

OSCILLATION CRITERIA FOR TWO DIMENSIONAL LINEAR
NEUTRAL DELAY DIFFERENCE SYSTEMS

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Abstract. In this work, necessary and sufficient conditions for the oscillation of solutions of 2-dimensional linear neutral delay difference systems of the form

$$\Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

are established, where $m > 0$, $\alpha \geq 0$, $\beta \geq 0$ are integers and $a(n)$, $b(n)$, $c(n)$, $d(n)$, $p(n)$ are sequences of real numbers.

Keywords: oscillation; nonoscillation; system of neutral equations; Krasnoselskii's fixed point theorem

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1. INTRODUCTION

Consider the 2-dimensional difference system

$$(S_1) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix},$$

where $m > 0$, $\alpha \geq 0$, $\beta \geq 0$ are integers and $a(n)$, $b(n)$, $c(n)$, $d(n)$, $p(n)$ are sequences of real numbers for $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \dots\}$, $n_0 \geq 0$. If $\alpha = 0$, $\beta = 0$ and $p(n) \equiv 0$ for all n , then (S_1) reduces to

$$(S_2) \quad \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}.$$

In [17], Tripathy has studied the oscillatory behaviour of solutions of the system (S₂) along with the oscillatory behaviour of solutions of the system

$$(S_3) \quad \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a_1(n) & b_1(n) \\ c_1(n) & d_1(n) \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} + \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}.$$

Indeed, (S₁) and (S₂) are not viewed as the direct discrete analogue of their continuous counterparts, so the work [17] is challenging, being done with the help of the work [11]. In this work, the oscillation and nonoscillation criteria for (S₁) are established unlike to the work [17]. Of course, the study of (S₁) is not so much simple when $\alpha > 0$, $\beta > 0$ and $p(n) \neq 0$ for all n .

In [7], [8], [9], Graef and Thandapani, Jiang and Tang, and Li have studied the oscillatory and asymptotic behaviour of all vector solutions of the system of the form

$$(S_4) \quad \begin{bmatrix} \Delta x(n) \\ \Delta y(n-1) \end{bmatrix} = \begin{bmatrix} 0 & b(n) \\ -c(n) & 0 \end{bmatrix} \begin{bmatrix} f(x(n)) \\ g(y(n)) \end{bmatrix},$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$. We may note that (S₄) is a special case of (S₁), if we let $f(u) = u$ and $g(u) = u$. It is known that a similar kind of results can be obtained for

$$(S_5) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} 0 & b(n) \\ c(n) & 0 \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

as long as the works [7], [8] and [9] are concerned.

Consider a particular case of (S₁) as

$$(S_6) \quad \Delta \begin{bmatrix} x(n) + p(n)x(n-m) \\ y(n) + p(n)y(n-m) \end{bmatrix} = \begin{bmatrix} a(n) & 0 \\ 0 & d(n) \end{bmatrix} \begin{bmatrix} x(n-\alpha) \\ y(n-\beta) \end{bmatrix}$$

from which we find two first-order neutral delay difference equations

$$(1.1) \quad \Delta[x(n) + p(n)x(n-m)] - a(n)x(n-\alpha) = 0,$$

$$(1.2) \quad \Delta[y(n) + p(n)y(n-m)] - d(n)y(n-\beta) = 0.$$

A close observation reveals that the oscillation properties of (1.1) and (1.2) are studied by Parhi and Tripathy in their works [12] and [13] and hence the fact that (1.1) and (1.2) are oscillatory implies that (S₆) is oscillatory when $a(n)d(n) \neq 0$ for all n . Hence, we do not discuss the oscillation properties of (S₁) when either $a(n) = 0 = d(n)$ (as in (S₅)) or $b(n) = 0 = c(n)$ (as in (S₆)) for all n . In this work, our objective is to present the oscillatory behaviour of all vector solutions of (S₁) when $a(n) \neq 0$, $b(n) \neq 0$, $c(n) \neq 0$, $d(n) \neq 0$ for all n . Up to our best understanding, the present work is a new finding in the literature. However, there are some works

(see, e.g., [4], [5], [10], [14], [15], [16]) in which the authors have studied oscillation and nonoscillation properties of some kind of neutral and nonneutral systems of equations that are not in the closed forms like (S_1) , (S_2) and (S_3) . Concerning difference equations and systems of difference equations, we refer to the monographs by Agarwal et al. (see [3], [1]) and by Elyadi (see [6]).

Definition 1.1. By a solution of (S_1) we mean a vector $X(n) = [x(n), y(n)]^\top$ which satisfies (S_1) for $n \in \mathbb{N}(-\varrho) = \{-\varrho, -\varrho + 1, \dots, 0, 1, 2, \dots\}$, where $\varrho = \max\{m, \alpha, \beta\}$. We say that the solution $X(n)$ oscillates componentwise or simply oscillates or strongly oscillates, if each component oscillates. Otherwise, the solution $X(n)$ is called nonoscillatory. Therefore, a solution of (S_1) is nonoscillatory if it has a component which is eventually positive or eventually negative, and strongly nonoscillatory if both components of $X(n)$ are nonoscillatory. A vector solution $X(n)$ of (S_1) has the property that it oscillates or converges to zero as $n \rightarrow \infty$, if each component of $X(n)$ has this property.

Lemma 1.1 ([13]). *Let $f(n)$, $g(n)$ and $p(n)$ be real valued functions of discrete arguments defined for $n \geq n_0$ such that $f(n) = g(n) + p(n)g(n - m)$, $n \geq n_0 + m$, where $m \geq 0$ is an integer. Suppose that there exist real numbers b_1, b_2, b_3, b_4 such that $p(n)$ is in one of the following ranges:*

- (1) $-\infty < b_1 \leq p(n) \leq 0$,
- (2) $0 \leq p(n) \leq b_2 < 1$,
- (3) $1 < b_3 \leq p(n) \leq b_4 < \infty$.

If $g(n) > 0$ for $n \geq n_0$, $\liminf_{n \rightarrow \infty} g(n) = 0$, and $\lim_{n \rightarrow \infty} f(n) = L$ exists, then $L = 0$.

Theorem 1.1 ([2]). *Let X be a Banach space. Let Ω be a bounded closed convex subset of X and let T_1, T_2 be maps of Ω into X such that $T_1x + T_2y \in \Omega$ for every pair $x, y \in \Omega$. If T_1 is a contraction and T_2 is completely continuous, then the equation $T_1x + T_2x = x$ has a solution in Ω .*

2. OSCILLATION CRITERIA

In this section, necessary and sufficient conditions are established for the oscillation of all vector solutions of the system (S_1) .

Theorem 2.1. *Let $0 < p(n) \leq r < 1$ for large n . Assume that $a(n) < 0$, $b(n) > 0$, $c(n) > 0$, $d(n) < 0$ are for large n such that*

$$(A_1) \quad \sum_{n=0}^{\infty} b(n) < \infty, \quad \sum_{n=0}^{\infty} c(n) < \infty.$$

Then every bounded vector solution of (S_1) either strongly oscillates or converges to zero if and only if

$$(A_2) \quad \sum_{n=0}^{\infty} a(n) = -\infty, \quad \sum_{n=0}^{\infty} d(n) = -\infty.$$

Proof. On the contrary, let $X(n) = [x(n), y(n)]^T$ be a strongly nonoscillatory bounded vector solution of (S_1) such that $x(n) > 0$, $x(n-m) > 0$, $x(n-\alpha) > 0$, $x(n-\beta) > 0$ and $y(n) > 0$, $y(n-m) > 0$, $y(n-\alpha) > 0$, $y(n-\beta) > 0$ for $n \geq n_0 > \varrho$. Setting

$$K(n) = \sum_{i=n}^{\infty} b(i)y(i-\beta), \quad T(n) = \sum_{i=n}^{\infty} c(i)x(i-\alpha);$$

$$u(n) = x(n) + p(n)x(n-m), \quad v(n) = y(n) + p(n)y(n-m)$$

for (S_1) , we find that

$$(2.1) \quad \Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leq 0,$$

$$(2.2) \quad \Delta[v(n) + T(n)] = d(n)y(n-\beta) \leq 0$$

for $n \geq n_1 > n_0$. Hence, there exists $n_2 > n_1$ such that $[u(n)+K(n)]$ and $[v(n)+T(n)]$ are monotonic for $n \geq n_2$. Since $u(n) > 0$, $v(n) > 0$ and $\lim_{n \rightarrow \infty} K(n) < \infty$, $\lim_{n \rightarrow \infty} T(n) < \infty$, then $\lim_{n \rightarrow \infty} u(n)$ exists and $\lim_{n \rightarrow \infty} v(n)$ exists. We claim that $\liminf_{n \rightarrow \infty} x(n) = 0 = \liminf_{n \rightarrow \infty} y(n)$. If not, we can find $n_3 > n_2$ such that $x(n-\alpha) > \gamma$ and $y(n-\beta) > \eta$ for $n \geq n_3$. Therefore, summing (2.1) and (2.2) from n_3 to ∞ , we obtain contradictions to the hypothesis (A_2) . So, our claim holds. By Lemma 1.1, it follows that $\lim_{n \rightarrow \infty} u(n) = 0 = \lim_{n \rightarrow \infty} v(n)$. Ultimately, $u(n) \geq x(n)$ and $v(n) \geq y(n)$ implies that $\lim_{n \rightarrow \infty} x(n) = 0 = \lim_{n \rightarrow \infty} y(n)$. The above argument is analogous, if we assume that $x(n) < 0$, $x(n-m) < 0$, $x(n-\alpha) < 0$, $x(n-\beta) < 0$ and $y(n) < 0$, $y(n-m) < 0$, $y(n-\alpha) < 0$, $y(n-\beta) < 0$ for $n \geq n_0 > \varrho$.

Next, we consider the case when $x(n) > 0$, $x(n-m) > 0$, $x(n-\alpha) > 0$, $x(n-\beta) > 0$ and $y(n) < 0$, $y(n-m) < 0$, $y(n-\alpha) < 0$, $y(n-\beta) < 0$ for $n \geq n_0 > \varrho$. Then

$$(2.3) \quad \Delta[u(n) + K(n)] = a(n)x(n-\alpha) \leq 0,$$

$$(2.4) \quad \Delta[v(n) + T(n)] = d(n)y(n-\beta) \geq 0$$

and hence $[u(n) + K(n)]$ and $[v(n) + T(n)]$ are monotonic as well as bounded also for $n \geq n_2$. Consequently, $\lim_{n \rightarrow \infty} [u(n) + K(n)]$ and $\lim_{n \rightarrow \infty} [v(n) + T(n)]$ exist. Using the above argument, it is easy to see that $\lim_{n \rightarrow \infty} X(n) = [0, 0]^T$. The case $x(n) < 0$, $x(n-m) < 0$, $x(n-\alpha) < 0$, $x(n-\beta) < 0$ and $y(n) > 0$, $y(n-m) > 0$, $y(n-\alpha) > 0$, $y(n-\beta) > 0$ for $n \geq n_0 > \varrho$ is similar.

Conversely, let us assume that (A₂) fails to hold. Let \mathbf{B} denote the Banach space of all bounded sequences in \mathbb{R}^2 with the supremum norm, i.e., $\mathbf{B} = \left\{ X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty \right\}$. For a fixed real number $k > 0$, put

$$\Omega = \{X \in \mathbf{B}: x(n), y(n) \in I, n \in \mathbb{N}\},$$

where $I = [\frac{1}{3}k(1-r), k]$. Indeed, $\Omega \subset \mathbf{B}$ is closed, bounded and convex. Due to (A₁), we can find $n_1 > 0$ such that

$$\begin{aligned} \sum_{n=n_1}^{\infty} |a(n)| &< \frac{(1-r)}{6}, & \sum_{n=n_1}^{\infty} |b(n)| &< \frac{(1-r)}{6}, \\ \sum_{n=n_1}^{\infty} |c(n)| &< \frac{(1-r)}{6}, & \sum_{n=n_1}^{\infty} |d(n)| &< \frac{(1-r)}{6}. \end{aligned}$$

Let us define the maps $G, H: \Omega \rightarrow \mathbf{B}$ such that

$$(GX)(n) = \begin{bmatrix} \frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} a(s)x(s-\alpha) \\ \frac{(2+r)k}{3} - p(n)y(n-m) - \sum_{s=n}^{\infty} d(s)y(s-\beta) \end{bmatrix} \quad \text{for } n \geq n_1,$$

$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \begin{bmatrix} - \sum_{s=n}^{\infty} b(s)y(s-\beta) \\ - \sum_{s=n}^{\infty} c(s)x(s-\alpha) \end{bmatrix} \quad \text{for } n \geq n_1,$$

$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$

We rewrite G, H as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega$. Then for $n \geq n_1$,

$$\begin{aligned} (G_1X)(n) + (H_1Y)(n) &= \frac{(2+r)k}{3} - p(n)x(n-m) \\ &\quad - \sum_{s=n}^{\infty} a(s)x(s-\alpha) - \sum_{s=n}^{\infty} b(s)y(s-\beta) \\ &\leq \frac{(2+r)k}{3} + \sum_{s=n}^{\infty} |a(s)|x(s-\alpha) + \sum_{s=n}^{\infty} |b(s)|y(s-\beta) \\ &\leq \frac{(2+r)k}{3} + \frac{(1-r)k}{6} + \frac{(1-r)k}{6} = k \end{aligned}$$

and

$$\begin{aligned}
& (G_1X)(n) + (H_1Y)(n) \\
&= \frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} a(s)x(s-\alpha) - \sum_{s=n}^{\infty} b(s)y(s-\beta) \\
&\geq \frac{(2+r)k}{3} - p(n)x(n-m) - \sum_{s=n}^{\infty} |a(s)|x(s-\alpha) - \sum_{s=n}^{\infty} |b(s)|y(s-\beta) \\
&\geq \frac{(2+r)k}{3} - rk - \frac{(1-r)k}{6} - \frac{(1-r)k}{6} = \frac{k(1-r)}{3}.
\end{aligned}$$

A similar observation can be made for $(G_2X)(n) + (H_2Y)(n)$, $n \geq n_1$. Hence, $GX + HY \in \Omega$. For $X_1, X_2 \in \Omega$, it is easy to verify that

$$\begin{aligned}
|(G_1X_1)(n) - (G_1X_2)(n)| &\leq r|x_1(n-m) - x_2(n-m)| \\
&\quad + \sum_{s=n}^{\infty} |a(s)||x_1(s-\alpha) - x_2(s-\alpha)| \\
&\leq \left[r + \frac{(1-r)}{6} \right] \|x_1 - x_2\| = \frac{(5r+1)}{6} \|x_1 - x_2\|,
\end{aligned}$$

and

$$|(G_2X_1)(n) - (G_2X_2)(n)| \leq \frac{(5r+1)}{6} \|y_1 - y_2\|$$

for $n \geq n_1$ implies that

$$\|GX_1 - GX_2\| \leq \frac{(5r+1)}{6} \|X_1 - X_2\|,$$

that is, G is a contraction mapping.

Next, we show that H is continuous. Let $X_j = [x_j, y_j]^\top \in \Omega$ for any $j \in \mathbb{N}$. Let $X_j(n)$ be such that $x_j(n) \rightarrow x(n)$ and $y_j(n) \rightarrow y(n)$ as $j \rightarrow \infty$. If we choose $X = [x, y]^\top$, then $X_j \in \Omega$ implies that $X \in \Omega$ and hence $x(n), y(n) \in I$ for $n \geq n_1$. Therefore,

$$\begin{aligned}
|(H_1X_j)(n) - (H_1X)(n)| &\leq \sum_{s=n}^{\infty} |b(s)||y_j(s-\beta) - y(s-\beta)| \rightarrow 0 \quad \text{as } j \rightarrow \infty, \\
|(H_2X_j)(n) - (H_2X)(n)| &\leq \sum_{s=n}^{\infty} |c(s)||x_j(s-\alpha) - x(s-\alpha)| \rightarrow 0 \quad \text{as } j \rightarrow \infty
\end{aligned}$$

imply that

$$\|(HX_j) - (HX)\| \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

that is, H is continuous. To complete the proof of the theorem, we need to show that $H\Omega$ is uniformly Cauchy. Indeed, for $\varepsilon > \frac{2}{3}k(1-r) > 0$, we can find $n_2 > n_1$

such that for $n \geq n_2$

$$\sum_{s=n}^{\infty} |b(s)||y(s-\beta)| < \frac{\varepsilon}{2}, \quad \sum_{s=n}^{\infty} |c(s)||x(s-\alpha)| < \frac{\varepsilon}{2}.$$

Hence for $n_4 > n_3 > n_2$, it follows that

$$\begin{aligned} |(H_1X)(n_4) - (H_1X)(n_3)| &= \left| \sum_{s=n_4}^{\infty} b(s)y(s-\beta) - \sum_{s=n_3}^{\infty} b(s)y(s-\beta) \right| \\ &\leq \sum_{s=n_4}^{\infty} |b(s)||y(s-\beta)| + \sum_{s=n_3}^{\infty} |b(s)||y(s-\beta)| < \varepsilon \end{aligned}$$

and

$$\begin{aligned} |(H_2X)(n_4) - (H_2X)(n_3)| &= \left| \sum_{s=n_4}^{\infty} c(s)x(s-\alpha) - \sum_{s=n_3}^{\infty} c(s)x(s-\alpha) \right| \\ &\leq \sum_{s=n_4}^{\infty} |c(s)||x(s-\alpha)| + \sum_{s=n_3}^{\infty} |c(s)||x(s-\alpha)| < \varepsilon, \end{aligned}$$

that is, $H\Omega$ is uniformly Cauchy.

Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n) = [x(n), y(n)]^\top$ of (S_1) in Ω such that $(GX)(n) + (HX)(n) = X(n)$ for $n \geq n_1$. Keeping in view that

$$(G_1X)(n) + (H_1X)(n) = x(n), \quad (G_2X)(n) + (H_2X)(n) = y(n) \quad \text{for } n \geq n_1,$$

it is easy to verify that $X(n) = [x(n), y(n)]^\top$ is the required vector solution of (S_1) . This completes the proof of the theorem. \square

Theorem 2.2. *Let $1 < t \leq p(n) \leq t_1 \leq \frac{1}{2}t^2 < \infty$ for large n . If (A_1) holds, then the conclusion of Theorem 2.1 remains intact.*

Proof. The sufficient part of the proof is the same as in Theorem 2.1. For the necessary part, let \mathbf{B} denote the Banach space of all bounded sequences in \mathbb{R}^2 with the sup norm, i.e.,

$$\mathbf{B} = \left\{ X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty \right\}.$$

For a fixed real number $k > 0$, put

$$\Omega_1 = \{X \in \mathbf{B}: x(n), y(n) \in I_1, n \in \mathbb{N}\},$$

where $I_1 = [k(t-1)/(8tt_1), k]$. It is easy to see that $\Omega_1 \subset \mathbf{B}$ is closed, bounded and convex. Because of (A₁), we can find $n_1 > 0$ such that

$$\begin{aligned} \sum_{n=n_1}^{\infty} |a(n)| &< \frac{(t-1)}{4t}, & \sum_{n=n_1}^{\infty} |b(n)| &< \frac{(t-1)}{8t_1}, \\ \sum_{n=n_1}^{\infty} |c(n)| &< \frac{(t-1)}{8t_1}, & \sum_{n=n_1}^{\infty} |d(n)| &< \frac{(t-1)}{4t}. \end{aligned}$$

We define the maps $G, H: \Omega_1 \rightarrow \mathbf{B}$ as

$$(GX)(n) = \left[\begin{array}{l} \frac{(2t^2 + t - 1)k}{4tp(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ \frac{(2t^2 + t - 1)k}{4tp(n+m)} - \frac{y(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} d(s)y(s-\beta) \end{array} \right]$$

for $n \geq n_1$,

$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \left[\begin{array}{l} -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} c(s)x(s-\alpha) \end{array} \right] \quad \text{for } n \geq n_1,$$

$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$

We rewrite G, H as

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega_1$. Then for $n \geq n_1$,

$$\begin{aligned} (G_1X)(n) + (H_1Y)(n) &= \frac{(2t^2 + t - 1)k}{4tp(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ &\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |a(s)|x(s-\alpha) \\ &\quad + \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |b(s)|y(s-\beta) \\ &\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{(t-1)k}{8tt_1} + \frac{(t-1)k}{4t^2} \\ &\leq \frac{(2t^2 + t - 1)k}{4t^2} + \frac{(t-1)k}{8t^2} + \frac{(t-1)k}{4t^2} = k \frac{4t^2 + 5t - 5}{8t^2} < k \end{aligned}$$

and

$$\begin{aligned}
(G_1X)(n) + (H_1Y)(n) &= \frac{(2t^2 + t - 1)k}{4tp(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\
&\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\
&\geq \frac{(2t^2 + t - 1)k}{4tt_1} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |b(s)|y(s-\beta) \\
&\geq \frac{(2t^2 + t - 1)k}{4tt_1} - \frac{k}{t} - \frac{(t-1)k}{8tt_1} \\
&= k \frac{4t^2 + t - 8t_1 - 1}{8tt_1} > k \frac{t-1}{8tt_1}.
\end{aligned}$$

A similar observation can be obtained for $(G_2X)(n) + (H_2Y)(n)$, $n \geq n_1$. Hence, $GX + HY \in \Omega_1$. For $X_1, X_2 \in \Omega_1$, it is easy to verify that

$$\begin{aligned}
|(G_1X_1)(n) - (G_1X_2)(n)| &\leq \frac{1}{t} |x_1(n+m) - x_2(n+m)| \\
&\quad + \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |a(s)| |x_1(s-\alpha) - x_2(s-\alpha)| \\
&\leq \left[\frac{1}{t} + \frac{(t-1)}{4t} \right] \|x_1 - x_2\| = \frac{(3+t)}{4t} \|x_1 - x_2\|,
\end{aligned}$$

and

$$|(G_2X_1)(n) - (G_2X_2)(n)| \leq \frac{(3+t)}{4t} \|y_1 - y_2\|$$

for $n \geq n_1$ implies that

$$\|GX_1 - GX_2\| \leq \frac{(3+t)}{4t} \|X_1 - X_2\|,$$

that is, G is a contraction mapping.

Proceeding as in the proof of Theorem 2.1, we can show that H is continuous and $H\Omega_1$ is uniformly Cauchy. Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n) = [x(n), y(n)]^\top$ of (S_1) in Ω_1 such that $(GX)(n) + (HX)(n) = X(n)$ for $n \geq n_1$. Therefore, the theorem is proved. \square

Theorem 2.3. *Let $-1 < r_1 \leq p(n) \leq 0$ for large n . If (A_1) holds, then the conclusion of Theorem 2.1 remains intact.*

P r o o f. Proceeding as in the proof of Theorem 2.1, we can find an $n_2 > n_1$ such that $[u(n) + K(n)]$ and $[v(n) + T(n)]$ are monotonic for $n \geq n_2$. Since $\lim_{n \rightarrow \infty} K(n) < \infty$ and $\lim_{n \rightarrow \infty} T(n) < \infty$, then $\lim_{n \rightarrow \infty} u(n)$ exists and $\lim_{n \rightarrow \infty} v(n)$ exists. Using the same

argument as in the proof of Theorem 2.1, we can show that $\liminf_{n \rightarrow \infty} x(n) = 0 = \liminf_{n \rightarrow \infty} y(n)$. By Lemma 1.1, it follows that $\lim_{n \rightarrow \infty} u(n) = 0 = \lim_{n \rightarrow \infty} v(n)$. Therefore,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} u(n) = \limsup_{n \rightarrow \infty} (x(n) + p(n)x(n-m)) \geq \limsup_{n \rightarrow \infty} (x(n) + r_1x(n-m)) \\ &\geq \limsup_{n \rightarrow \infty} x(n) + \liminf_{n \rightarrow \infty} (r_1x(n-m)) = (1+r_1) \limsup_{n \rightarrow \infty} x(n) \end{aligned}$$

implies that $\lim_{n \rightarrow \infty} x(n) = 0$. Similarly, we can show that $\lim_{n \rightarrow \infty} y(n) = 0$. The rest of the sufficient part follows from Theorem 2.1.

Conversely, assume that (A_2) fails to hold. Let \mathbf{B} denote the Banach space of all bounded sequences in \mathbb{R}^2 with the supremum norm defined by $\mathbf{B} = \{X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty\}$. For a fixed real number $k > 0$, put

$$\Omega_2 = \{X \in \mathbf{B}: x(n), y(n) \in I_2, n \in \mathbb{N}\},$$

where $I_2 = [\frac{1}{12}k(1+r_1), k]$. Indeed, $\Omega_2 \subset \mathbf{B}$ is closed, bounded and convex. Due to (A_1) , we can find $n_1 > 0$ such that

$$\begin{aligned} \sum_{n=n_1}^{\infty} |a(n)| &< \frac{(1+r_1)}{24}, & \sum_{n=n_1}^{\infty} |b(n)| &< \frac{(1+r_1)}{24}, \\ \sum_{n=n_1}^{\infty} |c(n)| &< \frac{(1+r_1)}{24}, & \sum_{n=n_1}^{\infty} |d(n)| &< \frac{(1+r_1)}{24}. \end{aligned}$$

Let us define the maps $G, H: \Omega_2 \rightarrow \mathbf{B}$ such that

$$\begin{aligned} (GX)(n) &= \begin{bmatrix} \frac{(1+r_1)k}{6} - p(n)x(n-m) - \sum_{s=n}^{\infty} a(s)x(s-\alpha) \\ \frac{(1+r_1)k}{6} - p(n)y(n-m) - \sum_{s=n}^{\infty} d(s)y(s-\beta) \end{bmatrix} \quad \text{for } n \geq n_1, \\ (GX)(n) &= (GX)(n_1) \quad \text{for } 0 < n < n_1 \end{aligned}$$

and

$$\begin{aligned} (HX)(n) &= \begin{bmatrix} -\sum_{s=n}^{\infty} b(s)y(s-\beta) \\ -\sum_{s=n}^{\infty} c(s)x(s-\alpha) \end{bmatrix} \quad \text{for } n \geq n_1, \\ (HX)(n) &= (HX)(n_1) \quad \text{for } 0 < n < n_1. \end{aligned}$$

We note that

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

The rest of the proof follows from the proof of Theorem 2.1 and hence the details are omitted. \square

Theorem 2.4. *Let $-\infty < r_2 \leq p(n) \leq r_3 < -1$ for large n . If (A_1) holds, then the conclusion of Theorem 2.1 remains intact.*

Proof. The sufficient part of the proof is similar to that of Theorem 2.3. By Lemma 1.1, it follows that $\lim_{n \rightarrow \infty} u(n) = 0 = \lim_{n \rightarrow \infty} v(n)$. Hence,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} u(n) = \liminf_{n \rightarrow \infty} (x(n) + p(n)x(n-m)) \leq \liminf_{n \rightarrow \infty} (x(n) + r_3x(n-m)) \\ &\leq \limsup_{n \rightarrow \infty} x(n) + \liminf_{n \rightarrow \infty} (r_3x(n-m)) = (1+r_3) \limsup_{n \rightarrow \infty} x(n) \end{aligned}$$

implies that $\lim_{n \rightarrow \infty} x(n) = 0$. Similarly, we can show that $\lim_{n \rightarrow \infty} y(n) = 0$.

For the necessary part of the proof, let \mathbf{B} denote the Banach space of all bounded sequences in \mathbb{R}^2 with the sup norm, i.e., $\mathbf{B} = \left\{ X: \mathbb{N} \rightarrow \mathbb{R}^2: \|X\| = \sup_{n \in \mathbb{N}} |X| < \infty \right\}$. For a fixed real number $k > 0$, put

$$\Omega_3 = \{X \in \mathbf{B}: x(n), y(n) \in I_3, n \in \mathbb{N}\},$$

where $I_3 = [-kr_3/(M-r_3), Lk]$, and

$$M > \max \left\{ -r_2, r_3 + \frac{r_3}{1+r_3} \right\}, \quad L = \frac{2M - (M+1)r_3}{(r_3-M)(1+r_3)} > 0.$$

Indeed, $\Omega_3 \subset \mathbf{B}$ is closed, bounded and convex. Due to (A_1) , we can find $n_1 > 0$ such that

$$\begin{aligned} \sum_{n=n_1}^{\infty} |a(n)| &< \frac{-r_3}{(M-r_3)}, & \sum_{n=n_1}^{\infty} |b(n)| &< \frac{-r_3}{(M-r_3)}, \\ \sum_{n=n_1}^{\infty} |c(n)| &< \frac{-r_3}{(M-r_3)}, & \sum_{n=n_1}^{\infty} |d(n)| &< \frac{-r_3}{(M-r_3)}. \end{aligned}$$

Let us define the maps $G, H: \Omega_3 \rightarrow \mathbf{B}$ such that

$$(GX)(n) = \left[\begin{array}{l} \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{y(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} d(s)y(s-\beta) \end{array} \right]$$

for $n \geq n_1$,

$$(GX)(n) = (GX)(n_1) \quad \text{for } 0 < n < n_1$$

and

$$(HX)(n) = \left[\begin{array}{l} -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ -\frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} c(s)x(s-\alpha) \end{array} \right] \quad \text{for } n \geq n_1,$$

$$(HX)(n) = (HX)(n_1) \quad \text{for } 0 < n < n_1.$$

We note that

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

Let $X, Y \in \Omega_3$. Then for $n \geq n_1$,

$$\begin{aligned} (G_1X)(n) + (H_1Y)(n) &= \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{x(n+m)}{p(n+m)} \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ &\leq \frac{-(2-r_3)Mk}{(M-r_3)r_3} - \frac{x(n+m)}{p(n+m)} - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ &\leq \frac{-(2-r_3)Mk}{(M-r_3)r_3} - \frac{Lk}{r_3} + \frac{Lk}{(M-r_3)} \\ &= -k \left[\frac{L(M-r_3) + 2M - (M+1)r_3}{(M-r_3)r_3} \right] = kL \end{aligned}$$

and

$$\begin{aligned} (G_1X)(n) + (H_1Y)(n) &= \frac{-(2-r_3)Mk}{(M-r_3)p(n+m)} - \frac{x(n+m)}{p(n+m)} \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} a(s)x(s-\alpha) \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} b(s)y(s-\beta) \\ &\geq \frac{-(2-r_3)Mk}{(M-r_3)r_2} - \frac{kr_3}{(M-r_3)r_2} + \frac{1}{r_2} \sum_{s=n+m}^{\infty} |a(s)|x(s-\alpha) \\ &\geq \frac{-(2-r_3)Mk}{(M-r_3)r_2} - \frac{2kr_3}{(M-r_3)r_2} \geq -\frac{kr_3}{(M-r_3)}. \end{aligned}$$

A similar observation can be obtained for $(G_2X)(n) + (H_2Y)(n)$, $n \geq n_1$. Hence, $GX + HY \in \Omega_3$. For $X_1, X_2 \in \Omega_3$, it is easy to verify that

$$\begin{aligned} |(G_1X_1)(n) - (G_1X_2)(n)| &\leq -\frac{1}{r_3}|x_1(n+m) - x_2(n+m)| \\ &\quad - \frac{1}{p(n+m)} \sum_{s=n+m}^{\infty} |a(s)||x_1(s-\alpha) - x_2(s-\alpha)| \\ &\leq \left[-\frac{1}{r_3} + \frac{1}{M-r_3} \right] \|x_1 - x_2\| \end{aligned}$$

and

$$|(G_2X_1)(n) - (G_2X_2)(n)| \leq \left[-\frac{1}{r_3} + \frac{1}{M - r_3} \right] \|y_1 - y_2\|$$

for $n \geq n_1$ implies that

$$\|GX_1 - GX_2\| \leq \left[-\frac{1}{r_3} + \frac{1}{M - r_3} \right] \|X_1 - X_2\|,$$

that is, G is a contraction mapping.

Proceeding as in the proof of Theorem 2.3, we can show that H is continuous and $H\Omega_3$ is uniformly Cauchy. Hence by Krasnoselskii's fixed point theorem, there exists a solution $X(n) = [x(n), y(n)]^\top$ of (S_1) in Ω_3 such that $(GX)(n) + (HX)(n) = X(n)$ for $n \geq n_1$. Therefore, the theorem is proved. \square

Remark 2.1. It would be interesting to keep this work up for any solution of the system (S_1) (i.e., not necessarily the bounded solution).

Example 2.1. Consider a 2-dimensional linear neutral difference system of the form:

$$(S_7) \quad \Delta \begin{bmatrix} x(n) + e^{-n}x(n-2) \\ y(n) + e^{-n}y(n-2) \end{bmatrix} = \begin{bmatrix} -(2 + e^{-n} + 2e^{-(n+1)}) & e^{-(n+2)} \\ e^{-n} & -(2 + e^{-n} + 2e^{-(n+1)}) \end{bmatrix} \\ \times \begin{bmatrix} x(n-4) \\ y(n-6) \end{bmatrix} \quad \text{for } n > 6.$$

Clearly, (A_1) and (A_2) are satisfied for (S_7) . By Theorem 2.1, every bounded vector solution $X(n)$ of (S_7) is strongly oscillatory. Indeed, $X(n) = [(-1)^n, e(-1)^n]^\top$ is one of such solutions of (S_7) .

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