \tag{3.22}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \quad \text{strongly in } W_1(T). \tag{3.23}$$

Because of $u_m \in W(M,T)$, there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*},\\ u'_{m_j} \to u' & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*},\\ u''_{m_j} \to u'' & \text{in } L^2(0,T;V) \text{ weakly},\\ u\in W(M,T). \end{cases} \tag{3.24}$$

On the other hand, we have the estimation

$$\|\mu_m - \mu[u]\|_{L^\infty(0,T)} \leqslant \widetilde{K}_M(\mu)(1+q+4M)\|u_{m-1}-u\|_{W_1(T)}.$$

Therefore, (3.23) leads to

$$\mu_m \to \mu[u] \quad \text{strongly in } L^\infty(0,T). \tag{3.25}$$

Passing to limit in (3.2)–(3.3) as $m = m_j \to \infty$, it follows from (3.23)–(3.25) that there exists $u \in W(M,T)$ satisfying (2.3)–(2.5). Furthermore, (2.3) and $(3.24)_4$ imply that

$$u'' = \lambda\Delta u' + \mu[u](t)\Delta u - \int_0^t g(t-s)\Delta u(s)\,\mathrm{d}s + f \in L^\infty(0,T;L^2),$$

so we obtain $u \in W_1(M,T)$. The existence proof is completed. On the other hand, by (3.22)–(3.24), (3.17) follows.

(b) Uniqueness of the solution. Let $u_1, u_2 \in W_1(M,T)$ be two weak solutions of problem (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$(3.26) \qquad \begin{cases} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \mu_1(t) a(u(t), w) \\ \qquad = \bar{\mu}(t)\langle \Delta u_2(t), w\rangle = \displaystyle\int_0^t g(t-s) a(u(s), w)\,\mathrm{d}s \quad \forall\, w \in V, \\ u(0) = u'(0) = 0, \end{cases}$$

where $\bar{\mu}(t) = \mu_1(t) - \mu_2(t)$, $\mu_i(t) = \mu[u_i](t)$, $i = 1,2$.

Taking $w = u'(t)$ in $(3.26)_1$ and then integrating with respect to t, we have

$$(3.27) \quad \begin{aligned} \bar{\mu}_* \overline{X}(t) \leqslant & \int_0^t (\mu_1'(s) - 2g(0))\|u(s)\|_a^2\,\mathrm{d}s + 2\int_0^t g(t-s) a(u(s), u(t))\,\mathrm{d}s \\ & - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) a(u(s), u(\tau))\,\mathrm{d}s + 2\int_0^t \bar{\mu}(s)\langle \Delta u_2(s), u'(s)\rangle\,\mathrm{d}s, \end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and $\overline{X}(t) = \|u'(t)\|^2 + \|u(t)\|_a^2 + \int_0^t \|u'(s)\|_a^2\,\mathrm{d}s$. Similarly, we also have the estimate

$$(3.28) \qquad \overline{X}(t) \leqslant K_M^* \int_0^t \overline{X}(s)\,\mathrm{d}s,$$

where

$$\begin{aligned} K_M^* = & \frac{2}{\bar{\mu}_*}\Big(\widehat{\eta}_M + 2|g(0)| + \frac{2}{\bar{\mu}_*}\|g\|_{L^2(0,T^*)}^2 + 2\sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big) \\ & + \frac{4}{\bar{\mu}_*} M \widetilde{K}_M(\mu)(1 + q + 4M). \end{aligned}$$

Using Gronwall's lemma, it follows from (3.28) that $\overline{X}(t) \equiv 0$, i.e., $u_1 \equiv u_2$. Theorem 3.2 is proved. □

4. An approximate solution of the Kirchhoff-Carrier problem

In this section, we consider the couple of problems (P_q), (P_∞),

$$(\mathrm{P}_q) \quad \begin{cases} u_{tt} - \lambda u_{txx} - \mu\Big(t, \dfrac{1}{q}\displaystyle\sum_{i=0}^{q-1} u^2\Big(\dfrac{i}{q}, t\Big), \|u_x(t)\|^2\Big) u_{xx} + \displaystyle\int_0^t g(t-s) u_{xx}(x,s)\,\mathrm{d}s \\ \qquad = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x) \end{cases}$$

and

$$
(\mathrm{P}_\infty) \qquad \begin{cases} u_{tt} - \lambda u_{txx} - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2) u_{xx} + \displaystyle\int_0^t g(t-s) u_{xx}(x,s)\,\mathrm{d}s \\ \qquad = f(x,t), \quad 0 < x < 1,\ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x), \end{cases}
$$

where $\zeta \geqslant 0$, $\lambda > 0$ are constants and μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions.

We first note that, (P_q) is a special case of problem (1.1) with

$$
\mu \equiv \mu_q[u](t) = \mu\biggl(t, \frac{1}{q}\sum_{i=0}^{q-1} u^2\Bigl(\frac{i}{q}, t\Bigr), \|u_x(t)\|^2\biggr).
$$

On the other hand, for a.e. $t \in [0,T]$, by the fact that the function $y \mapsto u^2(y,t)$ is continuous on $[0,1]$, we have

$$
S_q[u^2](t) \equiv \frac{1}{q}\sum_{i=0}^{q-1} u^2\Bigl(\frac{i}{q}, t\Bigr) \to \|u(t)\|^2 \quad \text{as } q \to \infty,
$$

so $\mu_q[u](t)$ converges to $\mu[u](t) \equiv \mu(t, \|u(t)\|^2, \|u_x(t)\|^2)$ as $q \to \infty$ in the same sense as in Lemma 4.3 below.

Under suitable conditions, we next prove that the solution of (P_q) converges to the solution of the Kirchhoff-Carrier problem (P_∞) as $q \to \infty$. We make the assumptions:
$(\overline{\mathrm{H}}_3)$ $\mu \in C^1([0,T^*] \times \mathbb{R}_+^2)$ such that $\mu(t,y,z) \geqslant \mu_* > 0$ for all $(t,y,z) \in [0,T^*] \times \mathbb{R}_+^2$;
$(\overline{\mathrm{H}}_4)$ $f \in L^\infty(0,T^*;L^2)$ such that $f' \in L^2(0,T^*;L^2)$.

For each $M > 0$ given, we set

$$
\widetilde{K}_M(\mu) = \|\mu\|_{C^1(\widetilde{A}_M)} = \|\mu\|_{C^0(\widetilde{A}_M)} + \sum_{i=1}^{3} \|D_i\mu\|_{C^0(\widetilde{A}_M)},
$$

where $\|\mu\|_{C^0(\widetilde{A}_M)} = \sup\limits_{(t,y,z)\in\widetilde{A}_M} |\mu(t,y,z)|$ with $\widetilde{A}_M = [0,T^*] \times [0,M^2] \times [0,M^2]$.

Lemma 4.1. *Let* (H_1), (H_2), $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$ *hold. Then, there exist positive constants* M, T *independent of* q *such that* (P_q) *has a unique weak solution* $u_q \in W_1(M,T)$ *for all* $q \in \mathbb{N}$ *and* (P_∞) *has a unique weak solution* $u_\infty \in W_1(M,T)$.

Proof. By the assumptions (H_1), (H_2), $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$, based on the proof of Theorem 3.2, there exist positive constants M, T such that the problems (P_q) and (P_∞) have unique weak solutions $u_q \in W_1(M,T)$ for all $q \in \mathbb{N}$ and $u_\infty \in W_1(M,T)$, respectively.

It remains to show that, in this specific case, the positive constants M, T can be chosen independent of q. Indeed, as in (3.1)–(3.3) of linearization approximation, corresponding to $\mu_m(t) = \mu_q[u_{m-1}](t) = \mu(t, S_q[u_{m-1}^2](t), \|\nabla u_{m-1}(t)\|^2)$, we have

$$\begin{aligned}\mu_m'(t) = D_1\mu[u_{m-1}](t) + 2D_2\mu[u_{m-1}](t)\frac{1}{q}\sum_{i=0}^{q-1} u_{m-1}\Big(\frac{i}{q}, t\Big)u_{m-1}'\Big(\frac{i}{q}, t\Big)\\ + 2D_3\mu[u_{m-1}](t)\langle\nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}S_q[u_{m-1}^2](t) &= \frac{1}{q}\sum_{i=0}^{q-1} u_{m-1}^2\Big(\frac{i}{q}, t\Big)\\ &\leqslant \frac{1}{q}\sum_{i=0}^{q-1}\|\nabla u_{m-1}(t)\|^2 \leqslant \frac{1}{q}\sum_{i=0}^{q-1}\|u_{m-1}\|_{L^\infty(0,T;V)}^2 \leqslant M^2,\end{aligned}$$

therefore

(4.1)

$$\begin{aligned}|\mu_m'(t)| &\leqslant \widetilde{K}_M(\mu)\Big(1 + \frac{2}{q}\sum_{i=0}^{q-1}\Big|u_{m-1}\Big(\frac{i}{q}, t\Big)u_{m-1}'\Big(\frac{i}{q}, t\Big)\Big| + 2\|\nabla u_{m-1}(t)\|\|\nabla u_{m-1}'(t)\|\Big)\\ &\leqslant \widetilde{K}_M(\mu)\Big(1 + \frac{2}{q}\sum_{i=0}^{q-1}\|\nabla u_{m-1}(t)\|\|\nabla u_{m-1}'(t)\| + 2\|\nabla u_{m-1}(t)\|\|\nabla u_{m-1}'(t)\|\Big)\\ &= \widetilde{K}_M(\mu)(1 + 4\|\nabla u_{m-1}(t)\|\|\nabla u_{m-1}'(t)\|) \leqslant \widetilde{K}_M(\mu)(1 + 4M^2).\end{aligned}$$

Using (4.1), proving the estimates similar to (3.6)–(3.15) for (P_q), $\overline{S}_m^{(k)}(t)$ obtained satisfies the integral inequality

$$\overline{S}_m^{(k)}(t) \leqslant \overline{S}_{0m}^{(k)} + TD_1(f) + D_2(M)\int_0^t \overline{S}_m^{(k)}(s)\,\mathrm{d}s, \tag{4.2}$$

where

$$\begin{aligned}\overline{S}_{0m}^{(k)} &= \frac{1}{3\beta_*}\Big(S_m^{(k)}(0) + \Big(3|g(0)| + \frac{2}{\beta_*}\Big)\|\Delta\widetilde{u}_{0k}\|^2\Big)\\ &\quad + \frac{2}{3\beta_*}(\langle\Delta\widetilde{u}_{0k}, \Delta\widetilde{u}_{1k}\rangle + \langle f(0), \Delta\widetilde{u}_{1k}\rangle) + \frac{1}{3\beta_*}\Big(\frac{2}{\beta_*}\|f(0)\|^2 + \overline{K}_f^2\Big),\\ D_1(f) &= \frac{2}{3\beta_*}\Big(1 + \frac{1}{\beta_*}\Big)\overline{K}_f^2,\\ D_2(M) &= \frac{1}{3\beta_*}\Big(\frac{2}{\beta_*}T^* + \widetilde{K}_M(\mu)(1 + 4M^2) + 2(1 + T^*)|g(0)|\Big)\\ &\quad + \frac{4}{3\beta_*}\Big(1 + \frac{1}{\beta_*}\|g\|_{L^2(0,T^*)}^2 + \sqrt{T^*}\|g'\|_{L^2(0,T^*)}\Big).\end{aligned} \tag{4.3}$$

By (3.5), it follows from $(4.3)_1$ that

$$\overline{S}_{0m}^{(k)} \leqslant \frac{1}{2} M^2 \quad \forall\, m, k \in \mathbb{N}, \tag{4.4}$$

where M is a constant independent of q and depending only on μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$, λ, ζ. We choose $T \in (0, T^*]$ such that

$$\Big(\frac{1}{2} M^2 + T D_1(M)\Big) \exp(T D_2(M)) \leqslant M^2, \tag{4.5}$$

$$k_T = 12 \sqrt{\frac{2}{\bar{\mu}_*}} M^2 \widetilde{K}_M(\mu) \sqrt{T} \exp(T \widetilde{D}_2(M)) < 1,$$

where

$$\begin{aligned} \widetilde{D}_2(M) = {} & \frac{1}{\bar{\mu}_*} (1 + (1 + 4M^2) \widetilde{K}_M(\mu)) \\ & + \frac{2}{\bar{\mu}_*} \Big(|g(0)| + \frac{1}{\bar{\mu}_*} \|g\|^2_{L^2(0,T^*)} + \sqrt{T^*} \|g'\|_{L^2(0,T^*)} \Big). \end{aligned}$$

By using Gronwall's lemma, we deduce from (4.2), (4.4) and (4.5) that

$$\overline{S}_m^{(k)}(t) \leqslant M^2 \exp(-T D_2(M)) \exp(t D_2(M)) \leqslant M^2$$

for all $t \in [0, T]$, for all $m, k \in \mathbb{N}$. Obviously, M, T chosen as in (4.4), (4.5) are independent of q. Lemma 4.1 is proved. □

Because of $u_q \in W_1(M, T)$ for all $q \in \mathbb{N}$, there exists a subsequence of $\{u_q\}$ with the same symbol, such that

$$\begin{cases} u_q \to u & \text{in } L^\infty(0, T; H^2 \cap V) \text{ weakly*}, \\ u_q' \to u' & \text{in } L^\infty(0, T; H^2 \cap V) \text{ weakly*}, \\ u_q'' \to u'' & \text{in } L^2(0, T; V) \text{ weakly}. \end{cases} \tag{4.6}$$

Applying the compactness lemma of Aubin-Lions, there exists a subsequence of $\{u_q\}$, also with the same symbol, such that

$$\begin{cases} u_q \to u & \text{in } C([0, T]; V) \text{ strongly}, \\ u_q' \to u' & \text{in } C([0, T]; V) \text{ strongly}. \end{cases} \tag{4.7}$$

Because u_q is the unique weak solution of (P_q), we get

(4.8)

$$\begin{aligned} \int_0^T \langle u_q''(t), w \rangle \varphi(t) \, \mathrm{d}t + \lambda \int_0^T a(u_q'(t), w) \varphi(t) \, \mathrm{d}t + \int_0^T \mu_q[u_q](t) a(u_q(t), w) \varphi(t) \, \mathrm{d}t \\ = \int_0^T \Big(\int_0^t g(t - s) a(u_q(s), w) \, \mathrm{d}s \Big) \varphi(t) \, \mathrm{d}t + \int_0^T \langle f(t), w \rangle \varphi(t) \, \mathrm{d}t \end{aligned}$$

for all $w \in V$, for all $\varphi \in C_c^\infty(0,T)$. By $(4.6)_3$ and (4.7), it leads to

$$
\begin{aligned}
(4.9)\qquad & \int_0^T \langle u_q''(t), w\rangle \varphi(t)\,\mathrm{d}t \to \int_0^T \langle u''(t), w\rangle \varphi(t)\,\mathrm{d}t,\\
& \int_0^T a(u_q(t), w)\varphi(t)\,\mathrm{d}t \to \int_0^T a(u(t), w)\varphi(t)\,\mathrm{d}t,\\
& \lambda\int_0^T a(u_q'(t), w)\varphi(t)\,\mathrm{d}t \to \lambda\int_0^T a(u'(t), w)\varphi(t)\,\mathrm{d}t,\\
& \int_0^T \varphi(t)\,\mathrm{d}t \int_0^t g(t-s) a(u_q(s), w)\,\mathrm{d}s\\
&\qquad = \int_0^T a(u_q(s), w)\Big(\int_s^T \varphi(t) g(t-s)\,\mathrm{d}t\Big)\,\mathrm{d}s\\
&\qquad\quad \to \int_0^T a(u(s), w)\Big(\int_s^T \varphi(t) g(t-s)\,\mathrm{d}t\Big)\,\mathrm{d}s\\
&\qquad = \int_0^T \varphi(t)\,\mathrm{d}t \int_0^t g(t-s) a(u(s), w)\,\mathrm{d}s.
\end{aligned}
$$

We have to show that

$$
(4.10)\qquad \int_0^T \mu_q[u_q](t) a(u_q(t), w)\varphi(t)\,\mathrm{d}t \to \int_0^T \mu[u](t) a(u(t), w)\varphi(t)\,\mathrm{d}t.
$$

We use the following lemmas.

Lemma 4.2. *The following properties are fulfilled:*
(i) $\|S_q[u_q^2] - S_q[u^2]\|_{C([0,T])} \to 0$ *as* $q \to \infty$,
(ii) $\|S_q[u^2] - \|u(\cdot)\|^2\|^2_{L^2(0,T)} \to 0$ *as* $q \to \infty$,
(iii) $\|S_q[u_q^2] - \|u(\cdot)\|^2\|^2_{L^2(0,T)} \to 0$ *as* $q \to \infty$.

Proof. (i) Note that we have the estimation

$$
(4.11)\qquad
\begin{aligned}
|S_q[u_q^2](t) - S_q[u^2](t)| &\leqslant \frac{1}{q}\sum_{i=0}^{q-1}\Big|u_q^2\Big(\frac{i}{q}, t\Big) - u^2\Big(\frac{i}{q}, t\Big)\Big| \leqslant \|u_q^2(t) - u^2(t)\|_{C^0([0,1])}\\
&\leqslant 2M\|u_q(t) - u(t)\|_V \leqslant 2M\|u_q - u\|_{C([0,T];V)}.
\end{aligned}
$$

Then, by (4.7), we deduce from (4.11) that

$$
\|S_q[u_q^2] - S_q[u^2]\|_{C([0,T])} \leqslant 2M\|u_q - u\|_{C([0,T];V)} \to 0 \quad \text{as } q \to \infty.
$$

Thus (i) is valid.

(ii) For all $h \in C^0([0,1])$, we have

$$\frac{1}{q}\sum_{i=0}^{q-1} h\Big(\frac{i}{q}\Big) \to \int_0^1 h(y)\,\mathrm{d}y.$$

Since $u \in L^\infty(0,T;V) \hookrightarrow L^\infty(0,T;C^0(\overline{\Omega}))$, the function $y \mapsto u^2(y,t)$, a.e. $t \in [0,T]$, belongs to $C^0(\overline{\Omega})$. Then, as above, we obtain

$$S_q[u^2](t) = \frac{1}{q}\sum_{i=0}^{q-1} u^2\Big(\frac{i}{q},t\Big) \to \int_0^1 u^2(y,t)\,\mathrm{d}y = \|u(t)\|^2 \quad \text{as } q \to \infty.$$

Additionally,

$$\begin{aligned} |S_q[u^2](t)| &\leqslant \frac{1}{q}\sum_{i=0}^{q-1} u^2\Big(\frac{i}{q},t\Big) \leqslant \frac{1}{q}\sum_{i=1}^{q} \|u_x(t)\|^2 \leqslant M^2, \\ \|u(t)\|^2 &\leqslant \|u_x(t)\|^2 \leqslant M^2, \end{aligned}$$

so we get that

$$|S_q[u^2](t) - \|u(t)\|^2| \leqslant 2M^2 \quad \forall\, q \in \mathbb{N} \text{ and a.e. } t \in [0,T].$$

Applying the dominated convergence theorem, we confirm that

$$\|S_q[u^2] - \|u(\cdot)\|^2\|^2_{L^2(0,T)} = \int_0^T \Big| S_q[u^2](t) - \int_0^1 u^2(y,t)\,\mathrm{d}y \Big|^2 \,\mathrm{d}t \to 0 \quad \text{as } q \to \infty,$$

thus, (ii) holds.

(iii) It follows from (i), (ii) that

$$\begin{aligned} &\|S_q[u_q^2] - \|u(\cdot)\|^2\|_{L^2(0,T)} \\ &\quad \leqslant \|S_q[u_q^2] - S_q[u^2]\|_{L^2(0,T)} + \|S_q[u^2] - \|u(\cdot)\|^2\|_{L^2(0,T)} \\ &\quad \leqslant 2\sqrt{T}M\|u_q - u\|_{C([0,T];V)} + \|S_q[u^2] - \|u(\cdot)\|^2\|_{L^2(0,T)} \to 0 \quad \text{as } q \to \infty. \end{aligned}$$

Hence, (iii) also holds. Therefore, Lemma 4.2 is proved. □

Lemma 4.3. *The convergence*

$$\|\mu_q[u_q] - \mu[u]\|_{L^2(0,T)} \to 0 \quad \text{as } q \to \infty$$

holds.

P r o o f. Due to

$$
\begin{aligned}
(4.12)\quad |\mu_q[u_q](t)-\mu[u](t)| &= |\mu(t, S_q[u_q^2](t), \|u_{qx}(t)\|^2) - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2)| \\
&\leqslant \widetilde{K}_M(\mu)(|S_q[u_q^2](t)-\|u(t)\|^2| + |\|u_{qx}(t)\|^2 - \|u_x(t)\|^2|) \\
&\leqslant \widetilde{K}_M(\mu)(|S_q[u_q^2](t)-\|u(t)\|^2| + 2M\|u_{qx}(t)-u_x(t)\|) \\
&\leqslant \widetilde{K}_M(\mu)(|S_q[u_q^2](t)-\|u(t)\|^2| + 2M\|u_q-u\|_{C([0,T];V)}),
\end{aligned}
$$

it follows from $(4.7)_1$, (4.12) and Lemma 4.2 (iii) that

$$
\begin{aligned}
\|\mu_q[u_q]-\mu[u]\|_{L^2(0,T)} \leqslant{}& \widetilde{K}_M(\mu)\|S_q[u_q^2]-\|u(\cdot)\|^2\|_{L^2(0,T)} \\
&+ 2\sqrt{T}M\widetilde{K}_M(\mu)\|u_q-u\|_{C([0,T];V)} \to 0 \quad \text{as } q\to\infty.
\end{aligned}
$$

Thus, Lemma 4.3 is proved. □

Now, we continue the proof of (4.10). By the inequality

$$
\begin{aligned}
&\left|\int_0^T \mu_q[u_q](t)a(u_q(t),w)\varphi(t)\,\mathrm{d}t - \int_0^T \mu[u](t)a(u(t),w)\varphi(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \left|\int_0^T [\mu_q[u_q](t)-\mu[u](t)]a(u_q(t),w)\varphi(t)\,\mathrm{d}t\right| \\
&\quad\quad + \left|\int_0^T \mu[u](t)a(u_q(t)-u(t),w)\varphi(t)\,\mathrm{d}t\right| \\
&\quad\leqslant \int_0^T |\mu_q[u_q](t)-\mu[u](t)|\|u_q(t)\|_a\|w\|_a|\varphi(t)|\,\mathrm{d}t \\
&\quad\quad + \int_0^T \mu[u](t)\|u_q(t)-u(t)\|_a\|w\|_a|\varphi(t)|\,\mathrm{d}t \\
&\quad\leqslant M\|w\|_a\|\varphi\|_{L^2(0,T)}\|\mu_q[u_q]-\mu[u]\|_{L^2(0,T)} \\
&\quad\quad + \widetilde{K}_M(\mu)\|w\|_a\|\varphi\|_{L^1(0,T)}\|u_q-u\|_{C([0,T];V)} \to 0 \quad \text{as } q\to\infty,
\end{aligned}
$$

combining $(4.7)_1$ and Lemma 4.3, we get (4.10).

Finally, by (4.9) and (4.10), letting $q\to\infty$ in (4.8), we obtain that $u\in W(M,T)$ satisfies the equation

$$
\begin{aligned}
&\int_0^T \langle u''(t),w\rangle\varphi(t)\,\mathrm{d}t + \lambda\int_0^T a(u'(t),w)\varphi(t)\,\mathrm{d}t + \int_0^T \mu[u](t)a(u(t),w)\varphi(t)\,\mathrm{d}t \\
&\qquad = \int_0^T\left(\int_0^t g(t-s)a(u(s),w)\,\mathrm{d}s\right)\varphi(t)\,\mathrm{d}t + \int_0^T \langle f(t),w\rangle\varphi(t)\,\mathrm{d}t
\end{aligned}
$$

together with the initial conditions $u(0)=\widetilde{u}_0$, $u'(0)=\widetilde{u}_1$.

Consequently,

$$
(4.13)\qquad \begin{cases} \langle u''(t), w\rangle + \lambda a(u'(t), w) + \mu[u](t)a(u(t), w) \\ \qquad\qquad = \displaystyle\int_0^t g(t-s)a(u(s), w)\,\mathrm{d}s + \langle f(t), w\rangle \quad \forall w \in V, \\ u(0) = \widetilde{u}_0,\ u'(0) = \widetilde{u}_1, \end{cases}
$$

and $u \in W(M,T)$. Furthermore, $(4.13)_1$ implies that

$$
u'' = \lambda u'_{xx} + \mu[u](t)u_{xx} - \int_0^t g(t-s)u_{xx}(s)\,\mathrm{d}s + f \in L^\infty(0,T;L^2),
$$

so $u \in W_1(M,T)$, hence $u \in W_1(M,T)$ is a solution of (P_∞). By the uniqueness of (P_∞), we have $u = u_\infty$. Further, we note that the sequence $\{u_q\}$ converges to u in the same sense as in (4.6) and (4.7).

The above result leads to the following theorem.

Theorem 4.4. *Let* (H_1), (H_2), $(\overline{\mathrm{H}}_3)$, $(\overline{\mathrm{H}}_4)$ *hold. Then there exist positive constants* M, T *such that:*

(i) (P_∞) *has the unique weak solution* $u \in W_1(M,T)$.

(ii) *The solution sequence* $\{u_q\}$ *of* (P_q) *converges to the weak solution* u *of* (P_∞) *in the sense*

$$
\begin{cases} u_q \to u & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*}, \\ u'_q \to u' & \text{in } L^\infty(0,T;H^2\cap V) \text{ weakly*}, \\ u''_q \to u'' & \text{in } L^2(0,T;V) \text{ weakly}, \\ u_q \to u & \text{in } C^1([0,T];V) \text{ strongly}. \end{cases}
$$

(iii) *Furthermore, we have the estimation*

$$
\|u_q - u\|_{W_1(T)} \leqslant C_T\|S_q[u_q^2] - \|u(\cdot)\|^2\|_{L^2(0,T)} \quad \forall\, q \in \mathbb{N},
$$

where C_T *is a constant depending only on* T, μ, f, g, $\widetilde{u}_0$, $\widetilde{u}_1$, λ, ζ.

Proof. It remains to prove (iii). We set

$$
(4.14)\qquad v_q = u_q - u,\quad \bar{\mu}_q(t) = \mu_q[u_q](t) - \mu[u](t),\quad \widetilde{\mu}_q(t) = \mu_q[u_q](t),
$$

then $v_q \in \widetilde{V}_T$ satisfies the variational problem

$$
(4.15)\qquad \begin{cases} \langle v''_q(t), w\rangle + \lambda a(v'_q(t), w) + \widetilde{\mu}_q(t)a(v_q(t), w) \\ \qquad\qquad = -\bar{\mu}_q(t)a(u(t), w) + \displaystyle\int_0^t g(t-s)a(v_q(s), w)\,\mathrm{d}s \quad \forall\, w \in V, \\ v_q(0) = v'_q(0) = 0. \end{cases}
$$

Taking $w = v_q'(t)$ in $(4.15)_1$ and then integrating with respect to t, we have

(4.16)

$$\begin{aligned}\bar{\mu}_* \overline{X}_q(t) \leqslant & \int_0^t (\widetilde{\mu}_q'(s) - 2g(0)) \|v_q(s)\|_a^2 \,\mathrm{d}s + 2\int_0^t g(t-s) a(v_q(s), v_q(t)) \,\mathrm{d}s \\ & - 2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) a(v_q(s), v_q(\tau)) \,\mathrm{d}s + 2\int_0^t \bar{\mu}_q(s) \langle \Delta u_q(s), v_q'(s)\rangle \,\mathrm{d}s,\end{aligned}$$

where $\bar{\mu}_* = \min\{1, \mu_*, \lambda\}$ and $\overline{X}_q(t) = \|v_q'(t)\|^2 + \|v_q(t)\|_a^2 + \int_0^t \|v_q'(s)\|_a^2 \,\mathrm{d}s$.

Note that

$$\begin{aligned}\widetilde{\mu}_q'(t) = & \, D_1\mu(t, S_q[u_q^2](t), \|u_{qx}(t)\|^2) \\ & + 2D_2\mu(t, S_q[u_q^2](t), \|u_{qx}(t)\|^2) \frac{1}{q} \sum_{i=0}^{q-1} u\Big(\frac{i}{q}, t\Big) u'\Big(\frac{i}{q}, t\Big) \\ & + 2D_3\mu(t, S_q[u_q^2](t), \|u_{qx}(t)\|^2) \langle u_{qx}(t), u_{qx}'(t)\rangle \\ \leqslant & \, \widetilde{K}_M(\mu)(1 + 4\|u_{qx}(t)\| \|u_{qx}'(t)\|) \leqslant \widetilde{K}_M(\mu)(1 + 4M^2) \equiv \widetilde{\mu}_M; \\ |\bar{\mu}_q(t)| = & \, |\mu_q[u_q](t) - \mu[u](t)| \\ \leqslant & \, \widetilde{K}_M(\mu)(|S_q[u_q^2](t) - \|u(t)\|^2| + |\|u_{qx}(t)\|^2 - \|u_x(t)\|^2|) \\ \leqslant & \, \widetilde{K}_M(\mu)(|S_q[u_q^2](t) - \|u(t)\|^2| + 2M\|v_{qx}(t)\|) \\ \leqslant & \, \widetilde{K}_M(\mu)\Big(|S_q[u_q^2](t) - \|u(t)\|^2| + 2M\sqrt{\overline{X}_q(t)}\Big).\end{aligned}$$

Using similar estimations as in the proof of Theorem 3.2, we obtain

$$\begin{aligned}(4.17) \qquad & \int_0^t (\widetilde{\mu}_q'(s) - 2g(0)) \|v_q(s)\|_a^2 \,\mathrm{d}s \leqslant (\widetilde{\mu}_M + 2|g(0)|) \int_0^t \overline{X}_q(s) \,\mathrm{d}s; \\ & 2\int_0^t g(t-s) a(v_q(s), v_q(t)) \,\mathrm{d}s \leqslant \frac{\bar{\mu}_*}{2} \overline{X}_q(t) + \frac{2}{\bar{\mu}_*} \|g\|_{L^2(0,T^*)}^2 \int_0^t \overline{X}_q(s) \,\mathrm{d}s; \\ & -2\int_0^t \mathrm{d}\tau \int_0^\tau g'(\tau - s) a(v_q(s), v_q(\tau)) \,\mathrm{d}s \leqslant 2\sqrt{T^*} \|g'\|_{L^2(0,T^*)} \int_0^t \overline{X}_q(s) \,\mathrm{d}s; \\ & 2\int_0^t \bar{\mu}_q(s) \langle \Delta u_q(s), v_q'(s)\rangle \,\mathrm{d}s \\ & \quad \leqslant 2M\widetilde{K}_M(\mu) \int_0^t \Big(|S_q[u_q^2](s) - \|u(s)\|^2| + 2M\sqrt{\overline{X}_q(s)}\Big) \sqrt{\overline{X}_q(s)} \,\mathrm{d}s \\ & \quad \leqslant \widetilde{K}_M(\mu) \|S_q[u_q^2] - \|u(\cdot)\|^2\|_{L^2(0,T)}^2 + 3M^2 \widetilde{K}_M(\mu) \int_0^t \overline{X}_q(s) \,\mathrm{d}s.\end{aligned}$$

It follows from (4.16) and (4.17) that

$$(4.18) \qquad \overline{X}_q(t) \leqslant \overline{D}_1(M) \|S_q[u_q^2] - \|u(\cdot)\|^2\|_{L^2(0,T)}^2 + 2\overline{D}_2(M) \int_0^t \overline{X}_q(s) \,\mathrm{d}s,$$

where

$$\overline{D}_1(M) = \frac{2}{\bar{\mu}_*}\widetilde{K}_M(\mu), \tag{4.19}$$

$$\overline{D}_2(M) = \frac{1}{\bar{\mu}_*}(\widetilde{\mu}_M + 3M^2\widetilde{K}_M(\mu)) + \frac{2}{\bar{\mu}_*}\Big(|g(0)| + \sqrt{T^*}\|g'\|_{L^2(0,T^*)} + \frac{1}{\bar{\mu}_*}\|g\|^2_{L^2(0,T^*)}\Big).$$

Using Gronwall's lemma, it follows from (4.18) that

$$\overline{X}_q(t) \leqslant \overline{D}_1(M)\exp(2T\overline{D}_2(M))\|S_q[u_q^2] - \|u(\cdot)\|^2\|^2_{L^2(0,T)}. \tag{4.20}$$

We deduce from (4.20) that

$$\|u_q - u\|_{W_1(T)} \leqslant 3\sqrt{\overline{D}_1(M)}\exp(T\overline{D}_2(M))\|S_q[u_q^2] - \|u(\cdot)\|^2\|_{L^2(0,T)}.$$

Hence, (iii) holds. Therefore, Theorem 4.4 is proved. □

5. Remarks

In this section, we remark that methods similar to the above ones can be applied to obtain similar results.

Remark 5.1. When f has the general form

$$f \equiv f(x,t,u,u_t,u_x,u(0,t),u(\eta_1,t),\dots,u(\eta_q,t),\|u(t)\|^2,\|u_x(t)\|^2),$$

the existence and uniqueness of a local weak solution of problem (1.1) are also valid. It means that we also prove a sufficient condition for the solvablility of problem (1.1) in this case.

Remark 5.2.

(i) The methods used to prove the unique existence of a weak solution for problem (P_q) can be applied to problem $(\widetilde{\mathrm{P}}_q)$ in which $S_q[u^2](t)$ is replaced by the sum

$$\widetilde{S}_q[u^2](t) = \frac{1}{q}\sum_{i=0}^{q-1} u^2\Big(\frac{i+\theta_i}{q}, t\Big),$$

where $\theta_i \in [0,1)$, $i = 0,\dots,q-1$, are given constants.

(ii) The arguments used to consider the relationship of (P_q) and (P_∞) can be also applied to the problems $(\overline{\mathrm{P}}_q)$, $(\overline{\mathrm{P}})$, and then, the same results are also given

$$(\overline{\mathrm{P}}_q)\quad \begin{cases} u_{tt} - \lambda u_{txx} - \mu(t,\widetilde{S}_q[u^2](t),\widetilde{S}_q[u_x^2](t))u_{xx} = f(x,t), \quad 0<x<1,\ 0<t<T,\\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0,\\ u(x,0) = \widetilde{u}_0(x),\ u_t(x,0) = \widetilde{u}_1(x),\end{cases}$$

where $\lambda > 0$, $\zeta \geqslant 0$ are given constants, μ, f, $\widetilde{u}_0$, $\widetilde{u}_1$ are given functions and $\widetilde{S}_q[u^2](t) = q^{-1} \sum\limits_{i=0}^{q-1} u^2((i+\theta_i)/q, t)$, $\widetilde{S}_q[u_x^2](t) = q^{-1} \sum\limits_{i=0}^{q-1} u_x^2((i+\theta_i)/q, t)$, $\theta_i \in [0,1)$, $i = 0, \ldots, q-1$ are given constants,

$$(\overline{\mathrm{P}}) \quad \begin{cases} u_{tt} - \lambda u_{txx} - \mu(t, \|u(t)\|^2, \|u_x(t)\|^2) u_{xx} = f(x,t), \quad 0 < x < 1, \ 0 < t < T, \\ u_x(0,t) - \zeta u(0,t) = u(1,t) = 0, \\ u(x,0) = \widetilde{u}_0(x), \ u_t(x,0) = \widetilde{u}_1(x), \end{cases}$$

where $\|u(t)\|^2 = \int_0^1 u^2(y,t)\,\mathrm{d}y$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(y,t)\,\mathrm{d}y$.

A c k n o w l e d g m e n t. The authors wish to express their sincere thanks to the editor and the referees for their valuable comments and important remarks for the improvement of the paper.

References

[1] *R. P. Agarwal*: Boundary Value Problems for Higher Order Differential Equations. World Scientific, Singapore, 1986. zbl MR doi

[2] *F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. J. Toledo-Melero*: Nonlocal Diffusion Problems. Mathematical Surveys and Monographs 165. AMS, Providence, 2010. zbl MR doi

[3] *G. F. Carrier*: On the non-linear vibration problem of the elastic string. Q. Appl. Math. *3* (1945), 157–165. zbl MR doi

[4] *M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. S. Prates Filho, J. A. Soriano*: Existence and exponential decay for a Kirchhoff-Carrier model with viscosity. J. Math. Anal. Appl. *226* (1998), 40–60. zbl MR doi

[5] *M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano*: Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation. Adv. Differ. Equ. *6* (2001), 701–730. zbl MR doi

[6] *M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, J. S. Prates Filho*: Existence and asymptotic behaviour for a degenerate Kirchhoff-Carrier model with viscosity and nonlinear boundary conditions. Rev. Mat. Complut. *14* (2001), 177–203. zbl MR doi

[7] *M. Kafini, S. A. Messaoudi*: A blow-up result in a Cauchy viscoelastic problem. Appl. Math. Lett. *21* (2008), 549–553. zbl MR doi

[8] *M. Kafini, M. I. Mustafa*: Blow-up result in a Cauchy viscoelastic problem with strong damping and dispersive. Nonlinear Anal., Real World Appl. *20* (2014), 14–20. zbl MR doi

[9] *G. Kirchhoff*: Vorlesungen über mathematische Physik. Erster Band. Mechanik. B. G. Teubner, Leipzig, 1897. (In German.) zbl MR

[10] *N. A. Larkin*: Global regular solutions for the nonhomogeneous Carrier equation. Math. Probl. Eng. *8* (2002), 15–31. zbl MR doi

[11] *Q. Li, L. He*: General decay and blow-up of solutions for a nonlinear viscoelastic wave equation with strong damping. Bound. Value Probl. *2018* (2018), Article ID 153, 22 pages. zbl MR doi

[12] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gauthier-Villars, Paris, 1969. (In French.) zbl MR

[13] *N. T. Long, A. P. N. Dinh, T. N. Diem*: Linear recursive schemes and asymptotic expansion associated with the Kirchhoff-Carrier operator. J. Math. Anal. Appl. *267* (2002), 116–134. zbl MR doi

[14] *L. A. Medeiros*: On some nonlinear perturbation of Kirchhoff-Carrier operator. Comput. Appl. Math. *13* (1994), 225–233. zbl MR
[15] *L. A. Medeiros, J. Limaco, S. B. Menezes*: Vibrations of elastic strings: Mathematical aspects. I. J. Comput. Anal. Appl. *4* (2002), 91–127. zbl MR doi
[16] *L. A. Medeiros, J. Limaco, S. B. Menezes*: Vibrations of elastic strings: Mathematical aspects. II. J. Comput. Anal. Appl. *4* (2002), 211–263. zbl MR doi
[17] *S. A. Messaoudi*: Blow up and global existence in a nonlinear viscoelastic wave equation. Math. Nachr. *260* (2003), 58–66. zbl MR doi
[18] *S. A. Messaoudi*: General decay of the solution energy in a viscoelastic equation with a nonlinear source. Nonlinear Anal., Theory Methods Appl., Ser. A *69* (2008), 2589–2598. zbl MR doi
[19] *M. I. Mustafa*: General decay result for nonlinear viscoelastic equations. J. Math. Anal. Appl. *457* (2018), 134–152. zbl MR doi
[20] *N. H. Nhan, L. T. P. Ngoc, N. T. Long*: Existence and asymptotic expansion of the weak solution for a wave equation with nonlinear source containing nonlocal term. Bound. Value Probl. *2017* (2017), Article ID 87, 20 pages. zbl MR doi
[21] *J. Y. Park, J. J. Bae*: On coupled wave equation of Kirchhoff type with nonlinear boundary damping and memory term. Appl. Math. Comput. *129* (2002), 87–105. zbl MR doi
[22] *J. Y. Park, S. H. Park*: General decay for quasilinear viscoelastic equations with nonlinear weak damping. J. Math. Phys. *50* (2009), Article ID 083505, 10 pages. zbl MR doi
[23] *M. L. Santos, J. Ferreira, D. C. Pereira, C. A. Raposo*: Global existence and stability for wave equation of Kirchhoff type with memory condition at the boundary. Nonlinear Anal., Theory Methods Appl., Ser. A *54* (2003), 959–976. zbl MR doi
[24] *R. E. Showalter*: Hilbert Space Methods for Partial Differential Equations. Electronic Journal of Differential Equations. Monograph 1. Southwest Texas State University, San Marcos, 1994. zbl MR
[25] *N. Tatar, A. Zaraï*: Exponential stability and blow up for a problem with Balakrishnan-Taylor damping. Demonstr. Math. *44* (2011), 67–90. zbl MR doi
[26] *Y. Wang, Y. Wang*: Exponential energy decay of solutions of viscoelastic wave equations. J. Math. Anal. Appl. *347* (2008), 18–25. zbl MR doi
[27] *S.-T. Wu*: Exponential energy decay of solutions for an integro-differential equation with strong damping. J. Math. Anal. Appl. *364* (2010), 609–617. zbl MR doi

Authors' addresses: *Nguyen Vu Dzung*, Department of Mathematics and Computer Science, University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam, Vietnam National University, Ho Chi Minh City, Vietnam, Cao Thang Technical College, 65 Huynh Thuc Khang Str., Ben Nghe Ward, Dist. 1, Ho Chi Minh City, Vietnam, e-mail: `dzungngv@gmail.com`; *Le Thi Phuong Ngoc*, University of Khanh Hoa, 01 Nguyen Chanh Str., Nha Trang City, Vietnam, e-mail: `ngoc1966@gmail.com`; *Nguyen Huu Nhan*, Ho Chi Minh University of Foreign Languages and Information Technology, 828 Su Van Hanh Str., Dist. 10, Ho Chi Minh City, Vietnam, e-mail: `nhannh1@huflit.edu.vn`; *Nguyen Thanh Long* (corresponding author), Department of Mathematics and Computer Science, University of Science, 227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam, Vietnam National University, Ho Chi Minh City, Vietnam, e-mail: `longnt2@gmail.com`.