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DOES THE ENDOMORPHISM POSET P^P DETERMINE WHETHER A FINITE POSET P IS CONNECTED? AN ISSUE DUFFUS RAISED IN 1978

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Abstract. Duffus wrote in his 1978 Ph.D. thesis, "It is not obvious that P is connected and $P^P \cong Q^Q$ imply that Q is connected", where P and Q are finite nonempty posets. We show that, indeed, under these hypotheses Q is connected and $P \cong Q$.

Keywords: (partially) ordered set; exponentiation; connected

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1. Introduction

In the 1979 *Proceedings of the American Mathematical Society*, Duffus and Wille proved that $P^P \cong Q^Q$ implies $P \cong Q$ if P and Q are both finite, nonempty, and connected (see [8], Theorem, page 14). Duffus said in 1984 in [6], page 90, "It is still an open problem to show connectedness can be dropped." He notes in his 1978 thesis [4], page 53, "It is not obvious that P is connected and $P^P \cong Q^Q$ imply that Q is connected." (Note that, if this is the case, then the Duffus-Wille result implies $P \cong Q$.) We prove that, indeed, if P and Q are finite, nonempty posets such that $P^P \cong Q^Q$ and P is connected, then Q is connected.

In other words, we have resolved the issue from Duffus's 1978 thesis in that we have "half-dropped" the connectedness hypothesis used in the 1979 *Proceedings of the American Mathematical Society* paper, whereas the 1984 problem asked if it could be dropped entirely.

We assume the reader is familiar with the basic facts about the arithmetic of ordered sets, the basic consequences of Hashimoto's refinement theorem, and Professor

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Birkhoff's theorem on finite distributive lattices (e.g., [16], Propositions 3.1 and 4.1 and [3], Theorems 5.9 and 5.12 and also see [17]).

Let P and Q be posets. For $p, p' \in P$, we write $p \equiv p'$ if p and p' are in the same connected component of P; $\mathrm{h}(P)$ denotes the height of the finite, nonempty poset P, the largest value of $|C| - 1$ for C a chain in P, and $\mathrm{h}_P(p)$ denotes the height of an element p in P. Note that $\mathrm{h}(P \times Q) = \mathrm{h}(P) + \mathrm{h}(Q)$ for P and Q finite and nonempty (see [2], Chapter I, §9, Exercise 4 (a)).

Let P^Q denote the poset of order-preserving maps from Q to P, where, for $f, g \in P^Q$,

$$f \leqslant g \quad \text{if for all } q \in Q, \quad f(q) \leqslant_P g(q)$$

(see [1], page 312). For $p \in P$, denote the constant map $f(q) = p$ for all $q \in Q$ by $\langle p \rangle$. Let $\mathcal{D}(P^Q)$ denote

$$\{g \in P^Q \colon\ g \text{ is constant on each connected component of } Q\}$$

and for $Q \neq \emptyset$ let $\mathcal{C}(P^Q)$ denote

$$\{f \in P^Q \colon\ f \equiv g \text{ for some } g \in \mathcal{D}(P^Q)\}$$

(see [16], Definition 2.1). Note that, despite the notation, $\mathcal{C}(P^Q)$ is really a function of P and Q, not P^Q (as far as the author knows).

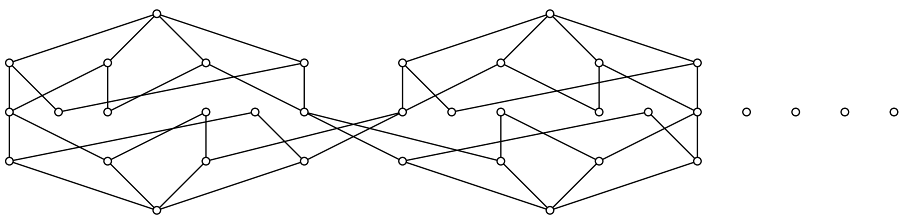

Figure 1. P^P where P is the 4-element crown, see [4], page 54.

The n-element chain is denoted $\mathbf{n}$. A poset P is *directly irreducible* if $|P| \neq 1$ and whenever $P \cong A \times B$ for posets A and B, then $|A| = 1$ or $|B| = 1$. A finite poset P is *absolutely* $\mathcal{C}$*-indecomposable* if it is connected, directly irreducible, and, whenever $P \cong \mathcal{C}(A^B)$ for posets A and B, $B \neq \emptyset$, we have $P \cong A$ and $|B| = 1$.

The following result comes from a remark McKenzie leaves to the reader after [16], Proposition 4.1, which nonetheless is true for finite, nonempty posets. (For a proof of (1), see [11], Theorem 2.8.)

Proposition 1. *Let P, Q, and R be nonempty posets. Then:*

(1) $\mathcal{C}(P^{Q\times R}) \cong \mathcal{C}(\mathcal{C}(P^Q)^R)$ *if P, Q, and R are finite;*
(2) $\mathcal{C}(P^{Q+R}) \cong \mathcal{C}(P^Q) \times \mathcal{C}(P^R)$*;*
(3) *if R is connected,* $\mathcal{C}((P+Q)^R) \cong \mathcal{C}(P^R) + \mathcal{C}(Q^R)$*;*
(4) $\mathcal{C}((Q\times R)^P) \cong \mathcal{C}(Q^P) \times \mathcal{C}(R^P)$*.*

We also use the structure theorems of McKenzie (see [16], Theorems 8.1, 9.1–9.2 and [15], Theorem 5.1):

Theorem 2.

(1) *Let A, B, C, and D be finite, nonempty, connected posets. Assume C and D are directly irreducible and $C \not\cong D$. Then if $\mathcal{C}(A^C) \cong \mathcal{C}(B^D)$, there exists a finite, nonempty, connected poset E such that $A \cong \mathcal{C}(E^D)$ and $B \cong \mathcal{C}(E^C)$. (Note that "A^Q" should be "E^Q" on* [16]*, page* 211.*)*
(2) *Let A, B, and C be finite, nonempty posets. Assume C is connected. Then $A \cong B$ if $\mathcal{C}(A^C) \cong \mathcal{C}(B^C)$.*
(3) *Let $n \in \mathbb{N}$. Let A and B be finite, nonempty, connected posets, and let $C_1, \ldots, C_n$ be posets such that $\mathcal{C}(A^B) \cong C_1 \times \ldots \times C_n$. Then there exist finite, nonempty posets $A_1, \ldots, A_n$ such that $C_i \cong \mathcal{C}(A_i^B)$ for $i = 1, \ldots, n$ and $A \cong A_1 \times \ldots \times A_n$.*
(4) *Let A be a finite, connected, directly irreducible poset. Then there exists an absolutely $\mathcal{C}$-indecomposable connected poset B and a finite, nonempty poset P such that $A \cong \mathcal{C}(B^P)$.*

We now extend parts of Theorem 2 to the analogue of [12], Theorem 8.2, using similar steps. The proofs of Lemma 3 and Theorem 4 first occurred in [9], but an editor asked that they be included as [10], Lemma 8 and Theorem 9.

Lemma 3. *Let A, B, C, and D be finite, nonempty, connected posets such that $\mathcal{C}(A^C) \cong \mathcal{C}(B^D)$. Assume that no nontrivial poset is isomorphic to both a direct factor of C and a direct factor of D. Then there exists a finite, nonempty, connected poset E such that $A \cong \mathcal{C}(E^D)$ and $B \cong \mathcal{C}(E^C)$.*

Theorem 4. *Let A, B, C, and D be finite, nonempty, connected posets such that $\mathcal{C}(A^C) \cong \mathcal{C}(B^D)$. Then there exist finite, nonempty, connected posets E, X, Y, and Z such that $A \cong \mathcal{C}(E^X)$, $B \cong \mathcal{C}(E^Y)$, $C \cong Y \times Z$, and $D \cong X \times Z$.*

P r o o f. Hashimoto's refinement theorem tells us we can find finite, nonempty, connected posets X, Y, and Z such that $C \cong Y \times Z$, $D \cong X \times Z$, and X and Y do not share a nontrivial direct factor. Thus, by Proposition 1 (1),

$$\mathcal{C}(\mathcal{C}(A^Y)^Z) \cong \mathcal{C}(A^{Y\times Z}) \cong \mathcal{C}(A^C) \cong \mathcal{C}(B^D) \cong \mathcal{C}(B^{X\times Z}) \cong \mathcal{C}(\mathcal{C}(B^X)^Z).$$

Now $\mathcal{C}(A^Y), \mathcal{C}(B^X) \neq \emptyset$ since A, B, X, and Y are nonempty and connected. Hence, by Theorem 2 (2), $\mathcal{C}(A^Y) \cong \mathcal{C}(B^X)$. By Lemma 3, there exists a finite, nonempty, connected poset E such that $A \cong \mathcal{C}(E^X)$ and $B \cong \mathcal{C}(E^Y)$. □

In [4], Theorem 3.2.6, Duffus proves an outstanding version of Hashimoto's refinement theorem for posets A, B, C, and D that are sums of connected posets with a finite maximal chain such that $A \times B \cong C \times D$ and A and C are connected. He never published this proof, although it is used in [8]. We only require one of the four posets to be connected. (We don't actually need this result, but the structure of its proof helps the reader understand the proof of the theorem we do need: this proof foreshadows the proof of Theorem 10.)

Theorem 5. *Let A, B, C, and D be finite posets. Let A be connected and nonempty. Assume that $A \times B \cong C \times D$. Then there exist posets W, X, Y, and Z such that $A \cong W \times X$, $B \cong Y \times Z$, $C \cong W \times Y$, and $D \cong X \times Z$.*

P r o o f. Note that $B = \emptyset$ if and only if either $C = \emptyset$ or $D = \emptyset$. First assume $B = \emptyset = C$. Then let $W = A$, $X = \mathbf{1}$, $Y = \emptyset$, and $Z = D$. Next, assume $B = \emptyset = D$. Then let $W = \mathbf{1}$, $X = A$, $Y = C$, and $Z = \emptyset$.

From now on, assume that B, C, and D are nonempty.

Let $B = \sum\limits_{b\in H} B_h$, $C = \sum\limits_{i\in I} C_i$, and $D = \sum\limits_{j\in J} D_j$ be the decompositions of B, C, and D, respectively, into connected components. Since $A \times B \cong \sum\limits_{h\in H} A \times B_h$ has the same number of components as $C \times D \cong \sum\limits_{i\in I,\, j\in J} C_i \times D_j$, there is a bijection $\Psi\colon I \times J \to H$ such that $C_i \times D_j \cong A \times B_{\Psi(i,j)}$ for all $i \in I$, $j \in J$.

Let $A_1, A_2, \ldots, A_r$ ($r \geqslant 0$) be the pairwise nonisomorphic connected, directly irreducible posets that could arise in the factorizations of any of A, B_h ($h \in H$), C_i ($i \in I$), and D_j ($j \in J$). Say $A \cong \prod\limits_{l=1}^{r} (A_l)^{k_l}$ where $k_l \geqslant 0$ ($l = 1, \ldots, r$). For each $l \in \{1, \ldots, r\}$, let $c_l \in \mathbb{N}_0$ be the highest power of A_l such that $A_l^{c_l}$ is a factor of all of the C_i ($i \in I$). Let $W := \prod\limits_{l=1}^{r} A_l^{\min\{c_l,k_l\}}$. Let $X := \prod\limits_{l=1}^{r} A_l^{k_l-\min\{c_l,k_l\}}$. Then $W \times X \cong \prod\limits_{l=1}^{r} A_l^{k_l} \cong A$. Clearly, for all $i \in I$, W is a factor of C_i. Hence, we may let $\widetilde{C}_i$ be C_i with W factored out, that is, $C_i \cong \widetilde{C}_i \times W$ ($i \in I$). Let $Y := \sum\limits_{i\in I} \widetilde{C}_i$.

Claim 6. *For each $j \in J$, X is a factor of D_j.*

P r o o f. Assume for a contradiction that X does not divide D_j for some $j \in J$. Then there exists $l \in \{1, \ldots, r\}$ such that $A_l^{k_l - c_l}$ does not divide D_j and so $k_l > c_l$. Pick $i \in I$ such that c_l is the highest power of A_l dividing C_i. Then the highest power of A_l dividing $C_i \times D_j$ is less than $c_l + k_l - c_l = k_l$, a contradiction, since $C_i \times D_j \cong A \times B_{\Psi(i,j)}$ and $A_l^{k_l}$ divides the right-hand side. $\square$

By Claim 6, we may let $\widetilde{D}_j$ be D_j with X factored out ($j \in J$). Let $Z = \sum_{j \in J} \widetilde{D}_j$.

Claim 7. *For all* $(i, j) \in I \times J$, $B_{\Psi(i,j)} \cong \widetilde{C}_i \times \widetilde{D}_j$.

P r o o f. We know

$$A \times B_{\Psi(i,j)} \cong C_i \times D_j \cong W \times X \times \widetilde{C}_i \times \widetilde{D}_j \cong A \times \widetilde{C}_i \times \widetilde{D}_j,$$

so $B_{\Psi(i,j)} \cong \widetilde{C}_i \times \widetilde{D}_j$. (See [13], (4.3).) $\square$

Thus $B \cong Y \times Z$. By definition, $C \cong W \times Y$ and $D \cong X \times Z$. $\square$

A "strong" or "strict" version of Theorem 5 would be useful (see [16], Proposition 3.1).

Lemma 8. *Let A be a finite, connected, directly irreducible poset. Let B and P be posets such that $P \neq \emptyset$. If $A \cong \mathcal{C}(B^P)$, then P is finite and connected, and B is finite, connected, and directly irreducible.*

P r o o f. We have $B \neq \emptyset$. Also $|B| \geqslant 2$. If $P = C + D$ for posets $C, D \neq \emptyset$, then, by Proposition 1 (2), $A \cong \mathcal{C}(B^C) \times \mathcal{C}(B^D)$. Therefore since A is directly irreducible, $\mathbf{1} \cong \mathcal{C}(B^C)$ or $\mathbf{1} \cong \mathcal{C}(B^D)$, a contradiction ($|\mathcal{C}(B^C)|$, $|\mathcal{C}(B^D)| \geqslant 2$ since $C, D \neq \emptyset$). Hence P is connected.

If $B = E + F$ for posets $E, F \neq \emptyset$, then by Proposition 1 (3), $A \cong \mathcal{C}(E^P) + \mathcal{C}(F^P)$, a contradiction. Hence B is connected. Thus there exist $x, y \in B$ such that $x < y$, so if P is infinite, then $|P| \leqslant |\mathbf{2}^P| \leqslant |\mathcal{C}(B^P)|$, a contradiction. Hence P is finite. Also, B is finite. If $B \cong G \times H$ for posets G and H, then by Proposition 1 (4), $A \cong \mathcal{C}(G^P) \times \mathcal{C}(H^P)$, therefore, by the direct irreducibility of A, without loss of generality $|\mathcal{C}(G^P)| = 1$, so $|G| = 1$. $\square$

Lemma 9. *Let A be a connected, finite, directly irreducible poset. Let B be a finite, nonempty, connected poset. Then:*

(1) *$\mathcal{C}(A^B)$ is connected and directly irreducible.*
(2) *Let C and D be nonempty posets. If $\mathcal{C}(C^D) \cong \mathcal{C}(A^B)$, then C is directly irreducible, finite, and connected, and D is finite and connected.*

(3) *There exist unique (up to isomorphism) nonempty posets E and F with the following two properties:*
(a) $\mathcal{C}(A^B) \cong \mathcal{C}(E^F)$ *and*
(b) *whenever C and D are nonempty posets such that $\mathcal{C}(A^B) \cong \mathcal{C}(C^D)$, then there exists a poset G such that $\mathcal{C}(E^G) \cong C$ and $G \times D \cong F$.*

We can choose as E any absolutely $\mathcal{C}$-indecomposable poset E such that $\mathcal{C}(E^J) \cong \mathcal{C}(A^B)$ for some nonempty poset J, and we can choose that J as our "F". If H is a finite, nonempty, connected poset, then E and $F \times H$ are the posets that work for $\mathcal{C}(A^{B\times H})$.

Proof. (1) The poset $\mathcal{C}(A^B)$ is nontrivial since A is nontrivial. By Theorem 2 (3), $\mathcal{C}(A^B)$ is directly irreducible. It is connected since A and B are connected.

(2) See Lemma 8.

(3) Using (1), take the poset E given by Theorem 2 (4). (We want E that is absolutely $\mathcal{C}$-indecomposable.) There exists a nonempty finite poset F such that $\mathcal{C}(E^F) \cong \mathcal{C}(A^B)$. By (2), F is connected.

If $\mathcal{C}(A^B) \cong \mathcal{C}(C^D)$, where C and D are nonempty posets, then by (2) C and D are finite and connected and $\mathcal{C}(E^F) \cong \mathcal{C}(C^D)$, so by Theorem 4, there exist nonempty, finite, connected posets U, G, W, and X such that

$$C \cong \mathcal{C}(U^G), \quad E \cong \mathcal{C}(U^W), \quad D \cong W \times X, \quad F \cong G \times X.$$

By absolute $\mathcal{C}$-indecomposability, $|W| = 1$ and $E \cong U$, so the result follows.

If E' and F' have the same property as E and F, then there exists a poset G' such that $\mathcal{C}(E'^{G'}) \cong E$ and $G' \times F \cong F'$. By absolute $\mathcal{C}$-indecomposability, $E' \cong E$ and $|G'| = 1$, so $F \cong F'$. □

Theorem 10. *Let A, B, C, and D be finite nonempty posets such that A is connected, B and D are nontrivial and connected, and $\mathcal{C}(B^A) \cong \mathcal{C}(D^C)$. Then there exist finite, nonempty posets W, X, Y, and Z such that Z is nontrivial and $A \cong W \times X$, $B \cong \mathcal{C}(Z^Y)$, $C \cong W \times Y$, and $D \cong \mathcal{C}(Z^X)$.*

Proof. Let $B = \prod_{h\in H} B_h$ be a representation of B as a product of connected, directly irreducible posets. Let $D = \prod_{j\in J} D_j$ be a similar product. Let $C = \sum_{i\in I} C_i$ be a decomposition into connected components. Let A_l $(l \in L)$ be the finitely many pairwise nonisomorphic connected, directly irreducible posets that could arise in the factorizations of A and C_i $(i \in I)$. Say $A \cong \prod_{l\in L} A_l^{k_l}$ where $k_l \geqslant 0$ $(l \in L)$. For each $l \in L$, let $c_l \in \mathbb{N}_0$ be the maximum power of A_l such that $A_l^{c_l}$ is a factor of all of the C_i $(i \in I)$. That means, of course, that it is the highest power of A_l in one of the C_i. Let $W = \prod_{l\in L} A_l^{\min\{c_l,k_l\}}$. Let $X = \prod_{l\in L} A_l^{k_l-\min\{c_l,k_l\}}$. Then

$$W \times X \cong \prod_{l\in L} A_l^{k_l} \cong A.$$

Claim 11. *For all $i \in I$, W is a factor of C_i.*

P r o o f. We made sure that $A_l^{c_l}$ is a factor of every C_i, and W is a product of all of those powers of A_l or even smaller powers. Since the different A_l are pairwise nonisomorphic and we are working with connected posets, by Hashimoto's refinement theorem we have "unique factorization", so $A_l^{c_l} \times A_{l'}^{c_{l'}}$ is a factor if and only if $A_l^{c_l}$ is a factor, and $A_{l'}^{c_{l'}}$ is a factor, when $l \neq l'$. $\square$

By Claim 11, we may let $\widetilde{C}_i$ be C_i with W factored out ($i \in I$). Let $Y = \sum_{i\in I} \widetilde{C}_i$. Then $W \times Y \cong \sum_{i\in I} W \times \widetilde{C}_i \cong \sum_{i\in I} C_i = C$.

Note that by Proposition 1 (1),

$$\mathcal{C}(B^A) \cong \mathcal{C}(\mathcal{C}(B^X)^W) \quad \text{and} \quad \mathcal{C}(D^C) \cong \mathcal{C}(\mathcal{C}(D^Y)^W),$$

and by Theorem 2 (2), since W, being a factor of a finite, nonempty, connected poset, is finite, nonempty, and connected, $\mathcal{C}(B^X) \cong \mathcal{C}(D^Y)$.

Claim 12. *X and Y have no nontrivial factor in common.*

P r o o f. Assume for a contradiction that X and Y do have a nontrivial factor in common. Then since X is a factor of A, we may assume the common factor is A_l for some $l \in L$. As A is connected, for A_l to be a factor of Y, it must be a factor of $\widetilde{C}_i$ for all $i \in I$. But then we would have pulled it out with W, as it were. To be precise, since A_l is a factor of X, we must have $c_l < k_l$, and so there is $i \in I$ such that $\widetilde{C}_i$ has no factor of A_l, a contradiction. $\square$

Write $\mathcal{C}(B^X) \cong \prod_{h\in H} \mathcal{C}(B_h^X)$ and $\mathcal{C}(D^Y) \cong \prod_{i\in I}\prod_{j\in J} \mathcal{C}(D_j^{\widetilde{C}_i})$. By Lemma 9 (1), $\mathcal{C}(B_h^X)$ ($h \in H$) and $\mathcal{C}(D_j^{\widetilde{C}_i})$ ($i \in I$, $j \in J$) are directly irreducible. As those factors are connected, by [5], Corollary 2 (cf. [14], Theorem 6.4) there is a bijection $\Psi\colon I\times J \to H$ such that $\mathcal{C}(D_j^{\widetilde{C}_i}) \cong \mathcal{C}(B_h^X)$ for all $(i,j) \in I \times J$. By Theorem 4 there are finite, nonempty, connected posets U_h, R_h, S_h, and T_h (where $h = \Psi(i,j)$) such that

$$B_h \cong \mathcal{C}(U_h^{R_h}), \quad D_j \cong \mathcal{C}(U_h^{S_h}), \quad X \cong S_h \times T_h, \quad \text{and} \quad \widetilde{C}_i \cong R_h \times T_h.$$

Moreover, by Lemma 8, U_h is directly irreducible.

Claim 13. *For each $j \in J$, choose some $i' \in I$ and let E_j and F_j be the posets corresponding to D_j (where $h = \Psi(i',j)$) given by part* (3) *of Lemma 9. Then X is a factor of F_j.*

P r o o f. By part (2) of Lemma 9, F_j is finite and connected. Assume for a contradiction that X does not divide F_j. Then there exists $l \in L$ such that $A_l^{k_l - c_l}$ does not divide F_j, and so $k_l > c_l$. Pick $i \in I$ such that c_l is the highest power of A_l dividing C_i. Consider

$$\mathcal{C}(D_j^{C_i}) \cong \mathcal{C}(D_j^{\widetilde{C}_i \times W}) \cong \mathcal{C}(\mathcal{C}(D_j^{\widetilde{C}_i})^W) \cong \mathcal{C}(\mathcal{C}(B_{\Psi(i,j)}^X)^W) \cong \mathcal{C}(B_{\Psi(i,j)}^{X \times W}) \cong \mathcal{C}(B_{\Psi(i,j)}^A).$$

By part (3) of Lemma 9, A is a factor of $F_j \times C_i$ and $A_l^{k_l}$ divides A, so $A_l^{k_l - c_l}$ divides F_j, a contradiction. □

By Claim 13, we may let $\widetilde{F}_j$ be F_j with X factored out ($j \in J$).

Claim 14. *For all* $(i,j) \in I \times J$, $B_{\Psi(i,j)} \cong \mathcal{C}(E_j^{\widetilde{C}_i \times \widetilde{F}_j})$.

P r o o f. Note that

$$\mathcal{C}(B_{\Psi(i,j)}^A) \cong \mathcal{C}(D_j^{C_i}) \cong \mathcal{C}(\mathcal{C}(E_j^{F_j})^{C_i}) \cong \mathcal{C}(\mathcal{C}(E_j^{X \times \widetilde{F}_j})^{W \times \widetilde{C}_i}) \cong \mathcal{C}(\mathcal{C}(E_j^{\widetilde{C}_i \times \widetilde{F}_j})^A)$$

so $B_{\Psi(i,j)} \cong \mathcal{C}(E_j^{\widetilde{C}_i \times \widetilde{F}_j})$ by Theorem 2 (2) since A is connected. Thus

$$B \cong \prod_{i \in I,\, j \in J} \mathcal{C}(\mathcal{C}(E_j^{\widetilde{F}_j})^{\widetilde{C}_i}) \cong \prod_{j \in J} \mathcal{C}(\mathcal{C}(E_j^{\widetilde{F}_j})^Y) \cong \mathcal{C}\Big(\Big(\prod_{j \in J} \mathcal{C}(E_j^{\widetilde{F}_j})\Big)^Y\Big).$$

Let $Z = \prod_{j \in J} \mathcal{C}(E_j^{\widetilde{F}_j})$. We know

$$D \cong \prod_{j \in J} D_j \cong \prod_{j \in J} \mathcal{C}(E_j^{F_j}) \cong \prod_{j \in J} \mathcal{C}(\mathcal{C}(E_j^{\widetilde{F}_j})^X) \cong \mathcal{C}\Big(\Big(\prod_{j \in J} \mathcal{C}(E_j^{\widetilde{F}_j})\Big)^X\Big) \cong \mathcal{C}(Z^X).$$

□

We also need a trivial extension of [15], Lemma 2.3 (although the added trivialities take up perhaps more space than is warranted).

Lemma 15. *Let A and B be finite posets. Then:*

(1) $A^B = \emptyset$ *if and only if* $A = \emptyset$ *and* $B \neq \emptyset$;

(2) A^B *is an antichain if and only if A is an antichain or $B = \emptyset$, in which case*

$$|A^B| = \begin{cases} |A|^c & \text{if } A \neq \emptyset \text{ or } B \neq \emptyset, \text{ where } c \text{ is the number} \\ & \text{of connected components of } B, \\ 1 & \text{if } A = B = \emptyset; \end{cases}$$

(3) *let $f, g \in A^B$ be such that $f \leqslant g$. Assume $A \neq \emptyset$. Then $\mathrm{h}_{A^B}([f,g]) = \mathrm{h}(A^B)$ if and only if $f, g \in \mathcal{D}(A^B)$ and $\mathrm{h}_A([f(b), g(b)]) = \mathrm{h}(A)$ for all $b \in B$;*

(4) *when $A \neq \emptyset$, $\mathrm{h}(A^B) = \mathrm{h}(A)|B|$.*

P r o o f. (1) This is clear.

(2) Assume A^B is an antichain. If A is not an antichain and $B \neq \emptyset$, then $\{\langle a \rangle : a \in A\} \cong A$ is a subposet of A^B, a contradiction. Conversely, if $B = \emptyset$, then $c = 0$ and $|A^B| = 1$, so A^B is an antichain. If $B \neq \emptyset$ but A is an antichain, then for all $f, g \in A^B$, if $f < g$, there exists $b \in B$ such that $f(b) < g(b)$ in A, a contradiction. Hence A^B is an antichain. For all $f \in A^B$ and for every connected component C of B, $|f[C]| = 1$, so $|A^B| = |A|^c$.

(3) This is [15], Lemma 2.3 (3) if $f < g$. Now assume $f = g$. If $\mathrm{h}_{A^B}([f, g]) = \mathrm{h}(A^B)$, then A^B is a nonempty antichain, so by (2) either $B = \emptyset$ and the forward implication is vacuous or $B \neq \emptyset$ and A is a nonempty antichain, so by the above, $f, g \in \mathcal{D}(A^B)$ and for all $b \in B$, $f(b) = g(b)$, so $\mathrm{h}_A([f(b), g(b)]) = 0 = \mathrm{h}(A)$.

Conversely, suppose $f, g \in \mathcal{D}(A^B)$ and $0 = \mathrm{h}_A([f(b), g(b)]) = \mathrm{h}(A)$ for all $b \in B$. If $B = \emptyset$ then $\mathrm{h}(A^B) = \mathrm{h}(\mathbf{1}) = 0 = \mathrm{h}_{A^B}([f, g])$. If $B \neq \emptyset$, let $b_0 \in B$. The fact $\mathrm{h}_A([f(b_0), g(b_0)]) = 0 = \mathrm{h}(A)$ means A is an antichain.

By (1) and (2), A^B is a nonempty antichain, so $\mathrm{h}(A^B) = 0 = \mathrm{h}_{A^B}([f, g])$.

(4) This is true by [7], Corollary 2.2 if $A, B \neq \emptyset$. If $A \neq \emptyset$ but $B = \emptyset$, then $\mathrm{h}(A^B) = \mathrm{h}(\mathbf{1}) = 0 = \mathrm{h}(A) \cdot 0$. □

Theorem 16. *Let P and Q be finite, nonempty posets such that P is connected. Assume $P^P \cong Q^Q$. Then Q is connected, and therefore $P \cong Q$.*

P r o o f. If Q is connected, then $P \cong Q$ by [8], Theorem. Assume now that Q is disconnected. Say $Q = Q_0 + D$, where $D \neq \emptyset$ and Q_0 is any component of Q of maximum height.

We show that Q_0 is the unique component of Q of height $\mathrm{h}(Q)$, and we will also show that Q_0 is a proper direct factor of P.

Claim 17. *P^P and P are not antichains. Hence $|Q_0| \neq 1$.*

P r o o f. By our assumption, $|Q| \geqslant 2$, therefore $|Q^Q| = |P^P| \geqslant 2$. Hence $|P| \geqslant 2$. As P is connected, P is not an antichain and, thus P^P is not an antichain. (Look at two constant maps, the images of which form a two-element chain.) If $|Q_0| = 1$, then $\mathrm{h}(Q) = 0$ and thus Q is an antichain, so $Q^Q \cong P^P$ is an antichain. □

Pick $q_0 \in Q_0$ of maximum height in Q. Using Lemma 15, $\langle q_0 \rangle$ corresponds to $\langle p_0 \rangle$ for some p_0 of maximum height in P. Since P is connected, $\mathcal{C}(P^P) = \{f \in P^P : f \equiv \langle p_0 \rangle\}$.

Claim 18. $\mathcal{C}(Q_0^Q) = \{g \in Q^Q : g \equiv \langle q_0 \rangle\}$.

P r o o f. Let $g \in Q^Q$ be such that $g \equiv \langle q_0 \rangle$. Then $g[Q] \subseteq Q_0$ and $g \in \mathcal{C}(Q_0^Q)$.

Let $h \in \mathcal{C}(Q_0^Q)$. Then $h \in Q^Q$, and, in Q_0^Q, $h \equiv k$ for some $k \in \mathcal{D}(Q_0^Q)$, and $k \equiv \langle q \rangle$ for some $q \in Q_0$. But Q_0 is connected, so $\langle q \rangle \equiv \langle q_0 \rangle$. Hence $h \equiv \langle q_0 \rangle$. □

We conclude that $\mathcal{C}(P^P) \cong \mathcal{C}(Q_0^Q)$ via the original isomorphism. This is because we have described both $\mathcal{C}(P^P)$ and $\mathcal{C}(Q_0^Q)$ in terms just involving P^P and Q^Q, respectively, and in the same way, up to the isomorphism, since the original isomorphism maps $\langle p_0 \rangle$ to $\langle q_0 \rangle$. Because the preceding argument can be repeated for any component of Q of maximum height, Q_0 must be the only component with height equal to that of Q.

By Proposition 1 (2), $\mathcal{C}(Q_0^Q) \cong \mathcal{C}(Q_0^{Q_0}) \times \mathcal{C}(Q_0^D) \cong \mathcal{C}(P^P)$. By Theorem 2 (3), there exist finite, nonempty, connected posets A_1 and A_2 such that $P \cong A_1 \times A_2$ and $\mathcal{C}(Q_0^{Q_0}) \cong \mathcal{C}(A_1^P)$. By Theorem 4, there exist nonempty, finite, connected posets E, X, Y, and Z such that

$$A_1 \cong \mathcal{C}(E^X), \quad Q_0 \cong \mathcal{C}(E^Y), \quad Q_0 \cong X \times Z, \quad P \cong Y \times Z.$$

Since E and Y are connected, there exists a finite, nonempty, connected poset F such that $X \cong \mathcal{C}(F^Y)$ by Theorem 2 (3) applied to the second and third isomorphisms above. Now suppose $|F| > 1$. Since F is connected, it contains $\mathbf{2}$ and thus we see $\mathcal{C}(F^Y)$ and hence X contains $\mathbf{2}^Y$. Since the dual of Y, Y^∂, can be embedded in $\mathbf{2}^Y$ ($\mathbf{2}^Y$ is order-isomorphic to the lattice of up-sets of Y, and its poset of join-irreducibles is given by the principal up-sets; under inclusion, these are ordered like the dual of Y), then $Y^\partial \times Z$ can be embedded in $\mathbf{2}^Y \times Z$ and thus in $X \times Z \cong Q_0$. Hence

$$\mathrm{h}(P) = \mathrm{h}(Y \times Z) = \mathrm{h}(Y) + \mathrm{h}(Z) = \mathrm{h}(Y^\partial) + \mathrm{h}(Z) = \mathrm{h}(Y^\partial \times Z) \leqslant \mathrm{h}(Q_0) = \mathrm{h}(Q)$$

and

$$|P| = |Y \times Z| = |Y||Z| = |Y^\partial||Z| = |Y^\partial \times Z| \leqslant |Q_0| < |Q|,$$

so by Lemma 15 (4) and Claim 17, $\mathrm{h}(P) \neq 0$, so $\mathrm{h}(P^P) = |P|\mathrm{h}(P) < |Q|\mathrm{h}(Q) = \mathrm{h}(Q^Q)$, a contradiction.

Thus $|F| = 1$, and $|X| = 1$, and so $P \cong Y \times Q_0$. If also $|Y| = 1$, then $\mathrm{h}(P) = \mathrm{h}(Q)$ but $|Q| > |P|$, so $\mathrm{h}(Q^Q) > \mathrm{h}(P^P)$, a contradiction stemming from Lemma 15 (4), unless $\mathrm{h}(P) = 0$, which would make P an antichain, contradicting Claim 17. Thus Q_0 is a proper direct factor of P.

We have that P and Q_0 are nonempty and connected, so Theorem 10 applies to the isomorphism we have already established, $\mathcal{C}(P^P) \cong \mathcal{C}(Q_0^Q)$: There exist finite, nonempty posets E', X', Y', and Z' such that

$$P \cong \mathcal{C}({E'}^{X'}), \quad Q_0 \cong \mathcal{C}({E'}^{Y'}), \quad P \cong Y' \times Z', \quad \text{and} \quad Q \cong X' \times Z'.$$

Since P is connected and nonempty, so are Y' and Z', and hence Z' divides every component of Q, in particular, Q_0—say, $Z' \times T \cong Q_0$ for some connected, nonempty poset T. Again, Y' is connected and so is E', by Proposition 1 (3), since Q_0 is connected. By Theorem 2 (3), since T is a factor of $Q_0 \cong \mathcal{C}(E'^{Y'})$, there exists a finite, nonempty poset H such that $T \cong \mathcal{C}(H^{Y'})$; H is connected by Proposition 1 (3).

Case 1. $|H| > 1$. Then since H is connected, it contains $\mathbf{2}$, and hence T contains $\mathbf{2}^{Y'}$, which contains Y'^{∂} (we get the dual for the same reason as before), and $Q_0 \cong Z' \times T$ contains $Z' \times Y'^{\partial}$, so $|Q_0| \geqslant |Z' \times Y'^{\partial}| = |Z'||Y'^{\partial}| = |Z'||Y'| = |Z' \times Y'| = |P|$. As we already know that P properly contains Q_0, we have a contradiction.

Case 2. $|H| = 1$. Then $|T| = 1$ because $H^{Y'}$ consists only of a constant map, and $Z' \cong Q_0$, because $Z' \times \mathbf{1} \cong Z'$ and $T \cong \mathbf{1}$, so Q_0 is a factor of Q, because $Q \cong X' \times Z'$ and $Z' \cong Q_0$. But then $\mathrm{h}(Q_0) = \mathrm{h}(Q) = \mathrm{h}(Q_0) + \mathrm{h}(X')$, so $\mathrm{h}(X') = 0$ and X' is an antichain. But if $|X'| > 1$, then Q has two components of maximum height, a contradiction.

Hence X' is a singleton, so $Q \cong X' \times Z' \cong Z' \cong Q_0$ and Q is connected, a contradiction. □

Perhaps one could prove that $P^P \cong Q^Q$ implies $P \cong Q$ if P and Q are finite and nonempty and P is directly irreducible. Of course, the remaining problem is to show that $P^P \cong Q^Q$ implies $P \cong Q$ if P and Q are finite and nonempty (or find a counterexample).

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