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NEW KINDS OF HYBRID FILTERS OF EQ-ALGEBRAS

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Abstract. The main goal of this paper is to introduce hybrid positive implicative and hybrid implicative (pre)filters of EQ-algebras. In the following, some characterizations of this hybrid (pre)filters are investigated and it is proved that the quotient algebras induced by hybrid positive implicative filters in residuated EQ-algebras are idempotent and residuated EQ-algebra. Moreover, the relationship between hybrid implicative prefilters and hybrid positive implicative prefilters are discussed and it is shown that these concepts coincide in good involutive EQ-algebras. Finally, it is proved that the quotient EQ-algebra respect to a hybrid positive implicative filter is involutive if and only if the hybrid filter is hybrid implicative filter.

Keywords: EQ-algebra; hybrid positive implicative (pre)filter; hybrid implicative (pre)filter

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1. Introduction

Recently Novák in [10] introduced a special algebra which is called EQ-algebra. The first motivation for studying this algebra comes from fuzzy type theory (FTT) (see [9]) that generalizes the system of classical type theory (see [1]), in which the sole basic connective is equality. Analogously, the basic connective in FTT should be fuzzy equality. Another motivation is from the equational style of proof in logic. EQ-algebras are interesting and important for studying and researching. In fact, EQ-algebras generalize non-commutative residuated lattices (see [3]). From the point of view of logic, the main difference between residuated lattices and EQ-algebras lies in the way the implication operation is obtained. While in residuated lattices it is obtained from (strong) conjunction, in EQ-algebras it is obtained from equivalence. Consequently, the two kinds of algebras differ in several essential points despite their many similar or identical properties. After the introduction of fuzzy sets by

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Zadeh (see [13]), this theory has been applied in different areas of computer and management sciences and computer engineering. Molodtsov in [8] introduced the concept of soft set to deal with uncertainties with avoiding the difficulties that appear with the usual theoretical approaches. The notion of hybrid structure was introduced by Jun, Song and Muhiuddin (see [5]) as a parallel circuit of fuzzy and soft sets. The notion of hybrid structure was introduced into a set of parameters on an initial universe set and it was applied to BCK/BCI algebras and linear spaces. Moreover, hybrid ideals in semigroups were introduced by Anis et al. in [2]. This motivated us to introduce the new kinds of hybrid (pre)filtres of EQ-algebras and study some properties of them.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1 ([11]). An EQ-algebra is an algebra $(L, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms:
(E1) $(L, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leqslant y$ if and only if $x \wedge y = x$,
(E2) $(L, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leqslant$,
(E3) $x \sim x = 1$ (reflexivity axiom),
(E4) $((x \wedge y) \sim z) \odot (s \sim x) \leqslant z \sim (s \wedge y)$ (substitution axiom),
(E5) $(x \sim y) \odot (s \sim t) \leqslant (x \sim s) \sim (y \sim t)$ (congruence axiom),
(E6) $(x \wedge y \wedge z) \sim x \leqslant (x \wedge y) \sim x$ (monotonicity axiom),
(E7) $x \odot y \leqslant x \sim y$ (boundedness axiom)
for all $s, t, x, y, z \in L$.

Let L be an EQ-algebra. Then for all $x, y \in L$ we put

$$x \to y = (x \wedge y) \sim x, \quad \widetilde{x} = x \sim 1.$$

The derived operation $\to$ is called implication. If an EQ-algebra L contains a bottom element 0, then we may define the unary operation $\neg$ on L by $\neg x = x \sim 0$.

Definition 2.2 ([3], [11]). Let L be an EQ-algebra. Then we say that it is *separated* if $x \sim y = 1$ implies $x = y$ for all $x, y \in L$, *good* if $\widetilde{x} = x$ for all $x \in L$, *residuated* if $(x \odot y) \wedge z = x \odot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$ for all $x, y, z \in L$, and *involutive* (IEQ-algebra) if $\neg\neg x = x$ for all $x \in L$, when L contains a bottom element 0.

Lemma 2.3 ([4], [11]). *Let L be an EQ-algebra. Then the following properties hold for all $x, y, z \in L$:*

(i) $x \sim y \leqslant x \to y$, $y \leqslant 1 \to y$, $x \odot y \leqslant x \wedge y \leqslant x, y$,

(ii) $x \to y \leqslant (x \wedge z) \to (y \wedge z)$,

(iii) $(x \to y) \leqslant (y \to z) \to (x \to z)$ *and* $(x \to y) \leqslant (z \to x) \to (z \to y)$,

(iv) *if* $x \leqslant y$, *then* $x \to y = 1$ *and* $y \to x = x \sim y$,

(v) *if* $x \leqslant y$, *then* $z \to x \leqslant z \to x$ *and* $y \to z \leqslant x \to z$,

(vi) $(x \sim y) \leqslant (x \wedge z) \sim (y \wedge z)$,

(vii) *if L is separated, then* $x \to y = 1$ *if and only if* $x \leqslant y$,

(vii) *if L is good, then* $x \odot (x \to y) \leqslant y$,

(viii) *if L is good, then* $x \to (y \to z) = y \to (x \to z)$.

Definition 2.4 ([6]). Let L be an EQ-algebra and $\emptyset \neq F \subseteq L$. Then

(i) F is called a *prefilter* of L if it satisfies:

(F1) $1 \in F$,

(F2) if $x \in F$, $x \to y \in F$, then $y \in F$ for all $x, y \in L$.

(ii) A prefilter F is said to be a *filter* if it satisfies:

(F3) If $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for all $x, y, z \in L$.

Prefilters and filters coincide in residuated EQ-algebas.

In the following, we shall use I for the unit interval, X for the set of parameters and $\mathbf{P}(\mathbf{U})$ for the power set of an initial universe set U.

Definition 2.5 ([5]). A hybrid structure in X over U is defined to be a mapping

$$\widetilde{\varphi}_\lambda := (\widetilde{\varphi}, \lambda)\colon\ X \to \mathbf{P}(\mathbf{U}) \times I,\ x \to (\widetilde{\varphi}(x), \lambda(x)),$$

where $\widetilde{\varphi}\colon X \to \mathbf{P}(\mathbf{U})$ and $\lambda\colon X \to I$ are mappings. Moreover, the set $\widetilde{\varphi}_\lambda[\alpha, t] := \{x \in X\colon \widetilde{\varphi}(x) \supseteq \alpha, \lambda(x) \leqslant t\}$ is called $[\alpha, t]$-*hybrid cut of* $\widetilde{\varphi}_\lambda$.

Definition 2.6 ([12]). Let L be an EQ-algebra. Then

(i) a hybrid structure $\widetilde{\varphi}_\lambda$ in L over U is called a *hybrid prefilter* of L over U if it satisfies the followings for any $x, y \in L$:

(HPF1) $\widetilde{\varphi}(1) \supseteq \widetilde{\varphi}(x)$, $\lambda(1) \leqslant \lambda(x)$,

(HPF2) $\widetilde{\varphi}(y) \supseteq \widetilde{\varphi}(x \to y) \cap \widetilde{\varphi}(x)$, $\lambda(y) \leqslant \lambda(x \to y) \vee \lambda(x)$.

(ii) $\widetilde{\varphi}_\lambda$ is called a *hybrid filter* of L over U if for any $x, y, z \in L$

(HF) $\widetilde{\varphi}(x \odot z \to y \odot z) \supseteq \widetilde{\varphi}(x \to y)$, $\lambda(x \odot z \to y \odot z) \leqslant \lambda(x \to y)$.

Theorem 2.7 ([12]). *Let $\widetilde{\varphi}_\lambda$ be a hybrid structure of EQ-algebra L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid (pre)filter of L over U.*

(ii) *For any $\alpha \in \mathbf{P}(\mathbf{U})$ and $t \in I$, the nonempty subsets $\widetilde{\varphi}_\lambda(\alpha)$ and $\widetilde{\varphi}_\lambda(t)$ are (pre)filters of L.*

Note that the concepts of hybrid prefilters and hybrid filters coincide when EQ-algebra L is residuated.

Proposition 2.8 ([12]). *Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of EQ-algebra L over U. Then for any $x, y, z \in L$:*
(i) *if $x \leqslant y$, then $\widetilde{\varphi}(y) \supseteq \widetilde{\varphi}(x)$, $\lambda(y) \leqslant \lambda(x)$,*
(ii) *$\widetilde{\varphi}(x \to z) \supseteq \widetilde{\varphi}(x \to y) \cap \widetilde{\varphi}(y \to z)$, $\lambda(x \to z) \leqslant \lambda(x \to y) \vee \lambda(y \to z)$.*

Consider hybrid filter $\widetilde{\varphi}_\lambda$ of EQ-algebra L over U and define a binary relation $\equiv_{\widetilde{\varphi}_\lambda}$ on L as follows:

$$x \equiv_{\widetilde{\varphi}_\lambda} y \quad \text{if and only if} \quad \widetilde{\varphi}(x \sim y) = \widetilde{\varphi}(1), \quad \lambda(x \sim y) = \lambda(1).$$

Then $\equiv_{\widetilde{\varphi}_\lambda}$ is a congruence relation on L. We denote the equivalence class x w.r.t $\equiv_{\widetilde{\varphi}_\lambda}$ by $[x]_{\widetilde{\varphi}_\lambda}$ ($[x]$ for short). Furthermore, we define quotient algebra $L/\widetilde{\varphi}_\lambda := \{[x] \colon x \in L\}$ and the operations $\sqcap$, $\simeq$, $\otimes$ for $[x], [y] \in L/\widetilde{\varphi}_\lambda$ as follows:

$$[x] \sqcap [y] := [x \wedge y], \quad [x] \approx [y] := [x \sim y], \quad [x] \otimes [y] := [x \odot y].$$

The top element is $[1]$ and $[x] \leqslant [y]$ if and only if $[x] \sqcap [y] = [x]$ if and only if $x \wedge y \equiv_{\widetilde{\varphi}_\lambda} x$ if and only if

$$\widetilde{\varphi}(x \wedge y \sim x) = \widetilde{\varphi}(1) \quad \text{and} \quad \lambda(x \wedge y \sim x) = \lambda(1)$$

if and only if

$$\widetilde{\varphi}(x \to y) = \widetilde{\varphi}(1) \quad \text{and} \quad \lambda(x \to y) = \lambda(1).$$

Theorem 2.9 ([12]). *Let $\widetilde{\varphi}_\lambda$ be a hybrid filter of EQ-algebra L over U. Then $(L/\widetilde{\varphi}_\lambda, \sqcap, \approx, \otimes, [1])$ is a separated EQ-algebra which is called quotient EQ-algebra respect to $\widetilde{\varphi}_\lambda$ and the mapping $f \colon L \to L/\widetilde{\varphi}_\lambda$ defined by $f(x) = [x]$ is an EQ-epimorphism.*

N o t e 2.10. From now on, in this paper we let L be an EQ-algebra, unless stated otherwise.

3. Hybrid positive implicative (pre)filters of EQ-algebras

In this section, hybrid positive implicative (pre)filters of EQ-algebras are introduced and some related results are provided.

Definition 3.1. Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then $\widetilde{\varphi}_\lambda$ is called a *hybrid positive implicative prefilter* of L over U if it satisfies the following for all $x, y, z \in L$:

(HPI) $\widetilde{\varphi}(x \to z) \supseteq \widetilde{\varphi}(x \to (y \to z)) \cap \widetilde{\varphi}(x \to y)$ and $\lambda(x \to z) \leqslant \lambda(x \to (y \to z)) \vee \lambda(x \to y)$.

Moreover, if $\widetilde{\varphi}_\lambda$ is a hybrid filter of L over U and satisfies (HPI), then $\widetilde{\varphi}_\lambda$ is called a *hybrid positive implicative filter* of L over U.

Theorem 3.2. *Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*

(ii) *$\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$.*

Proof. (i) $\Rightarrow$ (ii): Since by Lemma 2.3 (i) for any $x, y \in L$, $x \wedge (x \to y) \leqslant x$, $x \to y$, we get that $(x \wedge (x \to y)) \to x = 1$ and $(x \wedge (x \to y)) \to (x \to y) = 1$ and since $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U, we conclude that

$$\begin{aligned}\widetilde{\varphi}((x \wedge (x \to y)) \to y) &\supseteq \widetilde{\varphi}((x \wedge (x \to y)) \to (x \to y)) \cap \widetilde{\varphi}((x \wedge (x \to y)) \to x)\\ &= \widetilde{\varphi}(1) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}(1)\end{aligned}$$

and

$$\begin{aligned}\lambda((x \wedge (x \to y)) \to y) &\leqslant \lambda((x \wedge (x \to y)) \to (x \to y)) \vee \lambda((x \wedge (x \to y)) \to x)\\ &\leqslant \lambda(1) \vee \lambda(1) \leqslant \lambda(1).\end{aligned}$$

Therefore, by Definition 2.6, $\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$.

(ii) $\Rightarrow$ (i): Since by Lemma 2.3 (ii) for any $x, y, z \in L$, $x \to y \leqslant (x \wedge x) \to (x \wedge y) = x \to (x \wedge y)$ and $x \to (y \to z) \leqslant (x \wedge y) \to y \wedge (y \to z)$, by Proposition 2.8 (i), we conclude that

$$\widetilde{\varphi}(x \to (x \wedge y)) \supseteq \widetilde{\varphi}(x \to y) \quad \text{and} \quad \lambda(x \to (x \wedge y)) \leqslant \lambda(x \to y),$$
$$\widetilde{\varphi}((x \wedge y) \to y \wedge (y \to z)) \supseteq \widetilde{\varphi}(x \to (y \to z))$$

and

$$\lambda((x \wedge y) \to y \wedge (y \to z)) \leqslant \lambda(x \to (y \to z)).$$

Now, by (ii) and Proposition 2.8 (ii), we have:

$$\begin{aligned}\widetilde{\varphi}(x \to z) &\supseteq \widetilde{\varphi}(x \to (y \wedge (y \to z))) \cap \widetilde{\varphi}((y \wedge (y \to z)) \to z)\\ &= \widetilde{\varphi}(x \to (y \wedge (y \to z))) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}(x \to (y \wedge (y \to z)))\\ &\supseteq \widetilde{\varphi}(x \to (x \wedge y)) \cap \widetilde{\varphi}((x \wedge y) \to (y \wedge (y \to z)))\\ &\supseteq \widetilde{\varphi}(x \to y) \cap \widetilde{\varphi}(x \to (y \to z))\end{aligned}$$

and

$$\begin{aligned}\lambda(x \to z) &\leqslant \lambda(x \to (y \wedge (y \to z))) \vee \lambda((y \wedge (y \to z)) \to z)\\ &\leqslant \lambda(x \to (y \wedge (y \to z))) \vee \lambda(1) \leqslant \lambda(x \to (y \wedge (y \to z)))\\ &\leqslant \lambda(x \to (x \wedge y)) \vee \lambda((x \wedge y) \to (y \wedge (y \to z)))\\ &\leqslant \lambda(x \to y) \vee \lambda(x \to (y \to z)).\end{aligned}$$

Therefore, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. □

Corollary 3.3. *If $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U, then*

(i) $\widetilde{\varphi}((1 \to x) \to x) = \widetilde{\varphi}(1)$ *and* $\lambda((1 \to x) \to x) = \lambda(1)$ *for any* $x \in L$,

(ii) $\widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1)$ *and* $\lambda((x \odot (x \to y)) \to y) = \lambda(1)$ *for any* $x, y \in L$.

P r o o f. (i) It follows from Theorem 3.2, by considering $x = 1$ and $y = x$.

(ii) Since by Lemma 2.3 (i) and (v), $x \odot (x \to y) \leqslant x \wedge (x \to y)$ and so $(x \wedge (x \to y)) \to y \leqslant (x \odot (x \to y)) \to y$ for any $x, y \in L$, by Proposition 2.8 (i) and Theorem 3.2, we conclude that

$$\widetilde{\varphi}((x \odot (x \to y)) \to y) \supseteq \widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1),$$

and

$$\lambda((x \odot (x \to y)) \to y) \leqslant \lambda((x \wedge (x \to y)) \to y) = \lambda(1).$$

Therefore, by Definition 2.6,

$$\widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1) \quad \text{and} \quad \lambda((x \odot (x \to y)) \to y) = \lambda(1)$$

for any $x, y \in L$. □

E x a m p l e 3.4. Let $L = \{0, a, b, 1\}$ be a chain $0 < a < b < 1$ with Cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Table 1.

Routine calculation shows that $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra, see [6].

Now, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be an initial universe set and $\widetilde{\varphi}_\lambda = (\widetilde{\varphi}, \lambda)$ be a hybrid structure in L over U which is given by Table 2.:

L	$\widetilde{\varphi}$	λ
0	$\{u_1\}$	0.8
a	$\{u_1, u_2\}$	0.6
b	U	0.4
1	U	0.1

Table 2. Hybrid structure.

Then by direct calculations we verify that $\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$ and so by Theorem 3.2, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U.

Theorem 3.5. *Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative filter of L over U. Then $L/\widetilde{\varphi}_\lambda$ is a good EQ-algebra.*

Proof. Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative filter of L over U. Then by Theorem 2.9, $L/\widetilde{\varphi}_\lambda$ is a separated EQ-algebra. Moreover, by Corollary 3.3, $\widetilde{\varphi}((1 \to x) \to x) = \widetilde{\varphi}(1)$ and $\lambda((1 \to x) \to x) = \lambda(1)$ for any $x \in L$. Hence, $[1 \to x] \rightrightarrows [x] = [1]$ and since $L/\widetilde{\varphi}_\lambda$ is separated, by Lemma 2.3 (vii), we get that $[1] \rightrightarrows [x] = [1 \to x] \leqslant [x]$ and since by Lemma 2.3 (i), $[x] \leqslant [1] \rightrightarrows [x]$, we get that $[1] \rightrightarrows [x] = [x]$ and so $[1] \approx [x] = [x]$ for any $[x] \in L/\widetilde{\varphi}_\lambda$. Therefore, $L/\widetilde{\varphi}_\lambda$ is a good EQ-algebra. □

Theorem 3.6. *Let $\widetilde{\varphi}_\lambda$, $\widetilde{\psi}_\nu$ be two hybrid prefilters of L over U such that $\widetilde{\varphi}(1) = \widetilde{\psi}(1)$, $\lambda(1) = \nu(1)$ and $\widetilde{\varphi}_\lambda \ll \widetilde{\psi}_\nu$. If $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U, so is $\widetilde{\psi}_\nu$.*

Proof. Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative prefilter of L over U. Then by Theorem 3.2 for any $x, y \in L$, $\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ and since $\widetilde{\varphi}_\lambda \ll \widetilde{\psi}_\nu$, we conclude that $\widetilde{\psi}((x \wedge (x \to y)) \to y) \supseteq \widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1) = \widetilde{\psi}(1)$ and $\nu(1) = \lambda(1) = \lambda((x \wedge (x \to y)) \to y) \leqslant \nu((x \wedge (x \to y)) \to y)$. Hence, by Definition 2.6, we conclude that $\widetilde{\psi}((x \wedge (x \to y)) \to y) = \widetilde{\psi}(1)$ and $\nu((x \wedge (x \to y)) \to y) = \nu(1)$ for any $x, y \in L$. Therefore, by Theorem 3.2, $\widetilde{\psi}_\nu$ is a hybrid positive implicative prefilter of L over U. □

Theorem 3.7. *Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*

(ii) *$\widetilde{\varphi}(x \to y) \supseteq \widetilde{\varphi}(x \to (x \to y))$ and $\lambda(x \to y) \leqslant \lambda(x \to (x \to y))$ for any $x, y \in L$.*

P r o o f. (i) $\Rightarrow$ (ii): Let $x, y \in L$ and $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative perfilter of L over U. Then

$$\begin{aligned}\widetilde{\varphi}(x \to y) &\supseteq \widetilde{\varphi}(x \to (x \to y)) \cap \widetilde{\varphi}(x \to x)\\ &= \widetilde{\varphi}(x \to (x \to y)) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}(x \to (x \to y))\end{aligned}$$

and

$$\begin{aligned}\lambda(x \to y) &\leqslant \lambda(x \to (x \to y)) \vee \lambda(x \to x)\\ &\leqslant \lambda(x \to (x \to y)) \vee \lambda(1) \leqslant \lambda(x \to (x \to y)).\end{aligned}$$

(ii) $\Rightarrow$ (i): Let $x, y, z \in L$. Then by Lemma 2.3 (iii), $(x \to y) \leqslant (y \to z) \to (x \to z)$ and $x \to (y \to z) \leqslant ((y \to z) \to (x \to z)) \to (x \to (x \to z))$ and so by Proposition 2.8 (i),

$$\widetilde{\varphi}((y \to z) \to (x \to z)) \supseteq \widetilde{\varphi}(x \to y), \quad \lambda((y \to z) \to (x \to z)) \leqslant (x \to y)$$

and

$$\begin{aligned}&\widetilde{\varphi}(((y \to z) \to (x \to z)) \to (x \to (x \to z))) \supseteq \widetilde{\varphi}(x \to (y \to z)),\\ &\lambda(((y \to z) \to (x \to z)) \to (x \to (x \to z))) \leqslant \lambda(x \to (y \to z)).\end{aligned}$$

Now, by (ii), we conclude that

$$\begin{aligned}\widetilde{\varphi}(x \to z) &\supseteq \widetilde{\varphi}(x \to (x \to z))\\ &\supseteq \widetilde{\varphi}(((y \to z) \to (x \to z)) \to (x \to (x \to z))) \cap \widetilde{\varphi}((y \to z) \to (x \to z))\\ &\supseteq \widetilde{\varphi}(x \to (y \to z)) \cap \widetilde{\varphi}(x \to y)\end{aligned}$$

and

$$\begin{aligned}\lambda(x \to z) &\leqslant \lambda(x \to (x \to z))\\ &\leqslant \lambda(((y \to z) \to (x \to z)) \to (x \to (x \to z))) \vee \lambda((y \to z) \to (x \to z))\\ &\leqslant \lambda(x \to (y \to z)) \vee \lambda(x \to y).\end{aligned}$$

Therefore, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. □

Corollary 3.8. *If $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U, then for any $x, y \in L$,*

$$\widetilde{\varphi}(x \to y) \supseteq \widetilde{\varphi}(x \sim (x \to y)) \quad \textit{and} \quad \lambda(x \to y) \leqslant \lambda(x \sim (x \to y)).$$

P r o o f. It follows from Lemma 2.3 (i) and Theorem 3.7. □

Definition 3.9. Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then we say that $\widetilde{\varphi}_\lambda$ has weak hybrid exchange principle if for any $x, y, z \in L$, it satisfies

$$\widetilde{\varphi}(x \to (y \to z)) \supseteq \widetilde{\varphi}(y \to (x \to z)) \quad \text{and} \quad \lambda(x \to (y \to z)) \leqslant \lambda(y \to (x \to z)).$$

Example 3.10. Let $\widetilde{\varphi}_\lambda$ be a hybrid filter in Example 3.4. Then $\widetilde{\varphi}_\lambda$ has weak hybrid exchange principle. Note that in this example, L is not a good EQ-algebra because $1 \sim b = 1$.

Theorem 3.11. *Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U and satisfy the weak hybrid exchange principle. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*

(ii) *$\widetilde{\varphi}((x \to y) \to (x \to z)) \supseteq \widetilde{\varphi}(x \to (y \to z))$ and $\lambda((x \to y) \to (x \to z)) \leqslant \lambda(x \to (y \to z))$ for any $x, y, z \in L$.*

Proof. (i) $\Rightarrow$ (ii): Let $x, y, z \in L$. Then by Lemma 2.3 (iii) and (v), we have $(x \to y) \to y \leqslant (y \to z) \to ((x \to y) \to z)$ and so $x \to ((x \to y) \to y) \leqslant x \to ((y \to z) \to ((x \to y) \to z))$. Hence, by (i), Proposition 2.8 (i) and condition weak exchange principle we have:

$$\begin{aligned}\widetilde{\varphi}((x \to y) \to (x \to z)) &\supseteq \widetilde{\varphi}(x \to ((x \to y) \to z)) \\ &\supseteq \widetilde{\varphi}(x \to ((y \to z) \to ((x \to y) \to z))) \cap \widetilde{\varphi}(x \to (y \to z)) \\ &\supseteq \widetilde{\varphi}(x \to ((x \to y) \to y)) \cap \widetilde{\varphi}(x \to (y \to z)) \\ &\supseteq \widetilde{\varphi}((x \to y) \to (x \to y)) \cap \widetilde{\varphi}(x \to (y \to z)) \\ &= \widetilde{\varphi}(1) \cap \widetilde{\varphi}(x \to (y \to z)) = \widetilde{\varphi}(x \to (y \to z))\end{aligned}$$

and

$$\begin{aligned}\lambda((x \to y) \to (x \to z)) &\leqslant \lambda(x \to ((x \to y) \to z)) \\ &\leqslant \lambda(x \to ((y \to z) \to ((x \to y) \to z))) \vee \lambda(x \to (y \to z)) \\ &\leqslant \lambda(x \to ((x \to y) \to y)) \vee \lambda(x \to (y \to z)) \\ &\leqslant \lambda((x \to y) \to (x \to y)) \vee \lambda(x \to (y \to z)) \\ &= \lambda(1) \vee \lambda(x \to (y \to z)) = \lambda(x \to (y \to z)).\end{aligned}$$

(ii) $\Rightarrow$ (i): Since $\widetilde{\varphi}_\lambda$ is a hybrid prefilter of L over U, by (ii) we conclude that

$$\widetilde{\varphi}(x \to z) \supseteq \widetilde{\varphi}((x \to y) \to (x \to z)) \cap \widetilde{\varphi}(x \to y) \supseteq \widetilde{\varphi}(x \to (y \to z)) \cap \widetilde{\varphi}(x \to y)$$

and

$$\lambda(x \to z) \leqslant \lambda((x \to y) \to (x \to z)) \vee \lambda(x \to y) \leqslant \lambda(x \to (y \to z)) \vee \lambda(x \to y).$$

Therefore, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. □

Note that by Lemma 2.3 (viii), in any good EQ-algebra, for any $x, y, z \in L$, $x \to (y \to z) = y \to (x \to z)$. So if L is a good EQ-algebra, then every hybrid prefilter has weak hybrid exchange principle. Therefore, by Theorem 3.2, Theorem 3.7 and Theorem 3.11, we conclude the following corollary:

Corollary 3.12. *Let L be a good EQ-algebra and $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U.*

(ii) *$\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$.*

(iii) *$\widetilde{\varphi}(x \to y) \supseteq \widetilde{\varphi}(x \to (x \to y))$ and $\lambda(x \to y) \leqslant \lambda(x \to (x \to y))$ for any $x, y \in L$.*

(iv) *$\widetilde{\varphi}((x \to y) \to (x \to z)) \supseteq \widetilde{\varphi}(x \to (y \to z))$ and $\lambda((x \to y) \to (x \to z)) \leqslant \lambda(x \to (y \to z))$ for any $x, y, z \in L$.*

Proposition 3.13 ([4])**.** *The following properties are equivalent:*

(i) *an EQ-algebra L is residuated;*

(ii) *the EQ-algebra L is good and for any $x, y, z \in L$,*

$$(*) \qquad (x \odot y) \to z \leqslant x \to (y \to z).$$

Theorem 3.14. *Let L be an EQ-algebra and satisfy $(*)$, $\widetilde{\varphi}_\lambda$ be a hybrid filter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*

(ii) *$\widetilde{\varphi}(x \to x \odot x) = \widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda(x \to x \odot x) = \lambda((x \odot (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$.*

P r o o f. (i) $\Rightarrow$ (ii): Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative perfilter of L over U. Then by Corollary 3.3 (ii),

$$\widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1) \quad \text{and} \quad \lambda((x \odot (x \to y)) \to y) = \lambda(1)$$

for any $x, y \in L$.

Moreover, since $x \odot x \to x \odot x = 1$ and by condition $(*)$, $1 = x \odot x \to x \odot x \leqslant x \to (x \to (x \odot x))$, by Proposition 2.8 (i), we get that

$$\widetilde{\varphi}(x \to (x \to (x \odot x))) \supseteq \widetilde{\varphi}(1) \quad \text{and} \quad \lambda(x \to (x \to (x \odot x))) \leqslant \lambda(1).$$

Hence, by Theorem 3.7, $\widetilde{\varphi}(x \to (x \odot x)) \supseteq \widetilde{\varphi}(x \to (x \to (x \odot x)))$ and $\lambda(x \to (x \odot x)) \leqslant \lambda(x \to (x \to (x \odot x)))$. Therefore, $\widetilde{\varphi}(x \to (x \odot x)) = \widetilde{\varphi}(1)$ and $\lambda(x \to (x \odot x)) = \lambda(1)$.

(ii) $\Rightarrow$ (i): Let $x, y, z \in L$. Then by Proposition 2.8 (ii), we have

$$\begin{aligned}
\widetilde{\varphi}(x \to z) &\supseteq \widetilde{\varphi}(x \to (y \odot (y \to z))) \cap \widetilde{\varphi}((y \odot (y \to z)) \to z) \\
&\supseteq \widetilde{\varphi}(x \to (x \odot x)) \cap \widetilde{\varphi}((x \odot x) \to (y \odot (y \to z))) \cap \widetilde{\varphi}((y \odot (y \to z)) \to z) \\
&\supseteq \widetilde{\varphi}(1) \cap \widetilde{\varphi}((x \odot x) \to (y \odot (y \to z))) \cap \widetilde{\varphi}(1) \\
&= \widetilde{\varphi}((x \odot x) \to (y \odot (y \to z))) \\
&\supseteq \widetilde{\varphi}((x \odot x) \to x \odot y) \cap \widetilde{\varphi}((x \odot y) \to (y \odot (y \to z))) \\
&\supseteq \widetilde{\varphi}(x \to y) \cap \widetilde{\varphi}(x \to (y \to z))
\end{aligned}$$

and

$$\begin{aligned}
\lambda(x \to z) &\leqslant \lambda(x \to (y \odot (y \to z))) \vee \lambda((y \odot (y \to z)) \to z) \\
&\leqslant \lambda(x \to x \odot x) \vee \lambda((x \odot x) \to (y \odot (y \to z))) \vee \lambda((y \odot (y \to z)) \to z) \\
&\leqslant \lambda(1) \vee \lambda((x \odot x) \to (y \odot (y \to z))) \vee \lambda(1) \\
&= \lambda((x \odot x) \to (y \odot (y \to z))) \\
&\leqslant \lambda((x \odot x) \to (x \odot y)) \vee \lambda((x \odot y) \to (y \odot (y \to z))) \\
&\leqslant \lambda(x \to y) \vee \lambda(x \to (y \to z)).
\end{aligned}$$

Therefore, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. □

Theorem 3.15. *Let L be an EQ-algebra and satisfy $(*)$, $\widetilde{\varphi}_\lambda$ be a hybrid filter of L over U. Then the following statements are equivalent:*
(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*
(ii) *$L/\widetilde{\varphi}_\lambda$ is a residuated and idempotent EQ-algebra.*

Proof. (i) $\Rightarrow$ (ii): Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative perfilter of L over U. Then by Theorem 2.9, $L/\widetilde{\varphi}_\lambda$ is a good algebra and since by condition $(*)$, for any $x, y, z \in L$, $x \odot y \to z \leqslant x \to (y \to z)$, we conclude that

$$\widetilde{\varphi}((x \odot y \to z) \to (x \to (y \to z))) = \widetilde{\varphi}(1)$$

and

$$\lambda((x \odot y \to z) \to (x \to (y \to z))) = \lambda(1).$$

Hence, $[x] \otimes [y] \rightrightarrows [z] \leqslant [x] \rightrightarrows ([y] \rightrightarrows [z])$ and so $L/\widetilde{\varphi}_\lambda$ satisfies $(*)$. Thus, by Proposition 3.13, $L/\widetilde{\varphi}_\lambda$ is a residuated EQ-algebra. Moreover, since by Theorem 3.14, $\widetilde{\varphi}(x \to x \odot x) = \widetilde{\varphi}(1)$ and $\lambda(x \to x \odot x) = \lambda(1)$, for any $x \in L$ we conclude that $[x] \leqslant [x \odot x] = [x] \otimes [x]$ and since $[x] \otimes [x] \leqslant [x]$, we get that $[x]^2 = [x] \otimes [x] = [x]$ for any $[x] \in L/\widetilde{\varphi}_\lambda$. Therefore, $L/\widetilde{\varphi}_\lambda$ is an idempotent EQ-algebra.

(ii) $\Rightarrow$ (i): If $L/\widetilde{\varphi}_\lambda$ is a residuated and idempotent EQ-algebra, then by Porposition 3.13, it is a good EQ-algebra and so by Lemma 2.3 (vii), $[x \odot (x \to y)] = [x] \otimes ([x] \rightrightarrows [y]) \leqslant [y]$ and since $L/\widetilde{\varphi}_\lambda$ is idempotent, we get that $[x]^2 = [x] \otimes [x] = [x]$ for any $[x], [y] \in L/\widetilde{\varphi}_\lambda$. Hence, $\widetilde{\varphi}(x \to x \odot x) = \widetilde{\varphi}(1)$, $\widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \odot (x \to y)) \to y) = \lambda(1)$, $\lambda(x \to x \odot x) = \lambda(1)$ for any $x, y \in L$. Therefore, by Theorem 3.14, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. $\square$

Corollary 3.16. *Let L be a resiudated EQ-algebra and $\widetilde{\varphi}_\lambda$ be a hybrid filter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U,*

(ii) *$\widetilde{\varphi}((x \wedge y) \to z) \supseteq \widetilde{\varphi}((x \odot y) \to z)$ and $\lambda((x \wedge y) \to z) \leqslant \lambda((x \odot y) \to z)$ for any $x, y, z \in L$,*

(iii) *$\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ for any $x, y \in L$,*

(iv) *$\widetilde{\varphi}(x \to y) \supseteq \widetilde{\varphi}(x \to (x \to y))$ and $\lambda(x \to y) \leqslant \lambda(x \to (x \to y))$ for any $x, y \in L$,*

(v) *$\widetilde{\varphi}((x \to y) \to (x \to z)) \supseteq \widetilde{\varphi}(x \to (y \to z))$ and $\lambda((x \to y) \to (x \to z)) \leqslant \lambda(x \to (y \to z))$ for any $x, y, z \in L$,*

(vi) *$\widetilde{\varphi}(x \to x \odot x) = \widetilde{\varphi}(1)$ and $\lambda(x \to x \odot x) = \lambda(1)$ for any $x, y \in L$,*

(vii) *$L/\widetilde{\varphi}_\lambda$ is a residuated and idempotent EQ-algebra.*

P r o o f. (i) $\Rightarrow$ (ii): Since L is residuated, by Proposition 3.13 and Lemma 2.3 (ii), we conclude that $x \odot y \to z \leqslant x \to (y \to z) \leqslant x \wedge y \to (y \wedge (y \to z))$. Hence, by Proposition 2.8 (i) and (ii) and Theorem 3.2 we have:

$$\begin{aligned}\widetilde{\varphi}((x \wedge y) \to z) &\supseteq \widetilde{\varphi}((x \wedge y) \to (y \wedge (y \to z))) \cap \widetilde{\varphi}((y \wedge (y \to z)) \to z)\\ &= \widetilde{\varphi}((x \wedge y) \to (y \wedge (y \to z))) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}((x \wedge y) \to (y \wedge (y \to z)))\\ &\supseteq \widetilde{\varphi}((x \odot y) \to z)\end{aligned}$$

and

$$\begin{aligned}\lambda((x \wedge y) \to z) &\leqslant \lambda((x \wedge y) \to (y \wedge (y \to z))) \vee \lambda((y \wedge (y \to z)) \to z)\\ &= \lambda((x \wedge y) \to (y \wedge (y \to z))) \vee \lambda(1) = \lambda((x \wedge y) \to (y \wedge (y \to z)))\\ &\leqslant \lambda((x \odot y) \to z).\end{aligned}$$

(ii) $\Rightarrow$ (i): Since L is residuated, by Proposition 3.13 and Lemma 2.3 (iv) and (viii), we conclude that $(x \odot (x \to y)) \to y = 1$. Now, by (ii) for any $x, y \in L$, we have:

$$\begin{aligned}\widetilde{\varphi}((x \wedge (x \to y)) \to y) \supseteq \widetilde{\varphi}((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1),\\ \lambda((x \wedge (x \to y)) \to y) \supseteq \lambda((x \odot (x \to y)) \to y) = \widetilde{\varphi}(1).\end{aligned}$$

Hence, $\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1)$ and $\lambda((x \wedge (x \to y)) \to y) = \lambda(1)$ and so by Theorem 3.2, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative perfilter of L over U. $\square$

Definition 3.17. The characterstic hybrid structure in L over U is denoted by $\widetilde{\chi}_{\{1\},\lambda}$ and is defined by the following:

$$\widetilde{\chi}_{\{1\},\lambda} := (\widetilde{\chi}_{\{1\}}, \lambda_{\{1\}})\colon\ L \to \mathbf{P}(\mathbf{U}) \times I,$$
$$\widetilde{\chi}_{\{1\}}(x) := \begin{cases} U & \text{if } x = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$
$$\lambda_{\{1\}}(x) := \begin{cases} 0 & \text{if } x = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 3.18. *Let L be a residuated EQ-algebra. Then the following statements are equivalent:*
(i) *L is an idempotent EQ-algebra,*
(ii) *every hybrid filter of L over U is a hybrid positive implicative filter,*
(iii) *$\widetilde{\chi}_{\{1\},\lambda}$ is a hybrid positive implicative filter of L over U.*

Proof. (i) $\Rightarrow$ (ii): Let L be an idempotent residuated EQ-algebra and $\widetilde{\varphi}_\lambda$ be a hybrid filter of L over U. Then for any $x \in L$, $x \to x \odot x = 1$ and so $\widetilde{\varphi}(x \to (x \odot x)) = \widetilde{\varphi}(1)$ and $\lambda(x \to (x \odot x)) = \lambda(1)$. Hence, by Corollary 3.16, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative filter of L over U.

(ii) $\Rightarrow$ (iii): We prove that $\widetilde{\chi}_{\{1\},\lambda}$ is a hybrid filter of L over U. By definition of $\widetilde{\chi}_{\{1\},\lambda}$, we have $\widetilde{\chi}_{\{1\}}(1) = U \supseteq \widetilde{\chi}_{\{1\}}(x)$ and $\lambda_{\{1\}}(1) = 0 \leqslant \lambda_{\{1\}}(x)$ for any $x \in L$. If $y = 1$, then for any $x \in L$, $\widetilde{\chi}_{\{1\}}(y) = U \supseteq \widetilde{\chi}_{\{1\}}(x \to y) \cap \widetilde{\chi}_{\{1\}}(x)$ and $\lambda_{\{1\}}(y) = 0 \leqslant \lambda_{\{1\}}(x \to y) \vee \lambda_{\{1\}}(x)$. If $x = 1$ and $y \neq 1$, then for any $y \in L$,

$$\widetilde{\chi}_{\{1\}}(y) = \emptyset \supseteq \widetilde{\chi}_{\{1\}}(1 \to y) \cap \widetilde{\chi}_{\{1\}}(1) = \emptyset \cap U = \emptyset$$

and

$$\lambda_{\{1\}}(y) = 0 \leqslant \lambda_{\{1\}}(1 \to y) \vee \lambda_{\{1\}}(1) = 0 \vee 1 = 0.$$

If $x, y \neq 1$, then

$$\widetilde{\chi}_{\{1\}}(y) = \emptyset \supseteq \widetilde{\chi}_{\{1\}}(x \to y) \cap \widetilde{\chi}_{\{1\}}(x) = \widetilde{\chi}_{\{1\}}(x \to y) \cap \emptyset = \emptyset$$

and

$$\lambda_{\{1\}}(y) = 0 \leqslant \lambda_{\{1\}}(x \to y) \vee \lambda_{\{1\}}(1) = \lambda_{\{1\}}(x \to y) \vee 0 = \lambda_{\{1\}}(x \to y).$$

Therefore, $\widetilde{\chi}_{\{1\},\lambda}$ is a hybrid perfilter of L over U. Now, we show that $\widetilde{\chi}_{\{1\},\lambda}$ is a filter of L over U. Let $x, y, z \in L$. If $x \odot z \to y \odot z = 1$, then

$$\widetilde{\chi}_{\{1\}}(x \odot z \to y \odot z) = U \supseteq \widetilde{\chi}_{\{1\}}(x \to y) \ \text{ and } \ \lambda_{\{1\}}(x \odot z \to y \odot z) = 0 \leqslant \lambda_{\{1\}}(x \to y).$$

If $x \odot z \to y \odot z \neq 1$, then $x \odot z \nleqslant y \odot z$ and so $x \nleqslant y$, since if $x \leqslant y$, by (E2), we get that $x \odot z \leqslant y \odot z$, which is impossible. Hence, $x \nleqslant y$ and so $x \to y \neq 1$. Thus,

$$\widetilde{\chi}_{\{1\}}(x \odot z \to y \odot z) = \widetilde{\chi}_{\{1\}}(x \to y) = \emptyset \quad \text{and} \quad \lambda_{\{1\}}(x \odot z \to y \odot z) = \lambda_{\{1\}}(x \to y) = 0.$$

Therefore, $\widetilde{\chi}_{\{1\},\lambda}$ is a hybrid filter of L over U and so by (ii), a hybrid positive implicative of L over U.

(iii) $\Rightarrow$ (i): Since $\widetilde{\chi}_{\{1\},\lambda}$ is a hybrid positive implicative of L over U, by Corollary 3.16, we conclude that $L/\widetilde{\chi}_{\{1\},\lambda}$ is a residuated and idempotent EQ-algebra and so for any $[x] \in L/\widetilde{\chi}_{\{1\},\lambda}$, $[x] \otimes [x] = [x]$. Now, we consider two cases for any $x \in L$:

Case (I): If $[x] \neq [1]$, then $[x] \otimes [x] = [x]$ and so

$$\widetilde{\chi}_{\{1\}}(x \to x \odot x) = \widetilde{\chi}_{\{1\}}(1) = U.$$

Hence, by definition of $\widetilde{\chi}_{\{1\}}$, we conclude that $x \to x \odot x = 1$.

Case (II): If $[x] = [1]$, then $[x] \otimes [x] = [1]$ and

$$\widetilde{\chi}_{\{1\}}(x \odot x) = \widetilde{\chi}_{\{1\}}(1 \to x \odot x) = \widetilde{\chi}_{\{1\}}(1) = U.$$

Now, by Proposition 2.8 (ii), we have:

$$\begin{aligned} \widetilde{\chi}_{\{1\}}(x \to x \odot x) &\supseteq \widetilde{\chi}_{\{1\}}(x \to 1) \cap \widetilde{\chi}_{\{1\}}(1 \to x \odot x) \\ &= \widetilde{\chi}_{\{1\}}(1) \cap \widetilde{\chi}_{\{1\}}(1) = \widetilde{\chi}_{\{1\}}(1) = U. \end{aligned}$$

Hence, by definition of $\widetilde{\chi}_{\{1\}}$, we conclude that $x \to x \odot x = 1$. Therefore, for any $x \in L$ we have $x \to x \odot x = 1$ and since by Lemma 2.3 (i), $x \odot x \to x = 1$, we conclude that $x \odot x = x$ for any $x \in L$. Therefore, L is an idempotent EQ-algebra. □

4. Hybrid implicative (pre)filters in EQ-algebras

In this section we introduce the concept of hybrid implicative (pre)filters in EQ-algebras and we give some related results.

Definition 4.1. Let $\widetilde{\varphi}_\lambda$ be a hybrid structure in L over U. Then $\widetilde{\varphi}_\lambda$ is called a *hybrid implicative prefilter* of L over U if for any $x, y, z \in L$, it satisfies the followings:
(HPF1) $\widetilde{\varphi}(1) \supseteq \widetilde{\varphi}(x)$, $\lambda(1) \leqslant \lambda(x)$,
(HIF) $\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(z \to ((x \to y) \to x)) \cap \widetilde{\varphi}(z)$, $\lambda(x) \leqslant \lambda(z \to ((x \to y) \to x)) \vee \lambda(z)$.

Moreover, $\widetilde{\varphi}_\lambda$ is called a *hybrid implicative filter* of L over U if it satisfies (HF).

E x a m p l e 4.2. Let $L = \{0, a, b, c, d, 1\}$, where $0 \leqslant a, b \leqslant c \leqslant 1$, $0 \leqslant b \leqslant d \leqslant 1$, but a, b and respective c, d are incomparable. The production and fuzzy equality are defined as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Table 3.

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	b	c	0	a
b	c	b	1	d	c	b
c	b	c	d	1	b	c
d	a	0	c	b	1	d
1	0	a	b	c	d	1

Table 4.

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Table 5.

Then $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra (see [7]). Now, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be an initial universe set and $\widetilde{\varphi}_\lambda = (\widetilde{\varphi}, \lambda)$ be a hybrid structure in L over U, which is given by Table 6.:

L	$\widetilde{\varphi}$	λ
0	$\{u_2\}$	0.7
a	$\{u_1, u_2\}$	0.7
b	$\{u_2\}$	0.7
c	$\{u_1, u_2\}$	0.7
d	$\{u_2, u_3\}$	0.4
1	U	0.2

Table 6. Hybrid structure.

Then by direct calculations we verify that $\widetilde{\varphi}_\lambda$ is a hybrid implicative filter of L over U.

Proposition 4.3. *Let $\widetilde{\varphi}_\lambda$ be a hybrid implicative prefilter of L over U. If $x \leqslant y$, then $\widetilde{\varphi}(y) \supseteq \widetilde{\varphi}(x)$ and $\lambda(x) \leqslant \lambda(y)$ for any $x, y \in L$.*

P r o o f. Let $x, y \in L$ such that $x \leqslant y$. Then by Lemma 2.3 (iv), $x \to y = 1$ and since by Lemma 2.3 (i) and (v), $y \leqslant (y \to y) \to y$ and $x \to y \leqslant x \to ((y \to y) \to y)$, we get that $x \to ((y \to y) \to y) = 1$. Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, we conclude that

$$\widetilde{\varphi}(y) \supseteq \widetilde{\varphi}(x \to ((y \to y) \to y)) \cap \widetilde{\varphi}(x) = \widetilde{\varphi}(1) \cap \widetilde{\varphi}(x) = \widetilde{\varphi}(x),$$
$$\lambda(y) \leqslant \lambda(x \to ((y \to y) \to y)) \vee \lambda(x) = \lambda(1) \vee \lambda(x) = \lambda(x). \qquad \square$$

Proposition 4.4. *Every hybrid implicative prefilter of L over U is a hybrid prefilter.*

P r o o f. It follows from Lemma 2.3 (i) and Proposition 4.3. $\square$

Theorem 4.5. *Let $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U,*

(ii) *$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \leqslant \lambda((x \to y) \to x)$ for any $x, y \in L$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U and $x, y \in L$. Since by Lemma 2.3 (i), $(x \to y) \to x \leqslant 1 \to ((x \to y) \to x)$, by Proposition 2.8 (i), we conclude that

$$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(1 \to ((x \to y) \to x)) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}(1 \to ((x \to y) \to x)) \supseteq \widetilde{\varphi}((x \to y) \to x),$$
$$\lambda(x) \leqslant \lambda(1 \to ((x \to y) \to x)) \vee \lambda(1) = \lambda(1 \to ((x \to y) \to x)) \leqslant \lambda((x \to y) \to x).$$

(ii) $\Rightarrow$ (i): Since $\widetilde{\varphi}_\lambda$ is a hybrid prefilter of L over U, by (ii) for any $x, y, z \in L$ we get that

$$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}((x \to y) \to x) \supseteq \widetilde{\varphi}(z \to ((x \to y) \to x)) \cap \widetilde{\varphi}(z),$$
$$\lambda(x) \leqslant \lambda((x \to y) \to x) \leqslant \lambda(z \to ((x \to y) \to x)) \vee \lambda(z).$$

Therefore, $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. □

Theorem 4.6. *Let $\widetilde{\varphi}_\lambda$ be a hybrid implicative prefilter of L over U. Then $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U.*

Proof. Since by Lemma 2.3 (i) and (v), $x \wedge (x \to y) \leqslant x$ and $x \to y \leqslant (x \wedge (x \to y)) \to y$ and since $x \wedge (x \to y) \leqslant x \to y$, we get that $x \wedge (x \to y) \leqslant (x \wedge (x \to y)) \to y$. Hence, $((x \wedge (x \to y)) \to y) \to y \leqslant (x \wedge (x \to y)) \to y$ and so $(((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y) = 1$. Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, by Theorem 4.5, we conclude that

$$\begin{aligned}\widetilde{\varphi}((x \wedge (x \to y)) \to y) &\supseteq \widetilde{\varphi}((((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y)) \\ &= \widetilde{\varphi}(1)\end{aligned}$$

and

$$\begin{aligned}\lambda((x \wedge (x \to y)) \to y) &\leqslant \lambda((((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y)) \\ &= \lambda(1).\end{aligned}$$

Hence,

$$\widetilde{\varphi}((x \wedge (x \to y)) \to y) = \widetilde{\varphi}(1) \quad \text{and} \quad \lambda((x \wedge (x \to y)) \to y) = \lambda(1).$$

Therefore, by Theorem 3.2, $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U. □

Theorem 4.7. *Let $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative prefilter of L over U with hybrid weak exchange principle. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U,*

(ii) *$\widetilde{\varphi}((y \to x) \to x) \supseteq \widetilde{\varphi}((x \to y) \to y)$ and $\lambda((y \to x) \to x) \leqslant \lambda((x \to y) \to y)$ for any $x, y \in L$.*

Proof. (i) $\Rightarrow$ (ii): Since by Lemma 2.3 (iii), for any $x, y \in L$, $(x \to y) \to y \leqslant (y \to x) \to ((x \to y) \to x)$, by Proposition 2.8 (i), we get that

$$\widetilde{\varphi}((y \to x) \to ((x \to y) \to x)) \supseteq \widetilde{\varphi}((x \to y) \to y)$$

and

$$\lambda((y \to x) \to ((x \to y) \to x)) \leqslant \lambda((x \to y) \to y)$$

and since $\widetilde{\varphi}_\lambda$ has hybrid weak exchange principle, we conclude that

$$\widetilde{\varphi}((x \to y) \to ((y \to x) \to x)) \supseteq \widetilde{\varphi}((y \to x) \to ((x \to y) \to x))$$

and

$$\lambda((x \to y) \to ((y \to x) \to x)) \leqslant \lambda((y \to x) \to ((x \to y) \to x)).$$

Moreover, by Lemma 2.3 (i) and (v), we have $x \leqslant (y \to x) \to x$ and $((y \to x) \to x) \to y \leqslant x \to y$. Hence, $(x \to y) \to ((y \to x) \to x) \leqslant (((y \to x) \to x) \to y) \to ((y \to x) \to x)$ and so by Proposition 2.8 (i), we get

$$\begin{aligned}
&\widetilde{\varphi}((((y \to x) \to x) \to y) \to ((y \to x) \to x)) \supseteq \widetilde{\varphi}((x \to y) \to ((y \to x) \to x)),\\
&\lambda((((y \to x) \to x) \to y) \to ((y \to x) \to x)) \leqslant \lambda((x \to y) \to ((y \to x) \to x)).
\end{aligned}$$

Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, by Theorem 4.5, we conclude that

$$\begin{aligned}
\widetilde{\varphi}((y \to x) \to x) &\supseteq \widetilde{\varphi}((((y \to x) \to x) \to y) \to ((y \to x) \to x))\\
&\supseteq \widetilde{\varphi}((x \to y) \to ((y \to x) \to x))\\
&\supseteq \widetilde{\varphi}((y \to x) \to ((x \to y) \to x)) \supseteq \widetilde{\varphi}((x \to y) \to y)
\end{aligned}$$

and

$$\begin{aligned}
\lambda((y \to x) \to x) &\leqslant \lambda((((y \to x) \to x) \to y) \to ((y \to x) \to x))\\
&\leqslant \lambda((x \to y) \to ((y \to x) \to x))\\
&\leqslant \lambda((y \to x) \to ((x \to y) \to x)) \leqslant \lambda((x \to y) \to y).
\end{aligned}$$

(ii) $\Rightarrow$ (i): Since by Lemma 2.3 (i), (iii) and (v), for any $x, y \in L$, $y \leqslant x \to y$, $(x \to y) \to x \leqslant y \to x$ and $(x \to y) \to x \leqslant (x \to y) \to ((x \to y) \to y)$, by Proposition 2.8 (i), we conclude that

$$\widetilde{\varphi}(y \to x) \supseteq \widetilde{\varphi}((x \to y) \to x), \quad \lambda(y \to x) \leqslant \lambda((x \to y) \to x)$$

and

$$\begin{aligned} &\widetilde{\varphi}((x \to y) \to ((x \to y) \to y)) \supseteq \widetilde{\varphi}((x \to y) \to x), \\ &\lambda((x \to y) \to ((x \to y) \to y)) \leqslant \lambda((x \to y) \to x). \end{aligned}$$

Now, since $\widetilde{\varphi}_\lambda$ is a hybrid prefilter of L over U, by (ii) and Theorem 3.7, we have

$$\begin{aligned} \widetilde{\varphi}(x) &\supseteq \widetilde{\varphi}((y \to x) \to x) \cap \widetilde{\varphi}(y \to x) \supseteq \widetilde{\varphi}((x \to y) \to y) \cap \widetilde{\varphi}(y \to x) \\ &\supseteq \widetilde{\varphi}((x \to y) \to ((x \to y) \to y)) \cap \widetilde{\varphi}(y \to x) \\ &\supseteq \widetilde{\varphi}((x \to y) \to x) \cap \widetilde{\varphi}((x \to y) \to x) = \widetilde{\varphi}((x \to y) \to x) \end{aligned}$$

and

$$\begin{aligned} \lambda(x) &\leqslant \lambda((y \to x) \to x) \vee \lambda(y \to x) \leqslant \lambda((x \to y) \to y) \vee \lambda(y \to x) \\ &\leqslant \lambda((x \to y) \to ((x \to y) \to y)) \vee \lambda(y \to x) \\ &\leqslant \lambda((x \to y) \to x) \vee \lambda((x \to y) \to x) = \lambda((x \to y) \to x). \end{aligned}$$

Therefore, by Theorem 4.5 $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. □

Theorem 4.8. *Let $\widetilde{\varphi}_\lambda$, $\widetilde{\psi}_\nu$ be two hybrid prefilters of L over U such that $\widetilde{\varphi}(1) = \widetilde{\psi}(1)$, $\lambda(1) = \nu(1)$ and $\widetilde{\varphi}_\lambda \ll \widetilde{\psi}_\nu$. If $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U with hybrid weak exchange principle, then $\widetilde{\psi}_\nu$ is a hybrid implicative prefilter of L over U.*

Proof. For any $x, y \in L$ let $u = (x \to y) \to x$. Then by Lemma 2.3 (i) and (v), $x \leqslant u \to x$ and so

$$(u \to x) \to y \leqslant x \to y \quad \text{and so} \quad u = (x \to y) \to x \leqslant ((u \to x) \to y) \to x.$$

Hence, $u \to (((u \to x) \to y) \to x) = 1$ and since $\widetilde{\varphi}_\lambda$ has hybrid weak exchange principle, we get that

$$\begin{aligned} &\widetilde{\varphi}(((u \to x) \to y) \to (u \to x)) \supseteq \widetilde{\varphi}(u \to (((u \to x) \to y) \to x)) = \widetilde{\varphi}(1), \\ &\lambda(((u \to x) \to y) \to (u \to x)) \leqslant \lambda(u \to (((u \to x) \to y) \to x)) = \lambda(1). \end{aligned}$$

Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter, we conclude that

$$\widetilde{\varphi}(u \to x) \supseteq \widetilde{\varphi}(((u \to x) \to y) \to (u \to x)) \supseteq \widetilde{\varphi}(1),$$
$$\lambda(u \to x) \leqslant \lambda(((u \to x) \to y) \to (u \to x)) \leqslant \lambda(1).$$

Hence, $\widetilde{\varphi}(u \to x) = \widetilde{\varphi}(1)$ and $\lambda(u \to x) = \lambda(1)$ and since $\widetilde{\varphi}_\lambda \ll \widetilde{\psi}_\nu$, we get that $\widetilde{\psi}(u \to x) \supseteq \widetilde{\varphi}(u \to x) = \widetilde{\varphi}(1) = \widetilde{\psi}(1)$ and $\nu(u \to x) \leqslant \lambda(u \to x) = \lambda(1) = \nu(1)$. Hence, $\widetilde{\psi}(u \to x) = \widetilde{\psi}(1)$ and $\nu(u \to x) = \nu(1)$. Finally, since $\widetilde{\psi}_\nu$ is a hybrid prefilter of L over U, we conclude that

$$\widetilde{\psi(x)} \supseteq \widetilde{\psi}(u \to x) \cap \widetilde{\psi}(u) = \widetilde{\psi}(1) \cap \widetilde{\psi}(u) = \widetilde{\psi}((x \to y) \to x)$$

and

$$\nu(x) \leqslant \nu(u \to x) \vee \nu(x) \leqslant \nu(1) \vee \nu(x) = \nu(u) \leqslant \nu((x \to y) \to x).$$

Therefore, by Theorem 4.5, $\widetilde{\psi}_\nu$ is a hybrid implicative prefilter of L over U. □

Theorem 4.9. *Let L be an EQ-algebra with bottom element* 0 *and $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*
(i) *$\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U,*
(ii) *$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg x \to x)$ and $\lambda(x) \leqslant \lambda(\neg x \to x)$ for any $x \in L$.*

Proof. (i) ⇒ (ii): Let $\widetilde{\varphi}_\lambda$ be a hybrid implicative prefilter of L over U and $x \in L$. Then by Theorem 4.5,

$$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}((x \to 0) \to x) = \widetilde{\varphi}(\neg x \to x), \quad \lambda(x) \leqslant \lambda((x \to 0) \to x) = \lambda(\neg x \to x).$$

(ii) ⇒ (i): Since by Lemma 2.3 (v),

$$(x \to y) \to x \leqslant (x \to 0) \to x = \neg x \to x,$$

by (ii) and Proposition 2.8 (i), we have:

$$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg x \to x) \supseteq \widetilde{\varphi}((x \to y) \to x), \quad \lambda(x) \leqslant \lambda(\neg x \to x) \leqslant \lambda((x \to y) \to x).$$

Therefore, by Theorem 4.5, $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. □

Theorem 4.10. *Let L be a good EQ-algebra and $\widetilde{\varphi}_\lambda$ be a hybrid prefilter of L over U. Then the following statements are equivalent:*
(i) *$\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U,*
(ii) *$\widetilde{\varphi}(x \to z) \supseteq \widetilde{\varphi}(x \to (\neg z \to y)) \cap \widetilde{\varphi}(y \to z)$ and $\lambda(x \to z) \leqslant \lambda(x \to (\neg z \to y)) \vee \lambda(y \to z)$ for any $x, y, z \in L$.*

P r o o f. (i)⇒ (ii): Since by Lemma 2.3(iii) and (v), for any $x, y, z \in L$, $y \to z \leqslant (x \to y) \to (x \to z)$, $\neg z \to (x \to y) \leqslant ((x \to y) \to (x \to z)) \to (\neg z \to (x \to z))$ and $\neg z \to (x \to z) \leqslant \neg(x \to z) \to (x \to z)$, by Proposition 2.8 (i), we get that

$$\widetilde{\varphi}((x \to y) \to (x \to z)) \supseteq \widetilde{\varphi}(y \to z), \quad \lambda((x \to y) \to (x \to z)) \leqslant \lambda(y \to z)$$

and

$$\begin{aligned}
&\widetilde{\varphi}(((x \to y) \to (x \to z)) \to (\neg z \to (x \to z))) \supseteq \widetilde{\varphi}(\neg z \to (x \to y)),\\
&\lambda(((x \to y) \to (x \to z)) \to (\neg z \to (x \to z))) \leqslant \lambda(\neg z \to (x \to y))
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{\varphi}(\neg(x \to z) \to (x \to z)) \supseteq \widetilde{\varphi}(\neg z \to (x \to z)),\\
&\lambda(\neg(x \to z) \to (x \to z)) \leqslant \lambda(\neg z \to (x \to z)).
\end{aligned}$$

Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, by Lemma 2.3 (viii) and Theorem 4.9, we conclude that

$$\begin{aligned}
\widetilde{\varphi}(x \to z) &\supseteq \widetilde{\varphi}(\neg(x \to z) \to (x \to z)) \supseteq \widetilde{\varphi}(\neg z \to (x \to z))\\
&\supseteq \widetilde{\varphi}(((x \to y) \to (x \to z)) \to (\neg z \to (x \to z))) \cap \widetilde{\varphi}((x \to y) \to (x \to z))\\
&\supseteq \widetilde{\varphi}(\neg z \to (x \to y)) \cap \widetilde{\varphi}(y \to z) \supseteq \widetilde{\varphi}(x \to (\neg z \to y)) \cap \widetilde{\varphi}(y \to z)
\end{aligned}$$

and

$$\begin{aligned}
\lambda(x \to z) &\leqslant \lambda(\neg(x \to z) \to (x \to z)) \leqslant \lambda(\neg z \to (x \to z))\\
&\leqslant \lambda(((x \to y) \to (x \to z)) \to (\neg z \to (x \to z))) \vee \lambda((x \to y) \to (x \to z))\\
&\leqslant \lambda(\neg z \to (x \to y)) \vee \lambda(y \to z) \leqslant \lambda(x \to (\neg z \to y)) \vee \lambda(y \to z).
\end{aligned}$$

(ii) ⇒ (i): Since L is a good EQ-algebra, by (ii) we get that

$$\begin{aligned}
\widetilde{\varphi}(x) &= \widetilde{\varphi}(1 \to x) \supseteq \widetilde{\varphi}(1 \to (\neg x \to x)) \cap \widetilde{\varphi}(x \to x)\\
&= \widetilde{\varphi}(\neg x \to x) \cap \widetilde{\varphi}(1) = \widetilde{\varphi}(\neg x \to x)
\end{aligned}$$

and

$$\begin{aligned}
\lambda(x) &= \lambda(1 \to x) \leqslant \lambda(1 \to (\neg x \to x)) \vee \lambda(x \to x)\\
&\leqslant \lambda(\neg x \to x) \vee \lambda(1) = \lambda(\neg x \to x).
\end{aligned}$$

Therefore, by Theorem 4.9, $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. □

Theorem 4.11. *Let L be an EQ-algebra with bottom element 0 and $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative prefilter of L over U. Then the following statements are equivalent:*

(i) *$\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U,*

(ii) *$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg\neg x)$ and $\lambda(x) \leqslant \lambda(\neg\neg x)$ for any $x \in L$.*

Proof. (i) $\Rightarrow$ (ii): Since by Lemma 2.3 (v), for any $x \in L$, $\neg\neg x = \neg x \to 0 \leqslant \neg x \to x$, by Proposition 2.8 (i), we have $\widetilde{\varphi}(\neg x \to x) \supseteq \widetilde{\varphi}(\neg\neg x)$ and $\lambda(\neg x \to x) \leqslant \lambda(\neg\neg x)$ and since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, by Theorem 4.9, we get that $\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg x \to x)$ and $\lambda(x) \leqslant \lambda(\neg x \to x)$. Hence, $\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg\neg x)$ and $\lambda(x) \leqslant \lambda(\neg\neg x)$.

(ii) $\Rightarrow$ (i): Since by Lemma 2.3 (v), for any $x \in L$, $\neg x \to x \leqslant (x \to 0) \to (\neg x \to 0) = \neg x \to (\neg x \to 0)$, by Proposition 2.8 (i), we get that $\widetilde{\varphi}(\neg x \to (\neg x \to 0)) \supseteq \widetilde{\varphi}(\neg x \to x)$ and $\lambda(\neg x \to (\neg x \to 0)) \leqslant \lambda(\neg x \to x)$. Now, since $\widetilde{\varphi}_\lambda$ is a hybrid positive implicative prefilter of L over U, by Theorem 3.7, we conclude that

$$\widetilde{\varphi}(\neg x \to 0) \supseteq \widetilde{\varphi}(\neg x \to (\neg x \to 0)), \quad \lambda(\neg x \to 0) \leqslant \lambda(\neg x \to (\neg x \to 0)).$$

Hence,

$$\widetilde{\varphi}(\neg x \to 0) \supseteq \widetilde{\varphi}(\neg x \to x) \quad \text{and} \quad \lambda(\neg x \to 0) \leqslant \lambda(\neg x \to x)$$

and since by (ii), $\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg\neg x) = \widetilde{\varphi}(\neg x \to 0)$ and $\lambda(x) \leqslant \lambda(\neg\neg x) = \lambda(\neg x \to 0)$, we get that $\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg x \to x)$ and $\lambda(x) \leqslant \lambda(\neg x \to x)$. Therefore, by Theorem 4.9, $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. □

Corollary 4.12. *Let L be a good involutive EQ-algebra. Then the concepts of hybrid positive implicative prefilters and hybrid implicative prefilters coincide.*

Proof. It follows from Theorem 4.5 and Theorem 4.6. □

Theorem 4.13. *Let L be a good EQ-algebra with bottom element 0 and $\widetilde{\varphi}_\lambda$ be a hybrid positive implicative prefilter of L over U. Then $L/\widetilde{\varphi}_\lambda$ is an involutive EQ-algebra if and only if $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U.*

Proof. Let $L/\widetilde{\varphi}_\lambda$ be an involutive EQ-algebra. Then $\neg\neg[x] = [\neg\neg x] = [x]$ for any $[x] \in L/\widetilde{\varphi}_\lambda$ and so $\widetilde{\varphi}(\neg\neg x) = \widetilde{\varphi}(1)$ and $\lambda(\neg\neg x) = \lambda(1)$. Now, since $\widetilde{\varphi}_\lambda$ is a hybrid prefilter, we have:

$$\widetilde{\varphi}(x) \supseteq \widetilde{\varphi}(\neg\neg x \to x) \cap \widetilde{\varphi}(\neg\neg x) = \widetilde{\varphi}(1) \cap \widetilde{\varphi}(\neg\neg x) = \widetilde{\varphi}(\neg\neg x)$$

and

$$\lambda(x) \leqslant \lambda(\neg\neg x \to x) \vee \lambda(\neg\neg x) = \lambda(1) \vee \lambda(\neg\neg x) = \lambda(\neg\neg x).$$

Therefore, by Theorem 4.11, $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U. Conversely, by Lemma 2.3 (i), (iii) and (viii), for any $x \in L$, we have

$$x \leqslant ((x \to 0) \to 0) \to x \quad \text{and so} \quad (((x \to 0) \to 0) \to x) \to 0 \leqslant x \to 0.$$

Hence,

$$\begin{aligned}(x \to 0) \to (((x \to 0) \to 0) \to x) &\leqslant ((((x \to 0) \to 0) \to x) \to 0) \\ &\quad \to (((x \to 0) \to 0) \to x).\end{aligned}$$

Moreover, since

$$1 = 0 \to x \leqslant ((x \to 0) \to 0) \to ((x \to 0) \to x) = (x \to 0) \to (((x \to 0) \to 0) \to x),$$

we conclude that

$$\begin{aligned}\neg(\neg\neg x \to x) \to (\neg\neg x \to x) &= ((((x \to 0) \to 0) \to x) \to 0) \to (((x \to 0) \to 0) \to x) \\ &= 1.\end{aligned}$$

Now, since $\widetilde{\varphi}_\lambda$ is a hybrid implicative prefilter of L over U, by Theorem 4.11, we get that

$$\begin{aligned}\widetilde{\varphi}(\neg\neg x \to x) &\supseteq \widetilde{\varphi}(\neg(\neg\neg x \to x) \to (\neg\neg x \to x)) = \widetilde{\varphi}(1), \\ \lambda(\neg\neg x \to x) &\leqslant \lambda(\neg(\neg\neg x \to x) \to (\neg\neg x \to x)) = \lambda(1).\end{aligned}$$

Hence, $\widetilde{\varphi}(\neg\neg x \to x) = \widetilde{\varphi}(1)$ and $\lambda(\neg\neg x \to x) = \lambda(1)$ and so $\neg\neg[x] = [\neg\neg x] = [x]$. Therefore, $L/\widetilde{\varphi}_\lambda$ is an involutive EQ-algebra. □

5. Conclusion

The results of this paper are devoted to study the notion of hybrid positive implicative and hybrid implicative (pre)filters of EQ-algebras. By defining these notions, some characterizations of them were studied and some related properties of them were investigated. There are still some questions: How to define the notions of hybrid obstinate and fantastic (pre)filters of EQ-algebras? What is the relation between hybrid obstinate and hybrid fantastic (pre)filters and other types hybrid (pre)filters of EQ-algebras? These could be topics for further research.

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References

[1] *P. B. Andrews*: An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof. Applied Logic Series 27. Kluwer, Dordrecht, 2002. zbl MR doi

[2] *S. Anis, M. Khan, Y. B. Jun*: Hybrid ideals in semigroups. Cogent Math. *4* (2017), Article ID 1352117, 12 pages. zbl MR doi

[3] *M. El-Zekey*: Representable good EQ-algebras. Soft Comput. *14* (2010), 1011–1023. zbl doi

[4] *M. El-Zekey, V. Novák, R. Mesiar*: On good EQ-algebras. Fuzzy Sets Syst. *178* (2011), 1–23. zbl MR doi

[5] *Y. B. Jun, S. Z. Song, G. Muhiuddin*: Hybrid structures and applications. Ann. Commun. Math. *1* (2018), 11–25.

[6] *L. Liu, X. Zhang*: Implicative and positive implicative prefilters of EQ-algebras. J. Intell. Fuzzy Syst. *26* (2014), 2087–2097. zbl MR doi

[7] *N. Mohtashamnia, L. Torkzadeh*: The lattice of prefilters of an EQ-algebra. Fuzzy Sets Syst. *311* (2017), 86–98. zbl MR doi

[8] *D. Molodtsov*: Soft set theory—first results. Comput. Math. Appl. *37* (1999), 19–31. zbl MR doi

[9] *V. Novák*: On fuzzy type theory. Fuzzy Sets Syst. *149* (2005), 235–273. zbl MR doi

[10] *V. Novák*: EQ-algebras: Primary concepts and properties. Proc. Czech-Japan Seminar, Ninth Meeting. Kitakyushu and Nagasaki. Graduate School of Information, Waseda University, Waseda, 2006, pp. 219–223.

[11] *V. Novák, B. De Baets*: EQ-algebras. Fuzzy Sets Syst. *160* (2009), 2956–2978. zbl MR doi

[12] *A. Paad*: Hybrid structures on EQ-algebras. New Math. Nat. Comput. *18* (2022), 365–383. doi

[13] *L. A. Zadeh*: Fuzzy sets. Inf. Control *8* (1965), 338–353. zbl MR doi

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