PERIODIC SOLUTIONS FOR A CLASS OF NON-AUTONOMOUS HAMILTONIAN SYSTEMS WITH $p(t)$-LAPLACIAN

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Abstract. We investigate the existence of infinitely many periodic solutions for the $p(t)$-Laplacian Hamiltonian systems. By virtue of several auxiliary functions, we obtain a series of new super-p^+ growth and asymptotic-p^+ growth conditions. Using the minimax methods in critical point theory, some multiplicity theorems are established, which unify and generalize some known results in the literature. Meanwhile, we also present an example to illustrate our main results are new even in the case $p(t) \equiv p = 2$.

Keywords: auxiliary functions; $p(t)$-Laplacian systems; periodic solution; (C) condition; generalized mountain pass theorem

MSC 2020: 34C25, 35A15

1. Introduction and main results

Consider the $p(t)$-Laplacian systems

$$-(|u'(t)|^{p(t)-2}u'(t))' = \nabla F(t,u(t)) \quad \text{a.e. } t \in \mathbb{R}, \tag{1.1}$$

where $F(t,x)$ and $p(t)$ satisfy the following conditions:

(H0) $F\colon [0,T]\times\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is measurable and T-periodic ($T>0$) in its first variable for all $x\in\mathbb{R}^{\mathbb{N}}$, continuously differentiable in x for a.e. $t\in[0,T]$, and there exist $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| + |\nabla F(t,x)| \leqslant a(|x|)b(t)$$

for all $x\in\mathbb{R}^{\mathbb{N}}$ and a.e. $t\in[0,T]$;

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(P) $p(t) \in C(0,T;\mathbb{R}^+)$, $p(t)=p(t+T)$ and

$$1 < p^- := \min_{0\leqslant t\leqslant T} p(t) \leqslant p^+ := \max_{0\leqslant t\leqslant T} p(t) < \infty.$$

The interest in the study of problem (1.1) is twofold. On one hand, we have the physical motivations, since the $p(t)$-Laplacian systems can be applied to describe the physical phenomena with 'pointwise different properties' which first arose from the nonlinear elasticity theory, see [31] and the references therein. On the other hand, we have the purely mathematical interest in these types of problems, mainly regarding the existence of solutions as well as multiplicity results. For more general and recent works on $p(t)$-Laplacian systems, we refer the reader to [4], [10], [22], [26], [28].

As we have seen, if $p(t)\equiv p>1$, problem (1.1) reduces to the classical ordinary p-Laplacian systems

$$-(|u'(t)|^{p-2}u'(t))' = \nabla F(t,u(t)) \quad \text{a.e. } t\in\mathbb{R}. \tag{1.2}$$

In last decades, considerable attention has been drawn to the existence and multiplicity of periodic solutions for problem (1.2) under various conditions, see [8], [9], [11], [21], [25], [27], [30]. Specially, applying the generalized mountain pass theorem and some techniques of analysis, Ma and Zhang in [11] have proved problem (1.2) has infinitely many nontrivial periodic solutions. More precisely, for the super-p growth case, they established the following theorem.

Theorem 1.1 ([11]). *Assume that F satisfies* (H0) *and the following conditions:*

(H1) $F(t,x)\geqslant 0$ *for all* $(t,x)\in[0,T]\times\mathbb{R}^{\mathbb{N}}$*;*

(H2) $\lim\limits_{|x|\to 0} F(t,x)/|x|^p = 0$ *uniformly for a.e.* $t\in[0,T]$*;*

(H3) $\liminf\limits_{|x|\to\infty} F(t,x)/|x|^p > 0$ *uniformly for a.e.* $t\in[0,T]$*;*

(H4) $\limsup\limits_{|x|\to\infty} F(t,x)/|x|^r \leqslant M<\infty$ *uniformly for some* $M>0$ *and a.e.* $t\in[0,T]$*;*

(H5) $\liminf\limits_{|x|\to\infty} ((\nabla F(t,x),x)-pF(t,x))/|x|^\mu \geqslant \varrho>0$ *uniformly for some* $\varrho>0$ *and a.e.* $t\in[0,T]$,

where $r>p$, $\mu>r-p$ *and* $(\cdot,\cdot)$, $|\cdot|$ *are the Euclidean inner product and norm in* $\mathbb{R}^{\mathbb{N}}$*, respectively. Then problem* (1.2) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j\in\mathbb{N}$ *and* $k_j\to\infty$ *as* $j\to\infty$*.*

At the same time, for the asymptotic-p growth case, they obtained the following results.

Theorem 1.2 ([11]). *Assume that F satisfies* (H0)–(H3) *and the following conditions:*

(H4′) $\limsup_{|x|\to\infty} F(t,x)/|x|^p \leqslant M < \infty$ *uniformly for some* $M > 0$ *and a.e.* $t \in [0,T]$;

(H6) *there exists* $\gamma \in L^1(0,T;\mathbb{R}^+)$ *such that* $(\nabla F(t,x),x) - pF(t,x) \geqslant \gamma(t)$ *for all* $x \in \mathbb{R}^{\mathbb{N}}$ *and a.e.* $t \in [0,T]$;

(H7) $\lim_{|x|\to\infty} ((\nabla F(t,x),x) - pF(t,x)) = \infty$ *uniformly for a.e.* $t \in [0,T]$.

Then problem (1.2) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Theorem 1.3 ([11]). *Assume that* F *satisfies* (H0)–(H3), (H4′) *and the following conditions:*

(H6′) *there exists* $\gamma \in L^1(0,T;\mathbb{R}^+)$ *such that* $(\nabla F(t,x),x) - pF(t,x) \leqslant \gamma(t)$ *for all* $x \in \mathbb{R}^{\mathbb{N}}$ *and a.e.* $t \in [0,T]$;

(H7′) $\lim_{|x|\to\infty} ((\nabla F(t,x),x) - pF(t,x)) = -\infty$ *uniformly for a.e.* $t \in [0,T]$.

Then problem (1.2) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Motivated by the results of [6], [11], [19], [20], [23], [24], in the present paper, making use of auxiliary functions, we are interested in extending Theorem 1.1 to the $p(t)$-Laplacian systems (1.1) under more general hypotheses. In addition, we should stress that the different methods in [11] are used to ensure the compact conditions for the super-p growth situation and the asymptotic-p growth situation. Here, we attempt to propose a unified approach when the potential function $F(t,x)$ exhibits either an asymptotic-p^+ or a super-p^+ behaviour for problem (1.1). Hence, our results are new and improve recent results in the literature even in the case $p(t) \equiv p = 2$.

Now, we are in a position to state our main results. To begin with, for the super-p^+ growth case, we have:

Theorem 1.4. *Suppose that conditions* (H0), (P) *hold and* F *satisfies the following conditions:*

(F1) $F(t,x) \geqslant 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^{\mathbb{N}}$;

(F2) $\lim_{|x|\to 0} F(t,x)/|x|^{p^+} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(F3) $\liminf_{|x|\to\infty} F(t,x)/|x|^{p^+} > 0$ *uniformly for a.e.* $t \in [0,T]$;

(F4) *there exists* $h \in C(\mathbb{R}^+,\mathbb{R}^+)$ *such that*

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{h(|x|)} \leqslant M < \infty \text{ uniformly for some } M > 0 \text{ and a.e. } t \in [0,T];$$

(F5) *there exist* $M_1 > 0$, $\theta_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\lim\limits_{s\to\infty} h(s)/\theta_1(s)s^{p^-} = 0$ *and* $\theta_1(s)/h(s)$ *is non-increasing in* s *for all* $s \in \mathbb{R}^+$, *where* $h(s)$ *is defined in* (F4), *such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \geqslant \theta_1(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_1 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Remark 1.5. When comparing Theorem 1.4 to Theorem 1.1, the following four aspects must be pointed out:

(1) Condition (F1) is the same as (H1), conditions (F2) and (F3) can be regarded as natural generalizations of (H2) and (H3) about $p(t)$-Laplacian systems, respectively.

(2) We can find that conditions (F4) and (F5) cover the cases of assumptions (H4) and (H5) when (H1) (or (F1)) holds. We follow two steps to demonstrate this claim.

Step 1. We confirm that (H1) (or (F1)), (H4) and (H5) could imply $\mu \leqslant r$. It follows from (H4) that there exists $d_1 > 0$ such that

$$F(t,x) \leqslant M|x|^r \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant d_1 \text{ and for a.e. } t \in [0,T]. \tag{1.3}$$

By (H5), we can choose $d_2 > 0$ such that

$$(\nabla F(t,x), x) - pF(t,x) \geqslant \varrho|x|^\mu \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant d_2 \text{ and for a.e. } t \in [0,T]. \tag{1.4}$$

Let $d_3 := \max\{d_1, d_2\}$, taking account of (1.3), (1.4) and (H1) (or (F1)), we infer that

$$\begin{aligned} M|x|^r \geqslant F(t,x) &= \int_0^1 \frac{1}{s}(\nabla F(t,sx), sx)\,\mathrm{d}s + F(t,0) \geqslant \int_0^1 \frac{1}{s}(\varrho|sx|^\mu + pF(t,sx))\,\mathrm{d}s \\ &\geqslant \frac{1}{\mu}\varrho|x|^\mu \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant d_3 \text{ and for a.e. } t \in [0,T], \end{aligned}$$

which implies that $\mu \leqslant r$.

Step 2. We claim that (H4) and (H5) are special cases of (F4) and (F5), respectively, when (H1) (or (F1)) holds. We only need to take $p(t) \equiv p > 1$, $h(s) = s^r$, $\theta_1(s) = \varrho s^\mu$ and M_1 large enough, where $\varrho > 0$, $r > p$, $\mu > r - p$. In fact, we can check that $\lim\limits_{s\to\infty} h(s)/\theta_1(s)s^p = \lim\limits_{s\to\infty} s^{r-\mu-p}/\varrho = 0$ by $\mu > r - p$. Furthermore, $\theta_1(s)/h(s) = \varrho s^{\mu-r}$ is non-increasing in s for all $s \in \mathbb{R}^+$ by Step 1. Therefore, $h(s)$ and $\theta_1(s)$ satisfy all conditions of (F4) and (F5).

(3) There exist functions $F(t,x)$ satisfying Theorem 1.4 and not fulfilling the result of Theorem 1.1. The detailed example will be given in Section 5.

(4) Last but not the least, from the discussions of (1)–(3) we see that Theorem 1.4 significantly extends and improves Theorem 1.1.

Remark 1.6. In [23], the first author and Zhang are concerned with the existence of periodic solutions for the following damped vibration problem

$$(1.5)\qquad \begin{cases} \ddot u(t)+q(t)\dot u+\nabla F(t,u(t))=0 & \text{a.e. } t\in[0,T],\\ u(0)-u(T)=\dot u(0)-\dot u(T)=0, \end{cases}$$

where $q\in L^1(0,T;\mathbb{R})$, $\int_0^T q(t)\,\mathrm{d}t=0$. Theorem 1.1, the main result of [23], has introduced the following new non-quadratic conditions:

(F4*) $\limsup_{|x|\to\infty} \mathrm{e}^{Q(t)}F(t,x)/|x|^r\leqslant M<\infty$ uniformly for some $M>0$ and a.e. $t\in[0,T]$, where $Q(t):=\int_0^t q(s)\,\mathrm{d}s$, $r>2$;

(F5*) there exist $M_1>0$, $\mu>r-2$ and $k_1\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\lim_{s\to\infty}k_1(s)s^{\mu+2-r}=\infty$, $\lim_{s\to\infty}k_1^{r/\mu}(s)s^2=\infty$ and $k_1(s)$ is non-increasing in s for all $s\in\mathbb{R}^+$ such that

$$\mathrm{e}^{Q(t)}((\nabla F(t,x),x)-2F(t,x))\geqslant k_1(|x|)|x|^\mu\quad \forall\,x\in\mathbb{R}^{\mathbb{N}},\ |x|\geqslant M_1 \text{ and for a.e. } t\in[0,T].$$

From [23] we have known that (F4*) and (F5*) are more general than (H4) and (H5) when (H1) (or (F1)) holds. Here, if $q(t)\equiv 0$ and $\mu\leqslant r$, we emphasize that these new non-quadratic conditions (F4*) and (F5*) are also special cases of assumptions (F4) and (F5), respectively. Indeed, set $p(t)\equiv p=2$, $h(s)=s^r$, $\theta_1(s)=k_1(s)s^\mu$. Using the properties of $k_1(s)$ and $\mu\leqslant r$, a direct computation shows that $\lim_{s\to\infty}h(s)/\theta_1(s)s^2=\lim_{s\to\infty}1/k_1(s)s^{\mu+2-r}=0$; moreover, we have $g(s):=\theta_1(s)/h(s)=k_1(s)s^{\mu-r}$ is non-increasing in s for all $s\in\mathbb{R}^+$ since $g(s)/g(t)=(k_1(s)/k_1(t))(s/t)^{\mu-r}\geqslant 1$ for all $s\leqslant t$ by $\mu\leqslant r$. Therefore, in some sense, Theorem 1.4 also generalizes Theorem 1.1 with $q(t)\equiv 0$ of [23].

Theorem 1.7. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and the following condition:*

(F6) *there exist $M_2>0$, $\tau\geqslant 1$, $\theta_2\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\lim_{s\to\infty}s^{p^*}/\theta_2^{1/\tau}(s)=0$ and $\theta_2(s)/s^{p^+\tau}$ is non-increasing in s for all $s\in\mathbb{R}^+$, where $p^*:=p^+-p^-$ such that*

$$(\nabla F(t,x),x)-p^+F(t,x)\geqslant\theta_2(|x|)\Big(\frac{F(t,x)}{|x|^{p^+}}\Big)^\tau\quad\forall\,x\in\mathbb{R}^{\mathbb{N}},\ |x|\geqslant M_2 \text{ and for a.e. } t\in[0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j\in\mathbb{N}$ and $k_j\to\infty$ as $j\to\infty$.*

R e m a r k 1.8. We assert that (H1) (or (F1)), (H4) and (H5) could imply (F6) with $p(t) \equiv p > 1$. As a matter of fact, in view of (H4) and (H5), for all $x \in \mathbb{R}^{\mathbb{N}}$ and M_2 large enough we have

$$(\nabla F(t,x),x) - pF(t,x) \geqslant \varrho |x|^{\mu+p-r} \frac{|x|^r}{|x|^p} \geqslant \frac{\varrho}{M} |x|^{\mu+p-r} \frac{F(t,x)}{|x|^p}.$$

Select $p(t) \equiv p$, $\tau = 1$, $\theta_2(s) = \varrho s^{\mu+p-r}/M$, noticing $\mu > r-p$, then $\lim\limits_{s\to\infty} 1/\theta_2(s) = 0$ and $\theta_2(s)/s^p = \varrho s^{\mu-r}/M$ is non-increasing in s for all $s \in \mathbb{R}^+$ by Remark 1.5 (2), Step 1. So, (F6) holds. Therefore, Theorem 1.7 greatly unifies and generalizes Theorem 1.1.

Theorem 1.9. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and the following condition:*

(F7) *there exist $M_3 > 0$, $\sigma \geqslant p^-/(p^- - 1)$, $\theta_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim\limits_{s\to\infty} s^{p^*}/\theta_3^{1/\sigma}(s) = 0$ and $\theta_3(s)/s^{(p^+-1)\sigma}$ is non-increasing in s for all $s \in \mathbb{R}^+$, such that*

$$(\nabla F(t,x),x) - p^+ F(t,x) \geqslant \theta_3(|x|)\Big(\frac{|\nabla F(t,x)|}{|x|^{p^+-1}}\Big)^\sigma \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_3 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

R e m a r k 1.10. When $p(t) \equiv p = 2$, $\sigma > 1$, condition (F7) was originally due to [24].

Next, turning our attention to the asymptotic-p^+ growth case, from Theorem 1.4, Theorem 1.7 and Theorem 1.9, we can easily get the following results.

Corollary 1.11. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and the following conditions:*

(F4′) $\limsup\limits_{|x|\to\infty} F(t,x)/|x|^{p^+} \leqslant M < \infty$ *uniformly for some $M > 0$ and a.e. $t \in [0,T]$;*

(F5′) *there exist $M_1 > 0$, $\theta_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim\limits_{s\to\infty} s^{p^*}/\theta_1(s) = 0$ and $\theta_1(s)/s^{p^+}$ is non-increasing in s for all $s \in \mathbb{R}^+$ such that*

$$(\nabla F(t,x),x) - p^+ F(t,x) \geqslant \theta_1(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_1 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

R e m a r k 1.12. Corollary 1.11 seems like a new result. When $p(t) \equiv p$, in contrast to Theorem 1.2, Corollary 1.11 removes the assumption (H6) completely although it uses a few stronger condition (F5$'$) instead of (H7).

Corollary 1.13. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and* (F4$'$) *and the following condition:*

(F5$''$) *there exist $M_1 > 0$, $\theta_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim\limits_{s\to\infty} s^{p^*}/\theta_1(s) = 0$ and $\theta_1(s)/s^{p^+}$ is non-increasing in s for all $s \in \mathbb{R}^+$ such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \leqslant -\theta_1(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_1 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

Corollary 1.14. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and* (F4$'$) *and the following condition:*

(F6$'$) *there exist $M_2 > 0$, $\tau \geqslant 1$, $\theta_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim\limits_{s\to\infty} s^{p^*}/\theta_2^{1/\tau}(s) = 0$ and $\theta_2(s)/s^{p^+\tau}$ is non-increasing in s for all $s \in \mathbb{R}^+$ such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \geqslant \theta_2(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}}, |x| \geqslant M_2 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

Corollary 1.15. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and* (F4$'$) *and the following condition:*

(F6$''$) *there exist $M_2 > 0$, $\tau \geqslant 1$, $\theta_2 \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\lim\limits_{s\to\infty} s^{p^*}/\theta_2^{1/\tau}(s) = 0$ and $\theta_2(s)/s^{p^+\tau}$ is non-increasing in s for all $s \in \mathbb{R}^+$ such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \leqslant -\theta_2(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_2 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.*

Corollary 1.16. *Suppose that conditions* (H0), (P) *hold, F satisfies* (F1)–(F3) *and the following conditions:*

(F4$''$) $\limsup\limits_{|x|\to\infty} |\nabla F(t,x)|/|x|^{p^+-1} \leqslant M < \infty$ *uniformly for a.e. $t \in [0,T]$;*

(F7′) *there exist* $M_3 > 0$, $\sigma \geqslant p^-/(p^- - 1)$, $\theta_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\lim\limits_{s\to\infty} s^{p^*}/\theta_3^{1/\sigma}(s) = 0$ *and* $\theta_3(s)/s^{(p^+-1)\sigma}$ *is non-increasing in* s *for all* $s \in \mathbb{R}^+$ *such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \geqslant \theta_3(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}}, |x| \geqslant M_3 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Corollary 1.17. *Suppose that conditions* (H0), (P) *hold,* F *satisfies* (F1), (F2) *and* (F4″) *and the following conditions:*

(F3′) $\liminf\limits_{|x|\to\infty} (\nabla F(t,x), x)/|x|^{p^+} > 0$ *uniformly for a.e.* $t \in [0,T]$;

(F7″) *there exist* $M_3 > 0$, $\sigma \geqslant p^-/(p^- - 1)$, $\theta_3 \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\lim\limits_{s\to\infty} s^{p^*}/\theta_3^{1/\sigma}(s) = 0$ *and* $\theta_3(s)/s^{(p^+-1)\sigma}$ *is non-increasing in* s *for all* $s \in \mathbb{R}^+$ *such that*

$$(\nabla F(t,x), x) - p^+ F(t,x) \leqslant -\theta_3(|x|) \quad \forall\, x \in \mathbb{R}^{\mathbb{N}},\ |x| \geqslant M_3 \text{ and for a.e. } t \in [0,T].$$

Then problem (1.1) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Remark 1.18. Without loss of generality, we may assume that function b in assumption (H0) is T-periodic.

Finally, the rest of this paper is organized as follows. In Section 2, we set up the functional analytic framework needed to study problem (1.1) from the variational point of view. In Section 3, we find that all (C) sequences (see Definition 3.2 below) of the energy functional associated with systems (1.1) are bounded (see Lemmas 3.3–3.5 below), then we prove all the compact conditions (C) (see Definition 3.2 below) hold. In Section 4, we adopt the same ideas developed by Ma and Zhang in [11] to show our main results by generalized mountain pass theorem in [12]. At last, in Section 5, we give an example to illustrate our results are new even in the case $p(t) \equiv p = 2$.

2. Preliminaries

In this section, we first give some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We can refer the reader to [3] for more information.

Let k be a positive integer and $p(t)$ satisfy condition (P). Define

$$L^{p(t)}(0,kT;\mathbb{R}^{\mathbb{N}}) := \left\{ u \in L^1(0,kT;\mathbb{R}^{\mathbb{N}}) \colon \int_0^{kT} |u|^{p(t)}\,\mathrm{d}t < \infty \right\}$$

with the norm

$$|u|_{L^{p(t)}} = |u|_{p(t)} := \inf\left\{\lambda > 0\colon \int_0^{kT} \left|\frac{u}{\lambda}\right|^{p(t)} \mathrm{d}t \leqslant 1\right\}.$$

Define

$$C_{kT}^{\infty} = C_{kT}^{\infty}(\mathbb{R};\mathbb{R}^{\mathbb{N}}) := \{u \in C^{\infty}(\mathbb{R};\mathbb{R}^{\mathbb{N}})\colon\ u \text{ is } kT\text{-periodic}\}.$$

For $u \in L^1(0,kT;\mathbb{R}^{\mathbb{N}})$, if there exists $\nu \in L^1(0,kT;\mathbb{R}^{\mathbb{N}})$ satisfying

$$\int_0^{kT} \nu\varphi \,\mathrm{d}t = -\int_0^{kT} u\varphi' \,\mathrm{d}t \quad \forall\, \varphi \in C_{kT}^{\infty},$$

then ν is called the kT-weak derivative of u and is denoted by u'. Define

$$W_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}}) := \{u \in L^{p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})\colon\ u' \in L^{p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})\}$$

with the norm

$$\|u\|_{W_{kT}^{1,p(t)}} = \|u\| := |u|_{p(t)} + |u'|_{p(t)}.$$

For $u \in W_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})$, let

$$\overline{u} := \frac{1}{kT}\int_0^{kT} u(t)\,\mathrm{d}t, \quad \widetilde{u}(t) = u(t) - \overline{u}$$

and

$$\widetilde{W}_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}}) := \left\{u \in W_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})\colon\ \int_0^{kT} u(t)\,\mathrm{d}t = 0\right\},$$

then

$$W_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}}) = \widetilde{W}_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}}) \oplus \mathbb{R}^{\mathbb{N}}.$$

For the sake of convenience, in the following we use $L^{p(t)}$, $W_{kT}^{1,p(t)}$, $\widetilde{W}_{kT}^{1,p(t)}$ to denote $L^{p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})$, $W_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})$, $\widetilde{W}_{kT}^{1,p(t)}(0,kT;\mathbb{R}^{\mathbb{N}})$, respectively.

Proposition 2.1 ([4])**.** *For $u \in L^{p(t)}$, one has*

(1) $|u|_{p(t)} < 1\ (=1; >1) \Leftrightarrow \int_0^{kT} |u(t)|^{p(t)}\,\mathrm{d}t < 1\ (=1; >1)$;

(2) $|u|_{p(t)} > 1 \Rightarrow |u|_{p(t)}^{p^-} \leqslant \int_0^{kT} |u(t)|^{p(t)}\,\mathrm{d}t \leqslant |u|_{p(t)}^{p^+}$, $|u|_{p(t)} < 1 \Rightarrow |u|_{p(t)}^{p^+} \leqslant \int_0^{kT} |u(t)|^{p(t)}\,\mathrm{d}t \leqslant |u|_{p(t)}^{p^-}$;

(3) $|u|_{p(t)} \to 0 \Leftrightarrow \int_0^{kT} |u(t)|^{p(t)}\,\mathrm{d}t \to 0$, $|u|_{p(t)} \to \infty \Leftrightarrow \int_0^{kT} |u(t)|^{p(t)}\,\mathrm{d}t \to \infty$.

Proposition 2.2 ([4])**.** *The space $L^{p(t)}$ and $W_{kT}^{1,p(t)}$ are separable and reflexive Banach spaces when $p^- > 1$.*

Proposition 2.3 ([4])**.** *There is a continuous embedding $W_{kT}^{1,p(t)} \hookrightarrow C(0,kT;\mathbb{R}^{\mathbb{N}})$; when $p^- > 1$, it is a compact embedding.*

Proposition 2.4 ([4]). *For every $u \in \widetilde{W}_{kT}^{1,p(t)}$, there exists $c_k > 0$ such that $\|u\|_\infty \leqslant c_k |u'|_{p(t)}$, where $\|u\|_\infty := \max_{t\in[0,kT]} |u(t)|$.*

Proposition 2.5 ([4]). *Let $u = \overline{u} + \widetilde{u}(t) \in W_{kT}^{1,p(t)}$. Then the norm $|\widetilde{u}'|_{p(t)}$ is an equivalent norm on $\widetilde{W}_{kT}^{1,p(t)}$.*

Proposition 2.6 ([22]). *Let $J_k(u) := \int_0^{kT} |u'(t)|^{p(t)}/p(t)\,\mathrm{d}t$ for $u \in W_{kT}^{1,p(t)}$. Then $\langle J_k'(u), v\rangle = \int_0^{kT} (|u'(t)|^{p(t)-2}u'(t), v'(t))\,\mathrm{d}t$ for all $u, v \in W_{kT}^{1,p(t)}$, and J_k' is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle J_k'(u_n) - J'(u), u_n - u\rangle \leqslant 0$, then $\{u_n\}$ has a convergent subsequence in $W_{kT}^{1,p(t)}$.*

By assumption (H0), the functional

$$\varphi_k(u) := \int_0^{kT} \frac{1}{p(t)} |u'(t)|^{p(t)}\,\mathrm{d}t - \int_0^{kT} F(t, u(t))\,\mathrm{d}t \tag{2.1}$$

is continuously differentiable, and

$$\langle \varphi_k'(u), v\rangle = \int_0^{kT} |u'(t)|^{p(t)-2}(u'(t), v'(t))\,\mathrm{d}t - \int_0^{kT} (\nabla F(t, u(t)), v(t))\,\mathrm{d}t \tag{2.2}$$

for all $u, v \in W_{kT}^{1,p(t)}$ (see [22]).

We say that $u \in W_{kT}^{1,p(t)}$ is a weak solution of problem (1.1) if it satisfies

$$\langle \varphi_k'(u), v\rangle = 0 \quad \text{for any } v \in W_{kT}^{1,p(t)}.$$

Hence, the kT-solutions of problem (1.1) correspond to the critical points of the functional φ_k. In the whole paper, we denote various positive constants as C_i, $i = 1, 2, \ldots$

3. Compact conditions

Let us recall the following compact concepts, which can be found in [2], [12], [16].

Definition 3.1. Let E be a real Banach space. We say that $\{u_n\}$ in E is a Palais-Smale sequence ((PS) sequence) for φ_k if $\varphi_k(u_n)$ is bounded and $\varphi_k'(u_n) \to 0$ as $n \to \infty$. The functional $\varphi_k \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition ((PS) condition) if any Palais-Smale sequence contains a convergent subsequence.

Definition 3.2. Let E be a real Banach space. We say that $\{u_n\}$ in E is a Cerami sequence ((C) sequence) for φ_k if $\varphi_k(u_n)$ is bounded and $\varphi_k'(u_n)(1 + \|u_n\|) \to 0$ as $n \to \infty$. The functional $\varphi_k \in C^1(E, \mathbb{R})$ satisfies the Cerami condition ((C) condition) if any Cerami sequence contains a convergent subsequence.

In this section, we observe that although the energy functional of problem (1.1) may possess unbounded (PS) sequence, we can prove that all (C) sequences of this functional are bounded. Concretely speaking, we have

Lemma 3.3. *Assume that* (H0), (P), (F1), (F3)–(F5) *hold, then the functional* φ_k *satisfies condition* (C).

Proof. Let $E := W_{kT}^{1,p(t)}$. Suppose that $\{u_n\} \subset W_{kT}^{1,p(t)}$ is a (C) sequence of φ_k, then one has

$$|\varphi_k(u_n)| \leqslant C_1, \quad (1 + \|u_n\|)\|\varphi_k'(u_n)\|_{E^*} \leqslant C_1 \tag{3.1}$$

for all $n \in \mathbb{N}$, where E^* is the dual space of E.

In the first place, by (F4), there exists $M_4 > 0$ such that

$$F(t,x) \leqslant Mh(|x|)$$

for all $|x| \geqslant M_4$ and a.e. $t \in [0,T]$, which together with assumption (H0) yields

$$F(t,x) \leqslant Mh(|x|) + h_1(t) \tag{3.2}$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0,kT]$, where $h_1(t) := \max_{|x|\leqslant M_4} a(|x|)b(t) \geqslant 0$. It follows from (2.1), (3.1) and (3.2) that

$$\begin{aligned} C_1 \geqslant \varphi_k(u_n) &= \int_0^{kT} \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,\mathrm{d}t - \int_0^{kT} F(t,u_n(t))\,\mathrm{d}t \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - M\int_0^{kT} h(|u_n|)\,\mathrm{d}t - \int_0^{kT} h_1(t)\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

On the other hand, by (F5), one has

$$(\nabla F(t,x),x) - p^+F(t,x) \geqslant \theta_1(|x|) \tag{3.4}$$

for all $|x| \geqslant M_1$ and a.e. $t \in [0,T]$. Let $\Omega_{1n} := \{t \in [0,kT]\colon |u_n(t)| \geqslant M_1\}$, $\Omega_{1n}^c := \{t \in [0,kT]\colon |u_n(t)| < M_1\}$ and $h_2(t) := (p^+ + M_1)\max_{|x|\leqslant M_1} a(|x|)b(t) \geqslant 0$. It follows from (2.1), (2.2), (3.1) and (3.4) that

$$\begin{aligned} (p^+ + 1)C_1 \geqslant p^+\varphi_k(u_n) - \langle\varphi_k'(u_n),u_n\rangle &\geqslant \int_0^{kT} ((\nabla F(t,u_n),u_n) - p^+F(t,u_n))\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{1n}} \theta_1(|u_n|)\,\mathrm{d}t - \int_0^{kT} h_2(t)\,\mathrm{d}t \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$. Hence, we have

$$\int_0^{kT} \theta_1(|u_n|)\,\mathrm{d}t = \int_{\Omega_{1n}} \theta_1(|u_n|)\,\mathrm{d}t + \int_{\Omega_{1n}^c} \theta_1(|u_n|)\,\mathrm{d}t \leqslant C_2 \quad \forall\, n \in \mathbb{N}. \tag{3.6}$$

Next, from assumption (P) and Proposition 2.3, there exists $d > 0$ such that

$$\|u\|_\infty \leqslant d\|u\| \tag{3.7}$$

for all $u \in W_{kT}^{1,p(t)}$. By virtue of (3.3), (3.6), (3.7), and noticing that $h(s)/\theta_1(s)$ is non-decreasing in s for all $s \in \mathbb{R}^+$, we obtain that

(3.8)

$$\begin{aligned} C_1 \geqslant \varphi_k(u_n) &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - M\int_0^{kT} \frac{h(|u_n|)}{\theta_1(|u_n|)}\theta_1(|u_n|)\,\mathrm{d}t - \int_0^{kT} h_1(t)\,\mathrm{d}t \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - MC_2\frac{h(\|u_n\|_\infty)}{\theta_1(\|u_n\|_\infty)} - \int_0^{kT} h_1(t)\,\mathrm{d}t \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - MC_2\frac{h(d\|u_n\|)}{\theta_1(d\|u_n\|)} - \int_0^{kT} h_1(t)\,\mathrm{d}t. \end{aligned}$$

Finally, we claim $\{u_n\}$ is bounded, otherwise, going if necessary to a subsequence, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Set $v_n := u_n/\|u_n\| = \overline{u}_n/\|u_n\| + \widetilde{u}_n(t)/\|u_n\| = \overline{v}_n + \widetilde{v}_n(t)$, then $\{v_n\}$ is bounded in $W_{kT}^{1,p(t)}$. Hence, there exists a subsequence, again denoted by $\{v_n\}$, such that

$$v_n \rightharpoonup v_0 \quad \text{weakly in } W_{kT}^{1,p(t)}, \tag{3.9}$$

$$v_n \to v_0 \quad \text{strongly in } C(0, kT; \mathbb{R}^{\mathbb{N}}). \tag{3.10}$$

Then, by (3.10), one has

$$\overline{v}_n = \frac{1}{kT}\int_0^{kT} v_n(t)\,\mathrm{d}t \to \frac{1}{kT}\int_0^{kT} v_0(t)\,\mathrm{d}t = \overline{v}_0 \quad \text{as } n \to \infty. \tag{3.11}$$

Dividing both sides of (3.8) by $\|u_n\|^{p^-}$, in light of the properties of $\theta_1(s)$ and condition (P), we can find that

$$\begin{aligned} \frac{C_1}{\|u_n\|^{p^-}} &\geqslant \frac{1}{p^+}\int_0^{kT} \frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p^-}}\,\mathrm{d}t - MC_2\frac{h(d\|u_n\|)}{\theta_1(d\|u_n\|)\|u_n\|^{p^-}} - \frac{\int_0^{kT} h_1(t)\,\mathrm{d}t}{\|u_n\|^{p^-}} \\ &\geqslant \frac{1}{p^+}\int_0^{kT} \frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p(t)}}\,\mathrm{d}t - MC_2 d^{p^-}\frac{h(d\|u_n\|)}{\theta_1(d\|u_n\|)d^{p^-}\|u_n\|^{p^-}} - \frac{\int_0^{kT} h_1(t)\,\mathrm{d}t}{\|u_n\|^{p^-}} \\ &= \frac{1}{p^+}\int_0^{kT} |v_n'(t)|^{p(t)}\,\mathrm{d}t - MC_2 d^{p^-}\frac{h(d\|u_n\|)}{\theta_1(d\|u_n\|)d^{p^-}\|u_n\|^{p^-}} - \frac{\int_0^{kT} h_1(t)\,\mathrm{d}t}{\|u_n\|^{p^-}}, \end{aligned}$$

which implies that

$$\int_0^{kT} |v_n'(t)|^{p(t)}\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Then, by Proposition 2.1 (3), one has $|v_n'(t)|_{p(t)} \to 0$ as $n \to \infty$, which together with Proposition 2.4 and (3.10) yield

$$v_n \to \overline{v}_0 = v_0 \quad \text{as } n \to \infty. \tag{3.13}$$

Consequently, we get $v_0 \in \mathbb{R}^{\mathbb{N}}$ and $v_0 \neq 0$, which implies that

$$|u_n(t)| \to \infty \quad \text{as } n \to \infty \text{ uniformly for a.e. } t \in [0, kT]. \tag{3.14}$$

So, from (F1), (F3), (3.14) and Fatou's lemma, we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{\int_0^{kT} F(t, u_n(t))\,\mathrm{d}t}{\|u_n\|^{p^+}} &\geqslant \int_0^{kT} \liminf_{n\to\infty} \frac{F(t, u_n(t))}{|u_n(t)|^{p^+}} |v_n(t)|^{p^+}\,\mathrm{d}t \\ &= \int_0^{kT} \liminf_{|u_n(t)|\to\infty} \frac{F(t, u_n(t))}{|u_n(t)|^{p^+}} |v_0(t)|^{p^+}\,\mathrm{d}t > 0.\end{aligned} \tag{3.15}$$

However, by (2.1), (3.1) and (3.12), we have

$$\begin{aligned}\frac{\int_0^{kT} F(t, u_n(t))\,\mathrm{d}t}{\|u_n\|^{p^+}} &= \int_0^{kT} \frac{1}{p(t)} \frac{|u_n'(t)|^{p(t)}}{\|u_n\|^{p^+}}\,\mathrm{d}t - \frac{\varphi_k(u_n)}{\|u_n\|^{p^+}} \\ &\leqslant \frac{1}{p^-} \int_0^{kT} \Big|\frac{u_n'(t)}{\|u_n\|}\Big|^{p(t)}\,\mathrm{d}t - \frac{\varphi_k(u_n)}{\|u_n\|^{p^+}} \\ &= \frac{1}{p^-} \int_0^{kT} |v_n'(t)|^{p(t)}\,\mathrm{d}t - \frac{\varphi_k(u_n)}{\|u_n\|^{p^+}},\end{aligned}$$

which means that

$$\liminf_{n\to\infty} \frac{\int_0^{kT} F(t, u_n)\,\mathrm{d}t}{\|u_n\|^{p^+}} \leqslant 0,$$

which contradicts (3.15). Thus, $\{u_n\}$ is bounded in $W_{kT}^{1,p(t)}$.

By Proposition 2.2 and Proposition 2.3, $\{u_n\}$ has a subsequence, again denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_{kT}^{1,p(t)}, \tag{3.16}$$

$$u_n \to u \quad \text{strongly in } C(0, kT; \mathbb{R}^{\mathbb{N}}). \tag{3.17}$$

Now, we show that $\{u_n\}$ has a subsequence convergent strongly to u in $W_{kT}^{1,p(t)}$. From Proposition 2.6, it suffices to prove that $\limsup_{n\to\infty} \langle J_k'(u_n) - J_k'(u), u_n - u\rangle \leqslant 0$. It follows from (3.7) that

$$|u_n(t)| \leqslant C_3 \quad \forall\, t \in [0, kT]. \tag{3.18}$$

From (3.17), (3.18), (H0) and Remark 1.18, we get

$$\begin{aligned}(3.19)\quad \left|\int_0^{kT} (\nabla F(t,u_n(t)), u_n(t)-u(t))\,\mathrm{d}t\right| &\leqslant \int_0^{kT} |\nabla F(t,u_n(t))||u_n(t)-u(t)|\,\mathrm{d}t \\ &\leqslant \|u_n-u\|_\infty \int_0^{kT} a(|u_n(t)|)b(t)\,\mathrm{d}t \\ &\leqslant C_4\|u_n-u\|_\infty \int_0^{kT} b(t)\,\mathrm{d}t.\end{aligned}$$

Thus, from (3.17), we obtain

$$(3.20)\qquad \left|\int_0^{T} (\nabla F(t,u_n(t)), u_n(t)-u(t))\,\mathrm{d}t\right| \to 0 \quad \text{as } n\to\infty.$$

By (3.1) and (3.18), we also have

$$(3.21)\qquad \langle \varphi_k'(u_n), u_n-u\rangle \to 0 \quad \text{as } n\to\infty.$$

Then it follows from (3.20) and (3.21) that

(3.22)

$$\begin{aligned}\langle J_k'(u_n), u_n-u\rangle &= \int_0^{kT} (|u_n'(t)|^{p(t)-2}u_n'(t), u_n'(t)-u'(t))\,\mathrm{d}t \\ &= \langle \varphi_k'(u_n), u_n-u\rangle + \int_0^{kT} (\nabla F(t,u_n(t)), u_n(t)-u(t))\,\mathrm{d}t \to 0\end{aligned}$$

as $n\to\infty$. Moreover, since $J_k'(u)\in (W_{kT}^{1,p(t)})^*$, by (3.16), one has

$$(3.23)\qquad \langle J_k'(u), u_n-u\rangle \to 0 \quad \text{as } n\to\infty,$$

which combined with (3.22) implies that

$$\lim_{n\to\infty} \langle J_k'(u_n)-J_k'(u), u_n-u\rangle = 0.$$

Hence, from Proposition 2.6, $\{u_n\}$ has a subsequence convergent strongly to u in $W_{kT}^{1,p(t)}$. This concludes the proof of Lemma 3.3. □

Lemma 3.4. *Assume that* (H0), (P), (F1), (F3) *and* (F6) *hold. Then the functional* φ_k *satisfies condition* (C).

P r o o f. From the arguments of Lemma 3.3, we only need to prove that $\{u_n\}$ is bounded in $W^{1,p(t)}_{kT}$. It follows from assumptions (F6) and (H0) that

$$(\nabla F(t,x),x) - p^+F(t,x) \geqslant \theta_2(|x|)\Big(\frac{F(t,x)}{|x|^{p^+}}\Big)^{\tau} \tag{3.24}$$

for all $|x| \geqslant M_2$ and a.e. $t \in [0,kT]$. Let $\Omega_{2n} := \{t \in [0,kT]\colon |u_n(t)| \geqslant M_2\}$. By (2.1), (2.2), (3.1), (3.7), (3.24), (F1) and the properties of $\theta_2(s)$, we obtain

$$\begin{aligned}(p^+ + 1)C_1 &\geqslant p^+\varphi_k(u_n) - \langle \varphi_k'(u_n), u_n\rangle \\ &\geqslant \int_0^{kT} ((\nabla F(t,u_n), u_n(t)) - p^+F(t,u_n))\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{2n}} \theta_2(|u_n|)\Big(\frac{F(t,u_n)}{|u_n|^{p^+}}\Big)^{\tau}\,\mathrm{d}t - \int_0^{kT} h_3(t)\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{2n}} \theta_2(\|u_n\|_\infty)\frac{F^\tau(t,u_n)}{\|u_n\|_\infty^{p^+\tau}}\,\mathrm{d}t - \int_0^{kT} h_3(t)\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{2n}} \theta_2(d\|u_n\|)\frac{F^\tau(t,u_n)}{d^{p^+\tau}\|u_n\|^{p^+\tau}}\,\mathrm{d}t - \int_0^{kT} h_3(t)\,\mathrm{d}t,\end{aligned} \tag{3.25}$$

where $h_3(t) := (p^+ + M_2)\max\limits_{|x|\leqslant M_2} a(|x|)b(t) \geqslant 0$. Hence, we see that

$$\int_0^{kT} F^\tau(t,u_n(t))\,\mathrm{d}t \leqslant \frac{C_5}{\theta_2(d\|u_n\|)}\|u_n\|^{p^+\tau} + C_6 \tag{3.26}$$

for all $n \in \mathbb{N}$. Furthermore, by (F1), (3.1), (3.26) and Hölder's inequality, one derives

$$\begin{aligned}C_1 \geqslant \varphi_k(u_n) &= \int_0^{kT} \frac{1}{p(t)}|u_n'(t)|^{p(t)}\,\mathrm{d}t - \int_0^{kT} F(t,u_n(t))\,\mathrm{d}t \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - C_7\Big(\int_0^{kT} F^\tau(t,u_n(t))\,\mathrm{d}t\Big)^{1/\tau} \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - C_7\Big(\frac{C_5}{\theta_2(d\|u_n\|)}\|u_n\|^{p^+\tau} + C_6\Big)^{1/\tau} \\ &\geqslant \frac{1}{p^+}\int_0^{kT} |u_n'(t)|^{p(t)}\,\mathrm{d}t - \frac{C_8}{\theta_2^{1/\tau}(d\|u_n\|)}\|u_n\|^{p^+} - C_9 \quad \forall\, n \in \mathbb{N}.\end{aligned} \tag{3.27}$$

Now, we claim $\{u_n\}$ is bounded, otherwise, going if necessary to a subsequence, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. With the same manner of Lemma 3.3, dividing both sides of (3.27) by $\|u_n\|^{p^-}$, using the properties of $\theta_2(s)$, we conclude that $|u_n(t)| \to \infty$ as $n \to \infty$ uniformly for a.e. $t \in [0, kT]$. From (F1), (F3) and Fatou's lemma, we get

$$\liminf_{n\to\infty} \frac{\int_0^{kT} F(t,u_n)\,\mathrm{d}t}{\|u_n\|^{p^+}} \geqslant \int_0^{kT} \liminf_{|u_n|\to\infty} \frac{F(t,u_n)}{|u_n|^{p^+}} |v_0|^{p^+}\,\mathrm{d}t > 0. \tag{3.28}$$

On the other hand, noting that (3.1) and (3.12), we know that

$$\liminf_{n\to\infty} \frac{\int_0^{kT} F(t,u_n)\,\mathrm{d}t}{\|u_n\|^{p^+}} \leqslant 0,$$

which contradicts (3.28). Therefore, $\{u_n\}$ is bounded in $W_{kT}^{1,p(t)}$, and then φ_k satisfies condition (C). □

Lemma 3.5. *Assume that* (H0), (P), (F1), (F3) *and* (F7) *hold. Then the functional* φ_k *satisfies condition* (C).

Proof. Let $\Omega_{3n} := \{t \in [0, kT] \colon |u_n(t)| \geqslant M_3\}$. It follows from (3.1), (3.7), (H0), (F7) and the properties of $\theta_3(s)$ that

$$\begin{aligned} (p^+ + 1)C_1 &\geqslant p^+\varphi_k(u_n) - \langle \varphi_k'(u_n), u_n\rangle \\ &\geqslant \int_0^{kT} ((\nabla F(t,u_n), u_n) - p^+ F(t,u_n))\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{3n}} \theta_3(|u_n|) \frac{|\nabla F(t,u_n)|^\sigma}{|u_n|^{(p^+-1)\sigma}}\,\mathrm{d}t - \int_0^{kT} h_4(t)\,\mathrm{d}t \\ &\geqslant \int_{\Omega_{3n}} \theta_3(d\|u_n\|) \frac{|\nabla F(t,u_n)|^\sigma}{d^{(p^+-1)\sigma}\|u_n\|^{(p^+-1)\sigma}}\,\mathrm{d}t - \int_0^{kT} h_4(t)\,\mathrm{d}t \end{aligned} \tag{3.29}$$

for all $n \in \mathbb{N}$, where $h_4(t) := (p^+ + M_3) \max\limits_{|x|\leqslant M_3} a(|x|)b(t) \geqslant 0$. As a consequence, we have

$$\int_0^{kT} |\nabla F(t,u_n)|^\sigma\,\mathrm{d}t \leqslant \frac{C_{10}}{\theta_3(d\|u_n\|)} \|u_n\|^{(p^+-1)\sigma} + C_{11} \tag{3.30}$$

for all $n \in \mathbb{N}$. Let $1/\sigma + 1/\sigma' = 1$, since $\sigma \geqslant p^-/(p^- - 1)$, by simple computation, we get $p^- \geqslant \sigma'$. So, there exists continuous embedding $L^{p^-}(0, kT; \mathbb{R}) \hookrightarrow L^{\sigma'}(0, kT; \mathbb{R})$, where $L^\gamma(0, kT; \mathbb{R})$ is the usual L^γ space, $\gamma = p^-, \sigma'$. What is more, it is not difficult to see that there exists continuous embedding $L^{p(t)} \hookrightarrow L^{p^-}(0, kT; \mathbb{R})$, which together with (3.1), (3.30) and Hölder's inequality yields that

(3.31)
$$\begin{aligned}
C_1 \geqslant \langle \varphi_k'(u_n), u_n\rangle &= \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \int_0^{kT} (\nabla F(t,u_n), u_n)\,\mathrm{d}t \\
&\geqslant \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \Big(\int_0^{kT} |\nabla F(t,u_n)|^{\sigma}\,\mathrm{d}t\Big)^{1/\sigma} \Big(\int_0^{kT} |u_n|^{\sigma'}\,\mathrm{d}t\Big)^{1/\sigma'} \\
&\geqslant \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \Big(\frac{C_{10}}{\theta_3(d\|u_n\|)}\|u_n\|^{(p^+-1)\sigma} + C_{11}\Big)^{1/\sigma} C_{12}\Big(\int_0^{kT} |u_n|^{p^-}\,\mathrm{d}t\Big)^{1/p^-} \\
&\geqslant \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \Big(\frac{C_{10}}{\theta_3(d\|u_n\|)}\|u_n\|^{(p^+-1)\sigma} + C_{11}\Big)^{1/\sigma} C_{12}|u_n|_{p(t)} \\
&\geqslant \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \Big(\frac{C_{10}}{\theta_3(d\|u_n\|)}\|u_n\|^{(p^+-1)\sigma} + C_{11}\Big)^{1/\sigma} C_{12}\|u_n\| \\
&\geqslant \int_0^{kT} |u_n'|^{p(t)}\,\mathrm{d}t - \frac{C_{13}}{\theta_3^{1/\sigma}(d\|u_n\|)}\|u_n\|^{p^+} - C_{14}\|u_n\| \quad \forall\, n \in \mathbb{N}.
\end{aligned}$$

Finally, we claim $\{u_n\}$ is bounded, otherwise, going if necessary to a subsequence, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. In the same way as in the proof of Lemma 3.3, multiplying both sides of (3.31) by $\|u_n\|^{-p^-}$, we can obtain that $|u_n(t)| \to \infty$ as $n \to \infty$ uniformly for a.e. $t \in [0, kT]$, and then from (F1), (F3) and Fatou's lemma, we obtain that

$$\liminf_{n\to\infty} \frac{\int_0^T F(t,u_n)\,\mathrm{d}t}{\|u_n\|^{p^+}} > 0. \tag{3.32}$$

On the other hand, note that (3.1) and (3.12) imply that

$$\liminf_{n\to\infty} \frac{\int_0^T F(t,u_n)\,\mathrm{d}t}{\|u_n\|^{p^+}} \leqslant 0,$$

which contradicts (3.32). Thus, $\{u_n\}$ is bounded in $W_{kT}^{1,p(t)}$. Using the same arguments as in Lemma 3.3, we know that φ_k satisfies condition (C). □

4. Proofs of main results

In this section, we shall use the following generalized mountain pass theorem to prove our results.

Theorem 4.1 ([16]). *Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $\varphi_k \in C^1(E,\mathbb{R})$ satisfies the* (PS) *condition, and*

(i) *there exist* $\varrho_k, \alpha > 0$ *such that* $\varphi_k|_{\partial B_{\varrho_k} \cap X} \geqslant \alpha$, *where* $B_{\varrho_k} := \{u \in E\colon \|u\| < \varrho_k\}$, ∂B_{ϱ_k} *denotes the boundary of* B_{ϱ_k};

(ii) *there exist* $e_k \in \partial B_1 \cap X$ *and* $s_0 > \varrho_k$ *such that if* $Q_k :\equiv (\overline{B}_{s_0} \cap V) \oplus \{se_k\colon 0 \leqslant s \leqslant s_0\}$, *then* $\varphi|_{\partial Q_k} \leqslant 0$.

Then φ *possesses a critical value* $c \geqslant \alpha$ *which can be characterized as*

$$c := \inf_{h \in \Gamma} \max_{u \in Q} \varphi_k(h(u)),$$

where $\Gamma := \{h \in C(\overline{Q}_k, E)\colon h = \mathrm{id}$ *on* $\partial Q_k\}$; *here,* id *denotes the identity operator.*

R e m a r k 4.2. It is well known that Theorem 4.1 holds based on the deformation lemma (see [12] or [16]). As shown in [1], the deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition, and it turns out that the generalized mountain pass theorem holds true under condition (C).

Now we prove our main results. We only give the proofs of Theorem 1.4, Theorem 1.7, Theorem 1.9, Corollary 1.11, Corollary 1.14, Corollary 1.16 and Corollary 1.17. The other results can be proved similarly.

P r o o f of Theorem 1.4. Let $X := \widetilde{W}^{1,p(t)}_{kT}$, $V := \mathbb{R}^{\mathbb{N}}$, recalling $E = W^{1,p(t)}_{kT}$, then $E = V \oplus X$ and $\dim V < \infty$. From Lemma 3.3, we see that φ_k satisfies condition (C). By virtue of Theorem 4.1 and Remark 4.2, we only need to verify the assertions:

(a) $\inf\limits_{u \in S} \varphi_k(u) > 0$;

(b) $\sup\limits_{u \in Q_k} \varphi_k(u) < \infty$, $\sup\limits_{u \in \partial Q_k} \varphi(u) \leqslant 0$,

where $S := \widetilde{W}^{1,p(t)}_{kT} \cap \partial B_{\varrho_k}$, $Q_k := \{se_k\colon 0 \leqslant s \leqslant R_1, e_k(t) \in \widetilde{W}^{1,p(t)}_{kT}\} \oplus \{x \in \mathbb{R}^{\mathbb{N}}\colon |x| \leqslant R_2\}$ and $\varrho_k < R_1$. Firstly, by (F2) and (H0), we know that for any positive constant $\varepsilon_1 < \min\{c_k, 1/p^+ kT c_k^{p^+}\}$, there exists $\delta \in (0, \varepsilon_1)$ such that

$$F(t,x) \leqslant \varepsilon_1 |x|^{p^+} \quad \forall\, |x| \leqslant \delta \text{ and a.e. } t \in [0,kT]. \tag{4.1}$$

Let $\varrho_k \in (0, \delta/c_k)$ and by Proposition 2.5 set $S = \{u \in \widetilde{W}^{1,p(t)}_{kT}\colon |u'|_{p(t)} = \varrho_k\}$ for all $u \in S$, by Proposition 2.4, we get $|u(t)| \leqslant c_k |u'|_{p(t)} = c_k \varrho_k \leqslant \delta$. Since $0 < \varrho_k < 1$, then it follows from Proposition 2.1 (2), (4.1) and Proposition 2.4 that

$$\begin{aligned}
\varphi_k(u) &= \int_0^{kT} \frac{1}{p(t)} |u'(t)|^{p(t)} \,\mathrm{d}t - \int_0^{kT} F(t, u(t)) \,\mathrm{d}t \\
&\geqslant \frac{1}{p^+} \int_0^{kT} |u'(t)|^{p(t)} \,\mathrm{d}t - \varepsilon_1 \int_0^{kT} |u(t)|^{p^+} \,\mathrm{d}t \\
&\geqslant \frac{1}{p^+} |u'|^{p^+}_{p(t)} - \varepsilon_1 kT c_k^{p^+} |u'|^{p^+}_{p(t)} = \Big(\frac{1}{p^+} - \varepsilon_1 kT c_k^{p^+}\Big) \varrho_k^{p^+} \\
&= \alpha > 0 \quad \forall\, u \in S.
\end{aligned}$$

This establishes (a). Next, we check (b).

Let $f(t) := \lim\limits_{|x|\to\infty} F(t,x)/|x|^{p^+}$, $E_0 := \{t \in [0,T]\colon f(t) = 0\}$, then by (F3), $\operatorname{meas}(E_0) = 0$ and $f(t) > 0$ on $[0,T] \setminus E_0$. Let $E_m := \{t \in [0,T]\colon f(t) \geqslant 1/m\}$, $m \in \mathbb{N}$, then $[0,T] \setminus E_0 = \bigcup\limits_{m=1}^{\infty} E_m$, $\operatorname{meas}\Big(\bigcup\limits_{m=1}^{\infty} E_m\Big) = T$ and $\lim\limits_{m\to\infty} \operatorname{meas}(E_m) = T$. Therefore, for any $\varepsilon > 0$, there exist a small number $\varepsilon_2 = \varepsilon_2(\varepsilon) > 0$ (which is independent of k) and a subset $E_\varepsilon \subset \bigcup\limits_{m=1}^{\infty} E_m \subset [0,T]$ such that

$$T - \varepsilon \leqslant \operatorname{meas}(E_\varepsilon) \leqslant T \quad \text{and} \quad f(t) \geqslant 2\varepsilon_2 \text{ on } E_\varepsilon.$$

Let $E_\varepsilon^k := \bigcup\limits_{j=0}^{k-1} (E_\varepsilon + jT)$, where $E_\varepsilon + jT := \{t + jT\colon t \in E_\varepsilon\}$, $j = 0, 1, \ldots, k-1$. Then

$$kT - k\varepsilon \leqslant \operatorname{meas}(E_\varepsilon^k) \leqslant kT,$$

hence,

$$\operatorname{meas}([0,kT] \setminus E_\varepsilon^k) \leqslant k\varepsilon. \tag{4.2}$$

Furthermore, by the periodicity of $F(t,x)$ in t, we get

$$f(t) \geqslant 2\varepsilon_2 \quad \text{on } E_\varepsilon^k.$$

Again, by (F3) and (H0), there exists $M_5 = M_5(\varepsilon) > 0$ such that

$$F(t,x) \geqslant \varepsilon_2 |x|^{p^+} \quad \forall\, |x| \geqslant M_5 \text{ and } t \in E_\varepsilon^k.$$

Therefore, for all $x \in \mathbb{R}^{\mathbb{N}}$ and $t \in E_\varepsilon^k$ we have

$$F(t,x) \geqslant \varepsilon_2 |x|^{p^+} - \varepsilon_2 M_5^{p^+},$$

which combining with (F1) implies that for all $u \in W_{kT}^{1,p(t)}$,

$$\begin{aligned} \int_0^{kT} F(t,u)\,\mathrm{d}t &\geqslant \int_{E_\varepsilon^k} F(t,u)\,\mathrm{d}t \geqslant \varepsilon_2 \int_{E_\varepsilon^k} |u|^{p^+}\,\mathrm{d}t - \varepsilon_2 M_5^{p^+} kT \\ &= \varepsilon_2 \int_0^{kT} |u|^{p^+}\,\mathrm{d}t - \varepsilon_2 M_5^{p^+} kT - \varepsilon_2 \int_{[0,kT]\setminus E_\varepsilon^k} |u|^{p^+}\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

Choose $e_k := (\sin(k^{-1}\omega t), 0, \ldots, 0) \in \widetilde{W}_{kT}^{1,p(t)}$, where $\omega := 2\pi/T$. Let $\overline{W}_{kT}^{1,p(t)} := \mathbb{R}^{\mathbb{N}} \oplus \operatorname{span}\{e_k\}$. Since $\dim(\overline{W}_T^{1,p(t)}) = n+1$, one has

$$\Big(\int_0^T |u|^{p^+}\,\mathrm{d}t\Big)^{1/p^+} \geqslant C_{15} \Big(\int_0^T |u|^2\,\mathrm{d}t\Big)^{1/2} \quad \forall\, u \in \overline{W}_T^{1,p(t)}. \tag{4.4}$$

Let $|s|^{\widetilde{p}} := \max\{|s|^{p^-}, |s|^{p^+}\}$, $\omega^{\widetilde{p}^*} := \max\{\omega^{p^-}, \omega^{p^+}\}$, $A := (1/p^-)\omega^{\widetilde{p}^*}\int_0^T 1 \times |\cos \omega t|^{p(t)}\,\mathrm{d}t$. Bearing in mind that $Q_k = \{se_k\colon 0 \leqslant s \leqslant R_1\} \oplus \{x \in \mathbb{R}^{\mathbb{N}}\colon |x| \leqslant R_2\}$ with $R_1 \geqslant 1 > \varrho_k$ and $R_2 > 0$ being specified below, observe that by (4.3), (4.4), (P) and Hölder inequality, for $x + se_k \in \widetilde{W}^{1,p(t)}_{kT}$, one has

$$
\begin{aligned}
(4.5)\quad \varphi_k(x+se_k) &= \int_0^{kT} \frac{1}{p(t)}|se_k'|^{p(t)}\,\mathrm{d}t - \int_0^{kT} F(t, x+se_k)\,\mathrm{d}t \\
&\leqslant \frac{1}{p^-}k^{-p^-}\omega^{\widetilde{p}^*}|s|^{\widetilde{p}}\int_0^{kT} |\cos k^{-1}\omega t|^{p(t)}\,\mathrm{d}t - \varepsilon_2 \int_0^{kT} |x+se_k|^{p^+}\,\mathrm{d}t \\
&\quad + \varepsilon_2 M_5^{p^+} kT + \varepsilon_2 \int_{[0,kT]\setminus E_\varepsilon^k} |x+se_k|^{p^+}\,\mathrm{d}t \\
&\leqslant \frac{1}{p^-}k^{-p^-+1}\omega^{\widetilde{p}^*}|s|^{\widetilde{p}}\int_0^{T} |\cos \omega t|^{p(t)}\,\mathrm{d}t + \varepsilon_2 M_5^{p^+} kT \\
&\quad - \varepsilon_2 k \int_0^{T} |x+se_1|^{p^+}\,\mathrm{d}t + 2^{p^+}\varepsilon_2 \int_{[0,kT]\setminus E_\varepsilon^k} (|x|^{p^+} + |s|^{p^+})\,\mathrm{d}t \\
&\leqslant Ak^{-p^-+1}|s|^{\widetilde{p}} - \varepsilon_2 k C_{15}^{p^+}\Big(\int_0^T |x+se_1|^2\,\mathrm{d}t\Big)^{p^+/2} \\
&\quad + \varepsilon_2 M_5^{p^+} kT + 2^{p^+}\varepsilon_2 k\varepsilon(|x|^{p^+} + |s|^{p^+}) \\
&\leqslant Ak^{-p^-+1}|s|^{\widetilde{p}} - \varepsilon_2 k C_{15}^{p^+}\Big(\int_0^T (|x|^2 + |s|^2|e_1|^2)\,\mathrm{d}t\Big)^{p^+/2} \\
&\quad + \varepsilon_2 M_5^{p^+} kT + 2^{p^+}\varepsilon_2 k\varepsilon(R_1^{p^+} + R_2^{p^+}) \\
&\leqslant Ak^{-p^-+1}|s|^{\widetilde{p}} - \varepsilon_2 k C_{16}|x|^{p^+} - \varepsilon_2 k C_{16}|s|^{p^+} + \varepsilon_2 M_5^{p^+} kT \\
&\quad + 2^{p^+}\varepsilon_2 k\varepsilon(R_1^{p^+} + R_2^{p^+}).
\end{aligned}
$$

If $k \geqslant (2A/\varepsilon_2 C_{16})^{1/p^-}$, then we have

$$
\begin{aligned}
(4.6)\quad k^{-1}\varphi_k(x+se_k) &\leqslant Ak^{-p^-}|s|^{\widetilde{p}} - \varepsilon_2 C_{16}|s|^{p^+} + \varepsilon_2 M_5^{p^+} T + 2^{p^+}\varepsilon_2\varepsilon(R_1^{p^+} + R_2^{p^+}) \\
&\leqslant \frac{1}{2}\varepsilon_2 C_{16}|s|^{\widetilde{p}} - \varepsilon_2 C_{16}|s|^{p^+} + \varepsilon_2 M_5^{p^+} T + 2^{p^+}\varepsilon_2\varepsilon(R_1^{p^+} + R_2^{p^+})
\end{aligned}
$$

and

$$
\begin{aligned}
(4.7)\quad k^{-1}\varphi_k(x+se_k) &\leqslant Ak^{-p^-}|s|^{\widetilde{p}} - \varepsilon_2 C_{16}|x|^{p^+} + \varepsilon_2 M_5^{p^+} T + 2^{p^+}\varepsilon_2\varepsilon(R_1^{p^+} + R_2^{p^+}) \\
&\leqslant \frac{1}{2}\varepsilon_2 C_{16}|s|^{\widetilde{p}} - \varepsilon_2 C_{16}|x|^{p^+} + \varepsilon_2 M_5^{p^+} T + 2^{p^+}\varepsilon_2\varepsilon(R_1^{p^+} + R_2^{p^+}).
\end{aligned}
$$

Without loss of generality, we assume $\varepsilon < C_{16}/2^{p^+ + 3}$ and $M_5 = M_5(\varepsilon) \geqslant (C_{16}/4T)^{1/p^+}$. Then

$$R := M_5\Big(\frac{4T}{C_{16}}\Big)^{1/p^+} \geqslant 1.$$

Put $R_1 = R_2 = R$. Then for all $x + R_1 e_k \in \partial Q_k$, note $|s|^{\widetilde{p}} = |s|^{p^+}$ for $|s| \geqslant 1$. It follows from (4.6) that

$$\frac{1}{\varepsilon_2 k}\varphi_k(x+R_1e_k) \leqslant -\frac{1}{2}C_{16}R^{p^+} + M_5^{p^+}T + 2^{p^++1}R^{p^+}\varepsilon \leqslant -\frac{1}{4}C_{16}R^{p^+} + M_5^{p^+}T = 0,$$

and for all $x + se_k \in \partial Q_k$ with $|x| = R_2$ it follows from (4.7) that

$$\begin{aligned}\frac{1}{\varepsilon_2 k}\varphi_k(x+R_1e_k) &\leqslant \frac{1}{2}C_{16}|s|^{\widetilde{p}} - C_{16}|x|^{p^+} + M_5^{p^+}T + 2^{p^+}\varepsilon(R_1^{p^+} + R_2^{p^+})\\ &\leqslant -\frac{1}{2}C_{16}R^{p^+} + M_5^{p^+}T + 2^{p^++1}\varepsilon R^{p^+} \leqslant -\frac{1}{4}C_{16}R^{p^+} + M_5^{p^+}T = 0.\end{aligned}$$

It follows from (F1) that $\varphi_k(x) = -\int_0^{kT} F(t,x)\,\mathrm{d}t \leqslant 0$ for all $x \in \mathbb{R}^{\mathbb{N}}$. Hence, we get

$$\varphi_k(x+se_k) \leqslant 0 \quad \forall\, x+se_k \in \partial Q_k,$$

which implies that (b) holds.

Furthermore, for all $x + se_k \in Q_k$, by (F1) and (P), we have

$$\begin{aligned}\varphi_k(x+se_k) &= \int_0^{kT}\frac{1}{p(t)}|se_k'|^{p(t)}\,\mathrm{d}t - \int_0^{kT}F(t,x+se_k)\,\mathrm{d}t\\ &\leqslant \frac{1}{p^-}k^{-p^-}\omega^{\widetilde{p}^*}|s|^{\widetilde{p}}\int_0^{kT}|\cos k^{-1}\omega t|^{p(t)}\,\mathrm{d}t\\ &\leqslant Ak^{-p^-+1}|s|^{\widetilde{p}} \leqslant A|s|^{\widetilde{p}} \leqslant A\max\Big\{M_5\Big(\frac{4T}{C_{16}}\Big)^{1/p^+},1\Big\}^{\widetilde{p}}.\end{aligned}$$

Thus, for any positive integer $k \geqslant (2A/\varepsilon_2 C_{16})^{1/p^-}$, φ_k has at least one critical point u_k in $W_{kT}^{1,p(t)}$ and

$$\varphi_k(u_k) \leqslant A\max\Big\{M_5\Big(\frac{4T}{C_{16}}\Big)^{1/p^+},1\Big\}^{\widetilde{p}}. \tag{4.8}$$

For $k_1 \geqslant (2A/\varepsilon_2C_{16})^{1/p^-}$ we obtain a k_1T-periodic solution u_{k_1}. We claim that there exists a positive integer $k_2 > k_1$ such that $u_{kk_1} \neq u_{k_1}$ for all $kk_1 \geqslant k_2$. Otherwise, $\varphi_{kk_1}(u_{kk_1}) = k\varphi_{k_1}(u_{k_1}) \to \infty$ as $k \to \infty$, which contradicts (4.8). Repeating this process, we obtain a sequence $\{u_{k_j}\}$ of distinct nontrivial periodic solutions of problem (1.1). This completes the proof of Theorem 1.4. □

P r o o f of Theorem 1.7. From Lemma 3.4, using the same arguments of Theorem 1.4, we see that problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$. □

P r o o f of Theorem 1.9. Clearly, with the aid of Lemma 3.5 and the arguments of Theorem 1.4, we can easily see that problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$. □

P r o o f of Corollary 1.11. Taking $h(s)=s^{p^+}$, by Theorem 1.4, we can obtain that problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j\in\mathbb{N}$ and $k_j\to\infty$ as $j\to\infty$ immediately. $\square$

P r o o f of Corollary 1.14. From (F3) and (F4′) we conclude that $F(t,x)$ is positive and asymptotic-p^+ for all $|x|$ large enough and a.e. $t\in[0,T]$, thus, (F6) is equivalent to (F6′). By Theorem 1.7, problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j\in\mathbb{N}$ and $k_j\to\infty$ as $j\to\infty$. $\square$

P r o o f of Corollary 1.16. Applying (F3), (F4″) and (F7′), we have

$$M\geqslant\frac{|\nabla F(t,x)|}{|x|^{p^+-1}}\geqslant\frac{(\nabla F(t,x),x)}{|x|^{p^+}}\geqslant\frac{p^+F(t,x)}{|x|^{p^+}}>0 \tag{4.9}$$

for $|x|$ large enough and a.e. $t\in[0,T]$. From (4.9), we deduce that $|\nabla F(t,x)|$ is positive and asymptotic-p^+ for all $|x|$ large enough and a.e. $t\in[0,T]$, which means (F7) and (F7′) are equivalent. Then, by Theorem 1.9, problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j\in\mathbb{N}$ and $k_j\to\infty$ as $j\to\infty$. $\square$

P r o o f of Corollary 1.17. By (F3′) and (F7″), one has

$$\frac{F(t,x)}{|x|^{p^+}}\geqslant\frac{(\nabla F(t,x),x)}{p^+|x|^{p^+}}>0$$

for $|x|$ large enough and a.e. $t\in[0,T]$, which implies (F3) holds. Moreover, utilizing (F3′) and (F4″), we have

$$M\geqslant\frac{|\nabla F(t,x)|}{|x|^{p^+-1}}\geqslant\frac{(\nabla F(t,x),x)}{|x|^{p^+}}>0.$$

Therefore, we know that $|\nabla F(t,x)|$ is also positive and asymptotic-p^+ for all $|x|$ large enough and a.e. $t\in[0,T]$. From Corollary 1.16, we can deduce that problem (1.1) has a sequence of distinct periodic solutions with period k_jT satisfying $k_j\in\mathbb{N}$ and $k_j\to\infty$ as $j\to\infty$. $\square$

5. Example

In this section, we give an example to illustrate that our results are new even in the case $p(t)\equiv p=2$.

E x a m p l e 5.1. Let

$$\alpha(t):=\begin{cases}\sin(2\pi/T)t, & t\in[0,\frac12T],\\ 0, & t\in[\frac12T,T],\end{cases}$$

and $D(t) := 12 - \alpha(t)(4\ln(\mathrm{e}+4) + \sin 4 - \ln^2(\mathrm{e}+4)) > 0$. Consider

$$F(t,x) = \begin{cases} |x|^4, & |x| \leqslant 2, \\ \alpha(t)(|x|^2\ln(\mathrm{e}+|x|^2) + \sin|x|^2 - \ln^2(\mathrm{e}+|x|^2)) + |x|^2 + D(t), & |x| > 2. \end{cases}$$

It is easy to check that $F(t,x)$ is superquadratic at infinity when $t \in [0, \frac{1}{2}T]$ and asymptotically quadratic at infinity when $t \in [\frac{1}{2}T, T]$. Then, Theorem 1.2, Theorem 1.3 and the conclusions of [5]–[7], [13]–[15], [17]–[20], [29], [32] cannot treat this case. Furthermore, by simple computation, one has $\liminf_{|x|\to\infty}((\nabla F(t,x), x) - 2F(t,x))|x|^{-\lambda} = 0$ uniformly for a.e. $t \in [0, \frac{1}{2}T]$ and all $\lambda > 0$, which implies that $F(t,x)$ does not satisfy the results of Theorem 1.1. However for all $s \geqslant 0$, select $h(s) = s^2\ln(\mathrm{e}+s^2)$, $\theta_1(s) = \ln^2(\mathrm{e}+s^2)$, a direct computation shows that $F(t,x)$ satisfies all conditions of Theorem 1.4 with $p(t) \equiv p = 2$. Hence, problem (1.1) with $p(t) \equiv p = 2$ has a sequence of distinct periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

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