POSITIVE SOLUTION FOR INFINITELY IMPULSIVE SINGULAR THIRD-ORDER ϕ-LAPLACIAN BVPS ON THE HALF LINE WITH FIRST-ORDER DERIVATIVE DEPENDENCE

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Received July 11, 2024. Published online January 17, 2025.
Communicated by Leonid Berezansky

To the memory of Ahmed Benblidia

Abstract. We are concerned in this paper with the existence of positive solutions to the ϕ-Laplacian third-order boundary value problem

$$\begin{cases} -(\phi(u''))'(t) = f(t,u(t),u'(t)) \text{ for a.e. } t \in J, \\ u(0)=0,\ u'(0)=a,\ \lim\limits_{t\to\infty} u''(t)=0, \\ \Delta u(t_k) = I_{1,k}(u(t_k),u'(t_k)),\ k=1,2,\dots, \\ \Delta u'(t_k) = I_{2,k}(u(t_k),u'(t_k)),\ k=1,2,\dots, \\ -\Delta\phi(u'')(t_k) = I_{3,k}(u(t_k),u'(t_k)),\ k=1,2,\dots, \end{cases}$$

where $a \geqslant 0$, $J=(0,\infty)$, $0<t_1<t_2<\dots<t_k\dots$, $t_k\to\infty$ as $k\to\infty$, $\Delta u(t_k) = u(t_k^+)-u(t_k^-)$ and $J^* = J\setminus\{t_k\colon k\geqslant 1\}$. The function $\phi\colon \mathbb{R}\to\mathbb{R}$ is an increasing homeomorphism such that $\phi(0)=0$, $I_{i,k}\in C(I^2,[0,\infty))$ for $i=1,2,3$ and $k\geqslant 1$, and the nonlinearity $f\colon J^3\to\mathbb{R}^+$ is a Caratheodory function. By means of a Guo-Krasnoselskii type fixed point theorem, we prove an existence result for at least one positive solution.

Keywords: BVP on infinite intervals; ϕ-Laplacian; fixed point theory in cones

MSC 2020: 34A37, 34B16, 34B18, 34B40

The research has been supported by the Laboratory of Fixed Point Theory and Applications, École Normale Supérieure, BP 92 Kouba, 16050, Algiers, Algeria.

DOI: 10.21136/MB.2025.0081-24

1. Introduction and main results

Third-order differential equations arise in many physical and engineering applications such as the deflection of an elastic beam having constant or varying cross-section, three-layer beam, electromagnetic waves or gravity-driven flows, see [1], [2], [5], [10], [17] and the references therein. In infinite intervals, third-order BVPs can describe the evolution of physical phenomena, for example, some draining or coating fluid-flow problems, see [18]. Due to the unboundedness of the interval, the investigation for sufficient conditions for the solvability of BVPs is more complicated. Many papers and monographs considering such problems have appeared in the last few decades, we refer the reader to [1], [2], [12], [13], [14], [16], [19], [20] and references therein. In nature, physical phenomena are generally subject to some kind of perturbations that make them change their states suddenly. These perturbations are modeled as impulses. That's why many authors have been interested in the study of impulsive BVPs, see for instance [3], [4], [11], [15], [21] and references therein.

In the last three decades, many authors investigated the existence of positive solutions for such kind of problems involving p-Laplacian operator or its natural generalization, ϕ-Laplacian operator, see, for instance, [3], [7], [8], [9] and references therein. Physically speaking, often the solution to this type of problem refers to density, position, temperature, etc. That's why some of the above-cited references were interested in studying the existence of positive solutions.

This work deals with the existence of positive solutions to the ϕ-Laplacian third-order boundary value problem (bvp for short)

$$(1.1)\qquad \begin{cases} -(\phi(u''))'(t) = f(t, u(t), u'(t)) \text{ for a.e. } t \in J, \\ u(0) = 0,\ u'(0) = \alpha,\ \lim\limits_{t\to\infty} u''(t) = 0, \\ \Delta u(t_k) = I_{1,k}(u(t_k), u'(t_k)),\ k = 1, 2, \ldots, \\ \Delta u'(t_k) = I_{2,k}(u(t_k), u'(t_k)),\ k = 1, 2, \ldots, \\ -\Delta\phi(u'')(t_k) = I_{3,k}(u(t_k), u'(t_k)),\ k = 1, 2, \ldots, \end{cases}$$

where $\alpha \geqslant 0$, $J = (0, \infty)$, $0 < t_1 < t_2 < \ldots < t_k < \ldots$, $t_k \to \infty$ as $k \to \infty$ and $\Delta u^{(i)}(t_k) = u(t_k^+) - u(t_k^-)$ for any function u. The function $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$, $I_{i,k} \in C(\overline{J} \times \overline{J}, \overline{J})$ for $i = 1, 2, 3$ and $k \geqslant 1$, and the nonlinearity $f\colon J \times J \times J \to [0, \infty)$ is a Caratheodory function, that is

- ▷ $f(\cdot, u, v)$ is measurable for all $u, v \in J$ and
- ▷ $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$.

In all what follows, we use the following notations. We let $\overline{J} = [0, \infty)$, $J_j = (t_{j-1}, t_j)$ and $\overline{J}_j = [t_{j-1}, t_j]$ for all $j \geqslant 1$ with $t_0 = 0$ and $J^* = J \setminus \{t_k\colon k \geqslant 1\}$. We let also ψ to be the inverse function to ϕ and we suppose that

$$\text{(1.2)} \qquad \begin{cases} \text{there exist } p, q > 0 \text{ such that for all } x \geqslant 0 \\ \text{and } t \in (0,1),\ t^q \phi(x) \leqslant \phi(tx) \leqslant t^p \phi(x). \end{cases}$$

Note that for all $x \geqslant 0$ and $t \in (0,1)$, we have from (1.2) that

$$\text{(1.3)} \qquad t^{1/p}\psi(x) \leqslant \psi(tx) \leqslant t^{1/q}\psi(x).$$

Let ϕ^+, ϕ^-, ψ^+ and ψ^- be the functions defined on $\overline{J}$ by

$$\phi^-(x) = \begin{cases} x^q & \text{if } x \leqslant 1, \\ x^p & \text{if } x \geqslant 1, \end{cases} \qquad \phi^+(x) = \begin{cases} x^p & \text{if } x \leqslant 1, \\ x^q & \text{if } x \geqslant 1, \end{cases}$$

$$\psi^-(x) = \begin{cases} x^{1/q} & \text{if } x \leqslant 1, \\ x^{1/p} & \text{if } x \geqslant 1, \end{cases} \qquad \psi^+(x) = \begin{cases} x^{1/p} & \text{if } x \leqslant 1, \\ x^{1/q} & \text{if } x \geqslant 1 \end{cases}$$

and notice that

$$\text{(1.4)} \qquad \begin{aligned} &(\phi^+)^{-1} = \psi^-, \qquad (\phi^-)^{-1} = \psi^+, \\ &\phi^+\Big(\frac{1}{x}\Big) = \frac{1}{\phi^-(x)}, \quad \psi^+\Big(\frac{1}{x}\Big) = \frac{1}{\psi^-(x)} \quad \text{for all } x > 0. \end{aligned}$$

It follows from (1.3) that for all $t, x \geqslant 0$,

$$\text{(1.5)} \qquad \begin{aligned} \phi^-(t)\phi(x) \leqslant \phi(tx) \leqslant \phi^+(t)\phi(x), \\ \psi^-(t)\psi(x) \leqslant \psi(tx) \leqslant \psi^+(t)\psi(x). \end{aligned}$$

Equivalently, for all $t, x \geqslant 0$ we have

$$\text{(1.6)} \qquad \begin{aligned} \phi(\psi^-(t)x) \leqslant t\phi(x) \leqslant \phi(\psi^+(t)x), \\ \psi(\phi^-(t)x) \leqslant t\psi(x) \leqslant \psi(\phi^+(t)x). \end{aligned}$$

We obtain also, from (1.2) and (1.3), that for all $x, y \geqslant 0$,

$$\text{(1.7)} \qquad \begin{aligned} &\phi_*(\phi(x)+\phi(y)) \leqslant \phi(x+y) \leqslant \phi^*(\phi(x)+\phi(y)) \text{ and} \\ &\psi_*(\psi(x)+\psi(y)) \leqslant \psi(x+y) \leqslant \psi^*(\psi(x)+\psi(y)), \end{aligned}$$

where

$$\begin{aligned} \phi_* &= \min(1, 2^{p-1}), \qquad \phi^* = \max(1, 2^{q-1}), \\ \psi_* &= \min(1, 2^{(1-q)/q}), \quad \psi^* = \max(1, 2^{(1-p)/p}). \end{aligned}$$

Throughout this paper, we suppose that the following conditions hold.

$$(1.8)\qquad \begin{cases} I_{i,k}(u,v) \leqslant a_{i,k}u + b_{i,k}v + c_{i,k}\ \forall\, u,v \geqslant 0 \text{ for } i=1,2 \text{ and } k \geqslant 1,\\ I_{3,k}(u,v) \leqslant a_{3,k}\phi(u) + b_{3,k}\phi(v) + c_{3,k}\ \forall\, u,v \geqslant 0 \text{ for } k \geqslant 1,\\ a_1 = \sum_{k\geqslant 1} a_{1,k} < \infty,\ a_2 = \sum_{k\geqslant 1} a_{2,k}(1+t_k) < \infty,\\ a_3 = \sum_{k\geqslant 1} a_{3,k}(1+t_k)^{2q} < \infty,\\ b_1 = \sum_{k\geqslant 1} b_{1,k}(1+t_k)^{-1} < \infty,\ b_2 = \sum_{k\geqslant 1} b_{2,k} < \infty,\\ b_3 = \sum_{k\geqslant 1} b_{3,k}(1+t_k)^{q} < \infty,\\ c_1 = \sum_{k\geqslant 1} c_{1,k}(1+t_k)^{-2} < \infty,\ c_2 = \sum_{k\geqslant 1} c_{2,k}(1+t_k)^{-1} < \infty,\\ c_3 = \sum_{k\geqslant 1} c_{3,k} < \infty. \end{cases}$$

$$(1.9)\qquad \begin{cases} \text{For all } R>0 \text{ there exist two functions } \omega_R\colon I \to I \text{ and } \Psi_R\colon I^2 \to I\\ \text{such that } \Psi_R \text{ is nonincreasing following its two variables,}\\ f(t,(1+t)^2 w,(1+t)z) \leqslant \omega_R(t)\Psi_R(w,z) \text{ for all } t,w,z \geqslant 0\\ \ \text{with } |(w,z)| \leqslant R,\\ \displaystyle\int_s^\infty \omega_R(\tau)\Psi_R(r\widetilde{\gamma}(\tau), r\gamma(s))\,\mathrm{d}\tau < \infty \text{ for all } s \in I \text{ and } r \in (0,R], \text{ and}\\ \displaystyle\int_0^t \psi\Big(\int_s^\infty \omega_R(\tau)\Psi_R(r\widetilde{\gamma}(\tau), r\gamma(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s < \infty \text{ for all } t \in I \text{ and } r \in (0,R], \end{cases}$$

where $|(w,z)| = \sup(|w|,|z|)$,

$$\varrho(t) = \begin{cases} t & \text{if } t \in [0,1],\\ \dfrac{1}{t} & \text{if } t \geqslant 1, \end{cases}$$

$$\widetilde{\varrho}(t) = \int_0^t \varrho(s)\,\mathrm{d}s = \begin{cases} \dfrac{t^2}{2} & \text{if } t \in [0,1],\\ \dfrac{1}{2} + \ln t & \text{if } t \geqslant 1, \end{cases}$$

$$\gamma(t) = \frac{\varrho(t)}{1+t} = \begin{cases} \dfrac{t}{(1+t)} & \text{if } t \in [0,1],\\ \dfrac{1}{t(1+t)} & \text{if } t \geqslant 1, \end{cases}$$

$$\widetilde{\gamma}(t)=\frac{\widetilde{\varrho}(t)}{(1+t)^2}=\begin{cases}\dfrac{t^2}{(1+t)^2} & \text{if } t\in[0,1],\\ \dfrac{1+\ln t}{2(1+t)^2} & \text{if } t\geqslant 1.\end{cases}$$

We set throughout this work

$$\begin{aligned}A&=a_1+a_2+b_1+b_2+(\psi^*)^2\psi^+(a_3+b_3),\\ B&=\alpha+c_1+c_2+(\psi^*)^2\psi(c_3).\end{aligned}$$

R e m a r k 1.1. Observe that the case when the nonlinearity f satisfies the polynomial growth condition

$$f(t,u,v)\leqslant C(1+u^\sigma+v^\mu),$$

where $C,\sigma,\mu>0$, is a particular case when condition (1.9) is satisfied.

By a positive solution to bvp (1.1), we mean a function u in $C^2(\overline{J})$ such that $u>0$ in $(0,\infty)$, $u(0)=\lim\limits_{t\to\infty}u''(t)=0$, $\phi(u'')$ belongs to $W^{1,1}([\tau,\infty))$ for all $\tau>0$, and u satisfies the differential equation in (1.1).

Our approach to bvp (1.1) consists in a fixed point formulation. Since the nonlinearity f is supposed here to be nonnegative, the existence of a positive solution will be proved by means of Guo-Krasnoselskii's version of the expansion and compression of a cone in a Banach space principle.

The statement of the main result in this paper needs to introduce some additional notations. Let

$$\begin{aligned}\Pi=\{m\colon\ J\to\overline{J}\ &\text{measurable such that }\int_s^\infty m(\tau)\,\mathrm{d}\tau<\infty\\ &\text{for all } s>0 \text{ and }\int_0^t\psi^+\Big(\int_s^\infty m(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s<\infty \text{ for all } t>0\},\end{aligned}$$

and set for $m\in\Pi$, $\theta>1$, and $\nu=0,\infty$,

$$\begin{aligned}J_\theta&=[1/\theta,\theta],\\ f^\nu(m)&=\limsup_{|(w,z)|\to\nu}\Big(\sup_{t\geqslant 0}\frac{f(t,(1+t)^2w,(1+t)z)}{m(t)\phi(w+z)}\Big),\\ f_\nu(m,\theta)&=\liminf_{|(w,z)|\to\nu}\Big(\min_{t\in J_\theta}\frac{f(t,(1+t)^2w,(1+t)z)}{m(t)\phi(w+z)}\Big),\end{aligned}$$

where $|(w,z)| = |w| + |z|$,

$$\begin{aligned}
\varrho(m) &= 2\sup_{t\geqslant 1}\left(\frac{1}{1+t}\int_0^t \psi^+\left(\int_s^\infty m(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right),\\
\Gamma_0(m) &= \phi^+\left(\frac{2\psi^*\varrho(m)}{1-A}\right) \text{ and } \Gamma_\infty(m) = \phi^+\left(\frac{2(\psi^*)^2\varrho(m)}{1-A}\right) \quad \text{for } A>1,\\
\mu(m,\theta) &= \frac{1}{1+\theta}\psi^-\left(\int_{1/\theta}^\theta m(\tau)\phi^-(\widetilde{\gamma}(\tau)+\gamma(\tau))\,\mathrm{d}\tau\right),\\
\Theta(m,\theta) &= \phi^-(\mu(m,\theta)).
\end{aligned}$$

The following theorem is the main result of this work.

Theorem 1.2. *Assume that hypotheses* (1.2), (1.8) *and* (1.9) *hold and there exist* $\theta > 1$ *and two functions* $m_0, m_\infty \in \Pi$ *such that one of the conditions*

$$A<1,\ B=0,\ f^0(m_0)\Gamma_0(m_0) < 1 < \Theta(m_\infty,\theta)f_\infty(m_\infty,\theta) \tag{1.10}$$

and

$$A<1,\ f^\infty(m_\infty)\Gamma_\infty(m_\infty) < 1 < \Theta(m_0,\theta)f_0(m_0,\theta) \tag{1.11}$$

is satisfied. Then bvp (1.1) *has at least one positive solution* u *such that for all* $t>1$, $u(t) \geqslant \sigma_u(1/2+\ln t)$ *for some* $\sigma_u > 0$.

2. Example

Consider bvp (1.1) with $\phi(x) = |x|^{p-1}x + |x|^{q-1}x$ and

$$f(t,u,v) = \frac{t}{(1+t)^\xi}\left(\theta\left(\frac{u}{(1+t)^2}+\frac{v}{(1+t)}\right)^m + \sigma\left(\frac{u}{(1+t)^2}+\frac{v}{(1+t)}\right)^n\right),$$

where $\theta, \sigma, n > 0$, $q > p > 0$, $\xi > 2$ and $\max(-2-1/q, (2-\xi)/2) < m < 0$.

Clearly, the function ϕ satisfies (1.2) and for all $t, w, z > 0$ with $w + z < R$, we have

$$f(t,(1+t)^2w,(1+t)z) = \frac{t}{(1+t)^\xi}(\theta(w+z)^m + \sigma(w+z)^n) \leqslant \omega_R(t)\Psi_R(w,z),$$

where $\omega_R(t) = \omega(t) = t(1+t)^{-\xi}$ and $\Psi_R(w,z) = \theta w^m + \sigma R^n$.

Taking into account $\xi > 2$ and $m < 0$, straightforward calculations lead to

$$\omega(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau)) \sim \theta R^m (1+\tau)^{-\xi-2m+1} \quad \text{at } \infty \tag{2.1}$$

and

$$\omega(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau)) \sim \theta R^m \tau^{m+1} \quad \text{at } 0. \tag{2.2}$$

Noticing that $-\xi - 2m + 1 < -1$, we obtain from (2.1)

$$\int_s^\infty \omega_R(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau))\,\mathrm{d}\tau < \infty \quad \text{for all } s > 0.$$

We obtain also from (2.2) that at 0 we have

$$\psi^+\biggl(\int_s^\infty \omega_R(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau))\,\mathrm{d}\biggr)\tau \sim c \begin{cases} s^{(m+2)p} & \text{if } m > -2, \\ \ln s & \text{if } m = -2, \\ s^{m+2} & \text{if } -2 > m > -2 - 1/q. \end{cases} \tag{2.3}$$

Because of the estimate

$$\int_0^t \psi\biggl(\int_s^\infty \omega_R(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s \\ \leqslant \psi(1)\int_0^t \psi^+\biggl(\int_s^\infty \omega_R(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s,$$

we conclude that if $m > -2 - 1/q$, then

$$\int_0^t \psi\biggl(\int_s^\infty \omega_R(\tau)\Psi_R(R\widetilde{\gamma}(\tau), R\gamma(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s < \infty.$$

This shows that hypothesis (1.9) is fulfilled.

Set $m_0(t) = m_\infty(t) = \omega(t)$, straightforward computations lead to

$$f_0(\omega, \theta) = \infty,$$
$$f^\infty(\omega) = \begin{cases} 0 & \text{if } n < q, \\ \sigma & \text{if } n = q, \\ \infty & \text{if } n > q. \end{cases}$$

Suppose that $A < 1$, we conclude then from Theorem 1.2 and all the above calculations that for such a case, bvp (1.1) admits a positive solution if either $n < q$ or $n = q$ and $\sigma\Gamma_\infty(\omega) < 1$.

3. Abstract background

Let $(E, \|\cdot\|)$ be a Banach space and let K be a cone in E, i.e., K is a nonempty closed and convex subset of E, such that $K \cap (-K) = \emptyset$ and $tK \subset K$ for all $t \geqslant 0$. The main result of this work will be proved by means of the following theorem which can be proved in the same way as Theorems 2.3.3–2.3.5 in [11].

Theorem 3.1. *Let r_1, r_2 be two positive real numbers such that $r_1 < r_2$ and let $T\colon K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a compact mapping where $\Omega_i = \{u \in E, \|u\|_1 < r_i\}$ for $i = 1, 2$. If one of the conditions*

(1) *$\|Tu\| < \|u\|$ for all $u \in K \cap \partial\Omega_1$ and there exists $e \in K \setminus \{0\}$ such that $u \neq Tu + te$ for all $u \in K \cap \partial\Omega_2$ and all $t \geqslant 0$,*
(2) *there exists $e \in K \setminus \{0\}$ such that $u \neq Tu + te$ for all $u \in K \cap \partial\Omega_1$ and all $t \geqslant 0$ and $\|Tu\| < \|u\|$ for all $u \in K \cap \partial\Omega_2$,*

is satisfied, then T has at least a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$.

4. Fixed point formulation

Consider the functional spaces

$$\begin{aligned} PC(\overline{J}) &= \{u\colon \overline{J} \to \mathbb{R} \text{ continuous for } t \in J^*,\ u(t_k) = u(t_k^-) \\ &\qquad \text{and } u(t_k^+) \text{ exists for all } k \geqslant 1\}, \\ PC^1(\overline{J}) &= \{u \in PC(\overline{J})\colon \exists\, v \in PC(\overline{J}) \text{ such that } u'(t) = v(t) \text{ for all } t \in J^*\}. \end{aligned}$$

Let E and F be the linear spaces defined by

$$E = \Big\{u \in PC^1(\overline{J})\colon \sum_{k\geqslant 1} \frac{|\Delta u(t_k)|}{(1+t_k)^2} < \infty,\ \sum_{k\geqslant 1} \frac{|\Delta u'(t_k)|}{1+t_k} < \infty,\ \sup_{t\geqslant 0} \frac{|u(t)|}{(1+t)^2} < \infty, \\ \text{and } \sup_{t\geqslant 0} \frac{|u'(t)|}{1+t} < \infty\Big\}.$$

Equipped by the norm $\|\cdot\|$, where for $u \in E$

$$\begin{aligned} \|u\| &= \max(\|u\|_3, \|u\|_4, \|u\|_5, \|u\|_6), \\ \|u\|_3 &= \sum_{k\geqslant 1} \frac{|\Delta u(t_k)|}{(1+t_k)^2},\ \ \|u\|_4 = \sum_{k\geqslant 1} \frac{|\Delta u'(t_k)|}{1+t_k}, \\ \|u\|_5 &= \sup_{t\geqslant 0} \frac{|u(t)|}{(1+t)^2},\ \ \|u\|_6 = \sup_{t\geqslant 0} \frac{|u'(t)|}{1+t}, \end{aligned}$$

the linear space E becomes a Banach space.

Let E_1 and E_2 be the subspaces of E defined by

$$E_1 = \Big\{u\colon\ u(t) = \sum_{k=0}^{k=j-1} \gamma_k + \sum_{k=1}^{k=j}\Big(\sum_{k=0}^{i=k-1} \delta_i\Big)(t_k - t_{k-1})$$
$$+ \Big(\sum_{k=0}^{i=j} \delta_i\Big)(t - t_j) \text{ for } t \in (t_j, t_{j+1}),$$
$$(\gamma_i), (\delta_i) \subset \mathbb{R},\ \gamma_0 = \delta_0 = 0,\ \sum_{k\geqslant 1} \frac{|\gamma_k|}{(1+t_k)^2} < \infty \ \text{ and } \sum_{k\geqslant 1} \frac{|\delta_k|}{1+t_k} < \infty\Big\}$$

and

$$E_2 = \Big\{u \in C^1(\overline{J})\colon\ \lim_{t\to\infty} \frac{u'(t)}{1+t} = 0\Big\},$$

respectively.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the norms defined on E_1 and E_2 by

$$\|u\|_1 = \max(\|u\|_3, \|u\|_4) \text{ for all } u \in E_1$$
$$\text{and } \|u\|_2 = \max(\|u\|_5, \|u\|_6) \text{ for all } u \in E_2,$$

respectively. Since $\|u\| = \|u\|_i$ for $i = 1, 2$ and all $u \in E_i$, we conclude that for $i = 1, 2$, $(E_i, \|u\|_i)$ is a Banach space.

Let $l_\# = l_1^1 \times l_2^1$, where l_1^1 and l_2^1 are defined by

$$l_1^1 = \Big\{\gamma = (\gamma_k)_{k\geqslant 1}\colon\ |\gamma|_1 = \sum_{k\geqslant 1} \frac{|\gamma_k|}{(1+t_k)^2} < \infty\Big\},$$
$$l_2^1 = \Big\{\delta = (\delta_k)_{k\geqslant 1}\colon\ |\delta|_2 = \sum_{k\geqslant 1} \frac{|\delta_k|}{1+t_k} < \infty\Big\}.$$

Clearly for $i = 1, 2$ we have that $(l_i^1, |\cdot|_i)$ is a Banach space and $(E_2, \|\cdot\|_2)$ is isometric to $(l_\#, |\cdot|_\#)$ where $|(\gamma, \delta)|_\# = \max(|\gamma|_1, |\delta|_2)$ for all $(\gamma, \delta) \in l_\#$.

Throughout this paper, K is the cone of E given by

$$K = \{u \in E\colon\ u, u' \geqslant 0 \text{ on } J \text{ and } u' \text{ is concave in } J\}.$$

Lemma 4.1. *For all $u \in K$ and $t \geqslant 0$, we have*

$$u(t) \geqslant \widetilde{\varrho}(t)\|u\| \quad \text{and} \quad u'(t) \geqslant \varrho(t)\|u\|.$$

P r o o f. Let $u \in K \setminus \{0\}$ (the case $u = 0$ is obvious) and let us prove first that u' is nondecreasing on J. To this aim let $t_1, t_2 \in J$ with $t_1 < t_2$. By the concavity of u', for all $\eta > t_2$ we have

$$\frac{u'(\eta) - u'(t_2)}{\eta - t_2} \leqslant \frac{u'(\eta) - u'(t_1)}{\eta - t_1},$$

leading to

$$u'(t_2) \geqslant \frac{\eta - t_2}{\eta - t_1} u'(t_1) + \frac{(t_1 - t_2)(1+\eta)}{\eta - t_1} \frac{u'(\eta)}{1+\eta}.$$

Letting $\eta \to \infty$, we obtain $u'(t_2) \geqslant u'(t_1)$. We have proved that u' is nondecreasing on J.

Now, set $h(t) = u'(t)/(1+t)$. Since $h(0) = \alpha \geqslant 0$ and $\lim\limits_{t\to\infty} h(t) = 0$, we claim that there exists $t_0 \in \overline{J}$ such that $h(t_0^+) = \sup\limits_{t \in \overline{J}} h(t) = \|u\|_6$. Indeed, let $m = \sup\limits_{t\in\overline{J}} h(t)$. There exists a sequence (t_n) such that $\lim h(t_n) = m$. Because $\lim\limits_{t\to\infty} h(t) = 0$, the sequence (t_n) is bounded and converges (up to a subsequence) to some $t_0 \in \overline{J}$. Taking in account u' is nondecreasing on J yields

$$\lim h(t_n) = m \leqslant \limsup_{t \to t_0} h(t) = \frac{\limsup_{t\to t_0} u'(t)}{1+t_0} = \frac{u'(t_0^+)}{1+t_0} \leqslant m.$$

At this stage let $\theta \geqslant 1$. For arbitrary $t > t_0$ we have by the concavity of u'

$$\begin{aligned} u'\Big(\frac{1}{\theta}\Big) &= u'\Big(\frac{\theta - 1 + \theta t}{\theta + \theta t} \frac{1}{\theta - 1 + \theta t} + \frac{t}{\theta + \theta t}\Big) \\ &\geqslant \frac{\theta - 1 + \theta t}{\theta + \theta t} u'\Big(\frac{1}{\theta - 1 + \theta t}\Big) + \frac{1}{\theta + \theta t} u'(t) \\ &\geqslant \frac{1}{\theta} \frac{u'(t)}{1+t}. \end{aligned}$$

Letting $t \to t_0$ we obtain

$$u'\Big(\frac{1}{\theta}\Big) \geqslant \frac{1}{\theta} \frac{u'(t_0^+)}{1+t_0} = \frac{1}{\theta}\|u\|_6. \tag{4.1}$$

Thus, for $t \geqslant 0$ we distinguish the following cases.

(i) $t = 0$, in this case we have $u'(0) = \alpha \geqslant 0 = \varrho(0)\|u\|_6$.

(ii) $t \geqslant 1$, in this case taking in consideration u' is nondecreasing on J and (4.1), we obtain $u'(t) \geqslant u'(1/t) \geqslant (1/t)\|u\|_6$.

(iii) $0 < t < 1$, in this case we have from (4.1) that $u(t) = u(1/t^{-1}) \geqslant (1/t^{-1})\|u\|_6 = t\|u\|_6$.

Hence, we have proved that $u'(t) \geqslant \varrho(t)\|u\|_6$ for all $t \in J$.

At the end, because u is nondecreasing on J, we have $\Delta u(t_k) = u(t_k^+) - u(t_k^-) \geqslant 0$ for all $k \geqslant 1$. Therefore, we have

$$u(t) = \int_0^t u'(s)\,\mathrm{d}s + \sum_{t_k<t} \Delta u(t_k) \geqslant \int_0^t \varrho(s)\|u\|_6\,\mathrm{d}s = \widetilde{\varrho}(t)\|u\|_6.$$

□

Definition 4.2. A subset M in l^1 is said to be equiconvergent if for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\sum\limits_{k\geqslant k_\varepsilon} |x_k| \leqslant \varepsilon$ for all $(x_k)_{k\geqslant 1} \in M$.

Definition 4.3. Let M be a nonempty subset in E_2. The subset M is said to be:

(i) locally equicontinuous in E_2 if for all $a > 0$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t_1, t_2 \in [0, a]$ and all $u \in M$

$$|t_2 - t_1| \leqslant \delta \Longrightarrow \begin{cases} \left|\dfrac{u(t_2)}{(1+t_2)^2} - \dfrac{u(t_1)}{(1+t_1)^2}\right| \leqslant \varepsilon \text{ and} \\ \left|\dfrac{u'(t_2)}{1+t_2} - \dfrac{u'(t_1)}{1+t_1}\right| \leqslant \varepsilon, \end{cases}$$

(ii) equiconvergent if for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that for all $u \in M$ and all $t \geqslant t_\varepsilon$ we have

$$\left|\frac{u(t)}{(1+t)^2}\right| \leqslant \varepsilon, \quad \left|\frac{u'(t)}{1+t}\right| \leqslant \varepsilon.$$

The following lemma presents a compactness criterion in l^1.

Lemma 4.4. *Let M be a subset in l^1. If M is bounded and equivergent, then M is relatively compact in l^1.*

P r o o f. Set for any $l \in \mathbb{N}$,

$$E_l = \{(x_k)_{k\geqslant 1} \in l^1 \colon\ x_k = 0 \text{ for all } k \geqslant l+1\}.$$

Clearly, for any $l \in \mathbb{N}$, E_l is a closed subspace of l^1 with $\dim(E_l) = l$.

Let $\varepsilon > 0$, then there exits $k_\varepsilon \in \mathbb{N}$ such that $\sum\limits_{k\geqslant k_\varepsilon} |x_k| \leqslant \varepsilon/2$ for all $(x_k)_{k\geqslant 1} \in M$. Set now $\widetilde{X} = (x_1, x_2, \ldots, x_{k_\varepsilon - 1}, 0, 0, \ldots)$ and $\widetilde{M} = \{\widetilde{X} \colon X \in M\}$ for $X = (x_k)_{k\geqslant 1} \in M$. Hence, we have $\widetilde{M} \subset E_{k_\varepsilon}$ and $\widetilde{M}$ is bounded in E_{k_ε}. Indeed, for all $X = (x_k)_{k\geqslant 1} \in M$, we have

$$|\widetilde{X}|_1 = \sum_{k=1}^{k=k_\varepsilon - 1} |x_k| \leqslant \sum_{k\geqslant 1} |x_k| = |X|_1.$$

Now, we know that $\widetilde{M}$ is relatively compact in $E_{k_\varepsilon-1}$. So, there exist $a_1, a_2, \ldots, a_m$ belonging to E_{k_ε} such that

$$\widetilde{M} \subset \bigcup_{i=1}^{i=m} B\Big(a_i, \frac{\varepsilon}{2}\Big).$$

Let $X = (x_k)_{k\geqslant 1} \in M$. There exists $i_0 \in \{1, \ldots, m\}$ such that $\widetilde{X} \in B(a_{i_0}, \varepsilon/2)$. Therefore, if $a_{i_0} = (a_{i_0,1}, a_{i_0,2}, \ldots, a_{i_0,k_\varepsilon-1}, 0, 0, \ldots)$, we have

$$|X - a_{i_0}|_1 = \sum_{k=1}^{k=k_\varepsilon-1} |x_k - a_{i_0,k}| + \sum_{k\geqslant k_\varepsilon} |x_k| = |\widetilde{X} - a_{i_0}|_1 + \sum_{k\geqslant k_\varepsilon} |x_k| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $M \subset \bigcup_{i=1}^{i=m} B(a_i, \varepsilon)$ and M is relatively compact in l^1. □

Since the space E_1 is isometric to $l^1 \times l^1$, we deduce from the above lemma the following compactness criterion available in E_1.

Lemma 4.5. *A nonempty subset M of E_1 is relatively compact if the conditions* (a) *and* (b) *hold.*

(a) *M is bounded in E_1.*
(b) *For all $\varepsilon > 0$, there corresponds $k_\varepsilon > 0$ such that $\sum_{k\geqslant k_\varepsilon} |\Delta u(t_k)|/(1+t_k)^2 \leqslant \varepsilon$ and $\sum_{k\geqslant k_\varepsilon} |\Delta u'(t_k)|/(1+t_k) \leqslant \varepsilon$ for all $x \in M$.*

The following lemma is an adapted version for the case of the space E_2 of Corduneanu's compactness criterion ([6], p. 62). It will be used to prove that for $R > r > 0$, the operator T given by Lemma 4.9 maps $K \cap (B(0,R) \setminus B(0,r))$ into a relatively compact subset of E.

Lemma 4.6. *A nonempty subset M of F is relatively compact if the conditions* (a)–(c) *hold.*

(a) *M is bounded in F.*
(b) *The sets $\{u\colon u(t) = x(t)/(1+t)^2,\ x \in M\}$ and $\{u\colon u(t) = x'(t)/(1+t),\ x \in M\}$ are locally equicontinuous on $[0,\infty)$.*
(c) *The sets $\{u\colon u(t) = x(t)/(1+t)^2,\ x \in M\}$ and $\{u\colon u(t) = x'(t)/(1+t),\ x \in M\}$ are equiconvergent at ∞, that is, given $\varepsilon > 0$, there corresponds $T_\varepsilon > 0$ such that for all $x \in M$, $|x(t)/(1+t)^2| \leqslant \varepsilon$, $|x'(t)/(1+t)| \leqslant \varepsilon$ for any $t \geqslant T_\varepsilon$.*

Let for $u \in K \setminus \{0\}$ it holds $Tu = T_1u + T_2u$, where

$$T_1u(t) = \sum_{t_k<t} I_{1,k}(u(t_k), u'(t_k)) + \int_0^t \Bigl(\sum_{t_k<s} I_{2,k}(u(t_k), u'(t_k))\Bigr)\,\mathrm{d}s \text{ and}$$
$$T_2u(t) = \alpha t + \int_0^t (t-s)\psi\Bigl(\sum_{t_k>s} I_{3,k}(u(t_k), u'(t_k)) + \int_s^\infty f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau\Bigr)\,\mathrm{d}s.$$

The following lemma provides a fixed point formulation for bvp (1.1).

Lemma 4.7. *Assume that hypotheses* (1.5), (1.8) *and* (1.9) *hold, then* $T_1u \in K \cap E$ *for all* $u \in K$. *Moreover, the mapping* $T_1\colon K \to K \cap E_1$ *is completely continuous.*

Proof. Let $u \in K$. For any $t \in J_j = (t_j, t_{j+1})$ we have

$$\begin{aligned}
T_1u(t) &= \sum_{t_k<t} I_{1,k}(u(t_k), u'(t_k)) + \int_0^t \Bigl(\sum_{t_k<s} I_{2,k}(u(t_k), u'(t_k))\Bigr)\,\mathrm{d}s \\
&= \sum_{k=1}^{k=j} I_{1,k}(u(t_k), u'(t_k)) + \sum_{k=1}^{k=j}\Bigl[\int_{t_{k-1}}^{t_k}\Bigl(\sum_{i=1}^{i=k-1} I_{2,i}(u(t_i), u'(t_i))\Bigr)\,\mathrm{d}s\Bigr] \\
&\quad + \int_{t_j}^t \Bigl(\sum_{i=1}^{i=j} I_{2,i}(u(t_i), u'(t_i))\Bigr)\,\mathrm{d}s \\
&= \sum_{k=1}^{k=j} I_{1,k}(u(t_k), u'(t_k)) + \sum_{k=1}^{k=j}\Bigl[\Bigl(\sum_{i=1}^{i=k-1} I_{2,i}(u(t_i), u'(t_i))\Bigr)(t_k - t_{k-1})\Bigr] \\
&= \Bigl(\sum_{i=1}^{i=j} I_{2,i}(u(t_i), u'(t_i))\Bigr)(t - t_j).
\end{aligned}$$

This shows that $T_1u \in E_2$.

Furthermore, because that holds for all $t \in J_j$, we have

$$\begin{aligned}
(T_1u)'(t) &= \sum_{i=1}^{i=j} I_{2,i}(u(t_i), u'(t_i)), \\
(T_1u)''(t) &= 0
\end{aligned}$$

and for all $j \geqslant 1$ we have

$$\begin{aligned}
\Delta(T_1u)(t_j) &= I_{1,j}(u(t_j), u'(t_j)) \geqslant 0, \\
\Delta(T_1u)'(t_j) &= I_{2,j}(u(t_j), u'(t_j)) \geqslant 0,
\end{aligned}$$

thus we conclude that $T_1u \in K \cap E_2$.

Let $I_\#\colon K \to l_\#$ be the mapping defined by

$$I_\#(u) = \big((I_{1,k}(u(t_k), u'(t_k)))_{k\geqslant 1}, (I_{2,k}(u(t_k), u'(t_k)))_{k\geqslant 1}\big).$$

Since for all $u \in K$

$$\tag{4.2} \begin{aligned} \sum_{k\geqslant 1} \frac{|I_{1,k}(u(t_k), u'(t_k))|}{(1+t_k)^2} &\leqslant \sum_{k\geqslant 1}\bigg(a_{1,k}\frac{u(t_k)}{(1+t_k)^2} + b_{1,k}\frac{u'(t_k)}{(1+t_k)^2} + \frac{c_{1,k}}{(1+t_k)^2}\bigg) \\ &\leqslant \|u\|\bigg(\sum_{k\geqslant 1} a_{1,k} + \sum_{k\geqslant 1}\frac{b_{1,k}}{1+t_k}\bigg) + \sum_{k\geqslant 1}\frac{c_{1,k}}{(1+t_k)^2} < \infty \end{aligned}$$

and

$$\tag{4.3} \begin{aligned} \sum_{k\geqslant 1} \frac{|I_{2,k}(u(t_k), u'(t_k))|}{1+t_k} &\leqslant \sum_{k\geqslant 1}\bigg(a_{2,k}\frac{u(t_k)}{1+t_k} + b_{2,k}\frac{u'(t_k)}{1+t_k} + \frac{c_{2,k}}{1+t_k}\bigg) \\ &\leqslant \|u\|\bigg(\sum_{k\geqslant 1} a_{2,k}(1+t_k) + \sum_{k\geqslant 1} b_{2,k}\bigg) \\ &\quad + \sum_{k\geqslant 1}\frac{c_{2,k}}{1+t_k} < \infty, \end{aligned}$$

the mapping $I_\#$ is well defined. Moreover, the continuity of the functions $I_{1,k}$ and $I_{2,k}$ for all $k \geqslant 1$ makes of $I_\#$ a continuous mapping.

Hence, taking in consideration that $T_1 = \mathrm{Iso}\circ I_\#$, where Iso is the isometry between $l_\#$ and E_2, we deduce that T_1 is a continuous mapping.

Now, let M be a subset in K bounded by $R > 0$. From inequalities (4.2) and (4.3) we obtain that for all $u \in M$

$$\begin{aligned} \sum_{k\geqslant 1} \frac{|I_{1,k}(u(t_k), u'(t_k))|}{(1+t_k)^2} &\leqslant R\bigg(\sum_{k\geqslant 1} a_{1,k} + \sum_{k\geqslant 1}\frac{b_{1,k}}{1+t_k}\bigg) + \sum_{k\geqslant 1}\frac{c_{1,k}}{(1+t_k)^2} < \infty, \\ \sum_{k\geqslant 1} \frac{|I_{2,k}(u(t_k), u'(t_k))|}{(1+t_k)^2} &\leqslant R\bigg(\sum_{k\geqslant 1} a_{2,k}(1+t_k) + \sum_{k\geqslant 1} b_{2,k}\bigg) + \sum_{k\geqslant 1}\frac{c_{2,k}}{1+t_k} < \infty. \end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that for all $u \in M$ we have

$$\begin{aligned} \sum_{k\geqslant k_\varepsilon} \frac{|I_{1,k}(u(t_k), u'(t_k))|}{(1+t_k)^2} &\leqslant R\bigg(\sum_{k\geqslant k_\varepsilon} a_{1,k} + \sum_{k\geqslant k_\varepsilon}\frac{b_{1,k}}{1+t_k}\bigg) + \sum_{k\geqslant k_\varepsilon}\frac{c_{1,k}}{(1+t_k)^2} \leqslant \varepsilon, \\ \sum_{k\geqslant k_\varepsilon} \frac{|I_{2,k}(u(t_k), u'(t_k))|}{(1+t_k)^2} &\leqslant R\bigg(\sum_{k\geqslant k_\varepsilon} a_{2,k}(1+t_k) + \sum_{k\geqslant k_\varepsilon} b_{2,k}\bigg) + \sum_{k\geqslant k_\varepsilon}\frac{c_{2,k}}{1+t_k} \leqslant \varepsilon. \end{aligned}$$

Thus, the equiconvergence of $T_1(M)$ is proved and this concludes the proof of T_1 being completely continuous. □

Lemma 4.8. *Assume that hypotheses* (1.5), (1.8) *and* (1.9) *hold, then for all* $u \in K \setminus \{0\}$ *it is* $T_2 u \in K \cap E_2$. *Moreover, for all* r, R *with* $0 < r < R$, *the mapping* $T_2\colon\ K \cap (\overline{B}(0,R) \setminus B(0,r)) \to K \cap E_2$ *is compact.*

P r o o f. Let r, R be real numbers with $0 < r < R$ and set $\Omega = K \cap (B(0,R) \setminus B(0,r))$. Set for $k \geqslant 1$,

$$z_{3,k} = \phi(R)(a_{3,k}(1+t_k)^{2q} + b_{3,k}(1+t_k)^q) + c_{3,k}$$

and let $\Phi_{r,R}$ be the function defined by

$$\Phi_{r,R}(s) = \omega_R(s)\Psi_R\big(r\widetilde{\gamma}(s), r\gamma(s)\big),$$

where ω_R and Ψ_R are the functions given by hypothesis (1.9).

Claim 1. $T\colon\ \Omega \to K \cap E_2$ is well defined. For all $u \in \Omega$, we have by hypothesis (1.8),

$$\begin{aligned} I_{3,k}(u(t_k), u(t_k)) &\leqslant a_{3,k}\phi(u(t_k)) + b_{3,k}\phi(u'(t_k)) + c_{3,k} \\ &\leqslant a_{3,k}\phi^+((1+t_k)^2)\phi\bigg(\frac{u(t_k)}{(1+t_k)^2}\bigg) \\ &\quad + b_{3,k}\phi^+(1+t_k)\phi\bigg(\frac{u'(t_k)}{1+t_k}\bigg) + c_{3,k} \\ &\leqslant z_{3,k} \end{aligned}$$

and by hypothesis (1.9)

$$f(\tau, u(\tau), u'(\tau)) = f\bigg(\tau, (1+\tau)^2\frac{u(\tau)}{(1+\tau)^2}, (1+\tau)\frac{u'(\tau)}{1+\tau}\bigg) \leqslant \Phi_{r,R}(\tau).$$

Taking in account hypotheses (1.8) and (1.9), we obtain from the above estimates

$$\sum_{t_k > t} I_{3,k}(u(t_k), u(t_k)) \leqslant A_3 < \infty \text{ for all } t > 0$$

and

$$\int_s^\infty f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau \leqslant \int_s^\infty \Phi_{r,R}(\tau)\,\mathrm{d}\tau < \infty \text{ for all } s > 0,$$

where $A_3 = \sum\limits_{k\geqslant 1} z_{3,k} = \phi(R)(a_3 + b_3) + c_3$.

Therefore, the function

$$s \to \psi\Big(\int_s^\infty f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau + \sum_{t_k > s} I_{3,k}(u(t_k), u'(t_k)\Big)$$

belongs to $PC(J)$. Using (1.7), we obtain

$$\begin{aligned}\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{t_k>s}I_{3,k}(u(t_k),u'(t_k)\Big)\\ \leqslant\psi^*\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)+\psi^*\psi\Big(\sum_{t_k>s}I_{3,k}(u(t_k),u'(t_k)\Big)\\ \leqslant\psi^*\psi\Big(\int_s^\infty \Phi_{r,R}(\tau)\,\mathrm{d}\tau\Big)+\psi^*\psi(A_3).\end{aligned}$$

The above estimates and hypothesis (1.9) show that the function

$$s\to\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{t_k>s}I_{3,k}(u(t_k),u'(t_k)\Big)$$

is integrable on $[0,t]$ for all $t>0$ and T_2u is well defined on $\overline{J}$. Furthermore, it is easy to see that $T_2u\in C^1(\overline{J})$ with

$$(T_2u)'(t)=\alpha+\int_0^t\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{t_k>s}I_{3,k}(u(t_k),u'(t_k)\Big)\,\mathrm{d}s$$

for all $t\geqslant 0$ and $(T_2u)'$ is differentiable with

$$(T_2u)''(t)=\psi\Big(\int_t^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{k\geqslant j-1}I_{3,k}(u(t_k),u'(t_k)\Big)\text{ for all } t\in J_j,$$

$$(T_2u)''(t_j^-)=(T_2u)''(t_j)=\psi\Big(\int_{t_j}^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{k\geqslant j}I_{3,k}(u(t_k),u'(t_k)\Big),$$

and

$$(T_2u)''(t_j^+)=\psi\Big(\int_{t_j}^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau+\sum_{k\geqslant j+1}I_{3,k}(u(t_k),u'(t_k)\Big)\geqslant(T_2u)''(t_j^-).$$

The above identities show that $(T_2u)''$ is nonincreasing and $(T_2u)'$ is concave on $\overline{J}$.

It remains to show that for all $u\in\Omega$,

$$\lim_{t\to\infty}\frac{T_2u(t)}{(1+t)^2}=\lim_{t\to\infty}\frac{(T_2u)'(t)}{1+t}=0$$

to conclude that $T_2\Omega\subset K\cap E_2$. This needs to introduce some functions and establish some estimates. Set for all $t>0$,

$$U(t)=\int_0^t\psi\Big(\int_s^\infty\Phi_{r,R}(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s$$
$$\text{and } V(t)=\int_0^t\psi\Big(\sum_{t_k>s}z_{3,k}\Big)\,\mathrm{d}s.$$

Clearly, $U \in C^1(\overline{J})$ and by L'Hopital's rule

$$\lim_{t\to\infty} \frac{U(t)}{1+t} = \lim_{t\to\infty} U'(t) = \lim_{t\to\infty} \psi\biggl(\int_t^\infty \Phi_{r,R}(\tau)\,\mathrm{d}\tau\biggr) = 0.$$

Let $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that $\sum\limits_{t_k > t_\varepsilon} z_{3,k} \leqslant \phi(\varepsilon)$. Hence, for $t > t_\varepsilon$ we have

$$\begin{aligned} V(t) &= \int_0^{t_\varepsilon} \psi\biggl(\sum_{t_k > s} z_{3,k}\biggr)\,\mathrm{d}s + \int_{t_\varepsilon}^t \psi\biggl(\sum_{t_k > s \geqslant t_\varepsilon} z_{3,k}\biggr)\,\mathrm{d}s \\ &\leqslant \psi(A_3)t_\varepsilon + \varepsilon(t - t_\varepsilon) \end{aligned}$$

leading to

$$\limsup_{t\to\infty} \frac{V(t)}{1+t} \leqslant \lim_{t\to\infty} \frac{\psi(A_3)t_\varepsilon + \varepsilon(t-t_\varepsilon)}{1+t} = \varepsilon.$$

Since ε is arbitrary, we conclude that $\lim\limits_{t\to\infty} V(t)/(1+t) = 0$.

In the remainder of this proof, we let

$$U^* = \sup_{t\geqslant 0} \frac{U(t)}{1+t} \quad \text{and} \quad V^* = \sup_{t\geqslant 0} \frac{V(t)}{1+t}.$$

At this stage, for any $u \in \Omega$ we have

$$\frac{T_2u(t)}{(1+t)^2} = \frac{\int_0^t (T_2u)'(s)\,\mathrm{d}s}{(1+t)^2} \leqslant \frac{(T_2u)'(t)}{1+t} \leqslant \psi^* \frac{U(t)}{1+t} + \psi^* \frac{V(t)}{1+t}, \tag{4.4}$$

leading to

$$\lim_{t\to\infty} \frac{T_2u(t)}{(1+t)^2} = \lim_{t\to\infty} \frac{(T_2u)'(t)}{1+t} = 0.$$

This ends the proof of $T_2\Omega \subset K \cap E_2$.

Claim 2. $T\colon \Omega \to K \cap E_2$ is continuous. Let $(u_n)_n$ be a sequence in Ω with $\lim\limits_{n\to\infty} u_n = u$.

We start by proving that for all $\eta > 0$,

$$\sup_{t\in[0,\eta]} \frac{|(Tu_n)'(t) - (Tu)'(t)|}{1+t} \to 0 \quad \text{as } n \to \infty.$$

For any $s \in J$, we have

$$|f(\tau, u_n(\tau), u_n'(\tau) - f(\tau, u(\tau), u'(\tau))| \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for a.e. } \tau \in [s, \infty)$$

and

$$|f(\tau, u_n(\tau), u_n'(\tau) - f(\tau, u(\tau), u'(\tau))| \leqslant 2\Phi_{r,R}(\tau) \quad \text{for a.e. } \tau \in [s, \infty).$$

By means of the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_s^\infty f(\tau, u_n(\tau), u_n'(\tau))\,\mathrm{d}\tau = \int_s^\infty f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau \text{ for all } s \in I. \tag{4.5}$$

Moreover, since

$$|I_{3,k}(u_n(t_k), u_n'(t_k)) - I_{3,k}(u(t_k), u'(t_k))| \leqslant z_{3,k}$$

and $\sum\limits_{k=1}^{\infty} z_{3,k} < \infty$, the dominated convergence theorem for series gives

$$\lim_{n\to\infty} \sum_{t_k>s} I_{3,k}(u_n(t_k), u_n'(t_k)) = \sum_{t_k>s} I_{3,k}(u(t_k), u'(t_k)). \tag{4.6}$$

Set

$$\begin{aligned} g_n(s) = \bigg|\psi\bigg(\int_s^\infty f(\tau, u_n(\tau), u_n'(\tau))\,\mathrm{d}\tau + \sum_{t_k>s} I_{3,k}(u_n(t_k), u_n'(t_k))\bigg) \\ - \psi\bigg(\int_s^\infty f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau + \sum_{t_k>s} I_{3,k}(u(t_k), u'(t_k))\bigg)\bigg|. \end{aligned}$$

The continuity of ψ, (4.5) and (4.6) lead to $\lim g_n(s) = 0$ for all $s > 0$. Since

$$g_n(s) \leqslant 2\psi\bigg(\int_s^\infty \Phi_{r,R}(\tau)\,\mathrm{d}\tau + A_3\bigg) \quad \text{for all } n \in \mathbb{N},$$

the Lebesgue dominated convergence theorem leads to

$$\begin{aligned} \sup_{t\in[0,\eta]} \frac{|(Tu_n)'(t) - (Tu)'(t)|}{1+t} &\leqslant \sup_{t\in[0,\eta]} |(Tu_n)'(t) - (Tu)'(t)| \\ &\leqslant \int_0^\eta g_n(s)\,\mathrm{d}s \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Now, we have

$$\frac{|(T_2u_n)'(t) - (T_2u)'(t)|}{1+t} \leqslant 2\psi^* \frac{U(t)}{1+t} + 2\psi^* \frac{V(t)}{1+t} \to 0 \quad \text{as } t \to \infty.$$

Therefore, for all $\varepsilon > 0$ some $\eta_\varepsilon > 0$ corresponds such that for all $n \geqslant 1$

$$\sup_{t\in[\eta_\varepsilon,\infty)} \frac{|(T_2u_n)'(t) - (T_2u)'(t)|}{1+t} \leqslant \frac{\varepsilon}{2}.$$

Taking in consideration

$$\lim_{n\to\infty}\left(\sup_{t\in[0,\eta_\varepsilon]} \frac{|(T_2u_n)'(t) - (T_2u)'(t)|}{1+t}\right) = 0,$$

we conclude that there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geqslant n_\varepsilon$, we have

$$\begin{aligned}\|T_2u_n - T_2u\|_6 &\leqslant \sup_{t\in[0,\eta_\varepsilon]} \frac{|(T_2u_n)'(t) - (T_2u)'(t)|}{1+t} + \sup_{t\in[\eta_\varepsilon,\infty)} \frac{|(T_2u_n)'(t) - (T_2u)'(t)|}{1+t}\\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This ends the proof of Claim 2.

Claim 3. $T_2\Omega$ is relatively compact. Estimate (4.4) leads to

$$\|T_2u\|_6 \leqslant \psi^*(U^* + V^*) \quad \text{for all } u \in \Omega,$$

and estimate (4.4) with $\lim\limits_{t\to\infty} U(t)/(1+t) = \lim\limits_{t\to\infty} V(t)/(1+t) = 0$ shows that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for all $t \geqslant t_\varepsilon$ and all $u \in \Omega$, we have

$$\frac{Tu(t)}{(1+t)^2} \leqslant \frac{(Tu)'(t)}{1+t} \leqslant \varepsilon.$$

Hence, conditions (a) and (c) in Lemma 4.6 are satisfied and it remains to show that condition (b) is satisfied.

Let $[0,a] \subset \overline{J}$. For all $u \in \Omega$ and all $t \in [0,a]$ we have

$$T_2u(t) \leqslant (T_2u)'(t) \leqslant \psi^*(U(a) + V(a)) := c_*.$$

Let $t_1, t_2 \in [0,a]$. For all $u \in \Omega$, we obtain by means of the mean value theorem

$$\begin{aligned}\left|\frac{T_2u(t_2)}{(1+t_2)^2} - \frac{Tu(t_1)}{(1+t_1)^2}\right| &\leqslant (1+t_1)^2|T_2u(t_2) - T_2u(t_1)|\\ &\quad + |T_2u(t_1)|\,\bigl|(1+t_2)^2 - (1+t_1)^2\bigr|\\ &\leqslant ((1+a)^2 + 2(1+a))c_*|t_2 - t_1|\end{aligned}$$

and

$$\begin{aligned}\left|\frac{(T_2u)'(t_2)}{1+t_2} - \frac{(T_2u)'(t_1)}{1+t_1}\right| &\leqslant t(1+t_1)|(T_2u)'(t_2) - (T_2u)'(t_1)|\\ &\quad + |(T_2u)'(t_1)|\,\bigl|(1+t_2)^2 - (1+t_1)^2\bigr|\\ &\leqslant |\xi(t_2) - \xi(t_1)| + 2(1+a)c_*|t_2 - t_1|,\end{aligned}$$

where

$$\xi(t) = \int_0^t \psi\Big(\int_s^\infty \Phi_{r,R}(\tau)\,\mathrm{d}\tau + A_3\Big)\,\mathrm{d}s.$$

Taking into account that the function ξ is uniformly continuous on $[0,a]$, we conclude from the above calculations that condition (b) in Lemma 4.6 is fulfilled and $T_2\Omega$ is relatively compact. This ends the proof. □

Lemma 4.9. *Assume that Hypotheses* (1.5), (1.8) *and* (1.9) *hold. If* $u \in K\setminus\{0\}$ *is a fixed point of the mapping* $T = T_1 + T_2\colon K\setminus\{0\} \to K$, *then* u *is a positive solution to bvp* (1.1).

Proof. Let $u \in K\setminus\{0\}$ be a fixed point of T. We then have

$$\begin{aligned}
u(0) = Tu(0) = 0, \quad u'(0) &= (T_2u)'(0) = \alpha,\\
\Delta u(t_k) = \Delta T_1 u(t_k) &= I_{1,k}(u(t_k), u'(t_k)) \quad \text{for } k = 1,2,\ldots,\\
\Delta u'(t_k) = \Delta (T_1 u)'(t_k) &= I_{2,k}(u(t_k), u'(t_k)) \quad \text{for } k = 1,2,\ldots,\\
-\Delta\phi((u)''(t_j)) - \Delta\phi((T_2u)''(t_j)) &= I_{3,j}(u(t_j), u'(t_j)) \geqslant 0,\\
-(\phi((u)''))'(t) = -(\phi((T_2u)''))'(t) &= f(t,u(t),u'(t)) \quad \text{for a.e. } t \in J
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to\infty} u''(t) &= \lim_{t\to\infty} (T_2u)''(t)\\
&= \lim_{t\to\infty} \psi\Big(\int_t^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau + \sum_{t_k>t} I_{3,k}(u(t_k),u'(t_k))\Big)\,\mathrm{d}s = 0.
\end{aligned}$$

The lemma is proved. □

5. Proof of Theorem 1.2

Step 1. In this step we establish some estimates for the norm $\|\cdot\|$. Notice that for all $u \in K\setminus\{0\}$, we have

$$\|Tu\|_3 = \|T_1u\|_3 = \sum_{k\geqslant 1} \frac{I_{1,k}(u(t_k),u'(t_k))}{(1+t_k)^2} \leqslant \|u\|(a_1+b_1) + c_1,$$

$$\|Tu\|_4 = \|T_1u\|_4 = \sum_{k\geqslant 1} \frac{I_{2,k}(u(t_k),u'(t_k))}{1+t_k} \leqslant \|u\|(a_2+b_2) + c_2,$$

$$\|T_1u\|_5 = \sup_{t\geqslant 0} \sum_{t_k<t} \frac{I_{1,k}(u(t_k),u'(t_k))}{(1+t)^2} + \frac{\int_0^t \big(\sum_{t_k<s} I_{2,k}(u(t_k),u'(t_k))\big)\,\mathrm{d}s}{(1+t)^2}$$

$$
\begin{aligned}
&\leqslant \sum_{k\geqslant 1}\frac{I_{1,k}(u(t_k),u'(t_k))}{(1+t_k)^2}+\sum\nolimits_{k\geqslant 1}\frac{I_{2,k}(u(t_k),u'(t_k))}{1+t_k}=\|T_1u\|_3+\|T_1u\|_4\\
&\leqslant \|u\|(a_1+a_2+b_1+b_2)+(c_1+c_2),\\
\|T_1u\|_6 &= \sup_{t\geqslant 0}\frac{\sum_{t_k<t}I_{2,k}(u(t_k),u'(t_k))}{(1+t)^2}\leqslant \sum_{k\geqslant 1}\frac{I_{2,k}(u(t_k),u'(t_k))}{1+t_k}=\|T_1u\|_4,\\
\|T_2u\|_5 &\leqslant \|T_2u\|_6,\\
\|T_2u\|_6 &\leqslant \psi^*\|T_3u\|_6+\psi^*\sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\sum_{t_k>s}I_{2,k}(u(t_k),u'(t_k))\Big)\,\mathrm{d}s\\
&\leqslant \psi^*\|T_3u\|_6+\psi^*\psi(\phi(\|u\|)(a_3+b_3)+c_3),
\end{aligned}
$$

where

$$
T_3u(t)=\int_0^t\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\in K\cap E_2.
$$

Hence, for all $u\in K\setminus\{0\}$, we have

$$
\|T_3u\|=\|T_3u\|_6\leqslant\|Tu\|\leqslant\psi^*\|T_3u\|_6+A\|u\|+B. \tag{5.1}
$$

Step 2. Existence in the case when (1.10) holds

Let $\varepsilon>0$ be such that $(f^0(m_0)+\varepsilon)\Gamma_0(m_0)<1$. For such a positive real ε, there exists $R_1>0$ such that $f(t,(1+t)^2w,(1+t)z)\leqslant(f^0(m_0)+\varepsilon)m_0(t)\phi(w+z)$ for all w,z with $\sup(|w|,|z|)\leqslant R_1$.

Thus, for all $u\in K\cap\partial\Omega_1$, where $\Omega_1=\{u\in E,\ \|u\|<R_1\}$, the following estimates hold.

$$
\begin{aligned}
\|T_3u\|_6 &= \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\int_s^\infty f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\leqslant \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\int_s^\infty f(\tau,(1+\tau)^2\frac{u(\tau)}{(1+\tau)^2},(1+\tau)\frac{u'(\tau)}{(1+\tau)})\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\leqslant \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\int_s^\infty (f^0(m_0)+\varepsilon)m_0(\tau)\phi(2\|u\|)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.
\end{aligned}
$$

Using (1.6) twice, we get

$$
\begin{aligned}
\|T_3u\|_6 &\leqslant \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\phi(2\psi^+(f^0(m_0)+\varepsilon)\|u\|)\int_s^\infty m_0(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\leqslant \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\Big(\phi\Big(2\psi^+(f^0(m_0)+\varepsilon)\|u\|\psi^+\Big(\int_s^\infty m_0(\tau)\,\mathrm{d}\tau\Big)\Big)\Big)\,\mathrm{d}s\\
&= 2\psi^+(f^0(m_0)+\varepsilon)\varrho(m_0)\|u\|.
\end{aligned}
$$

Inserting the above estimate in (5.1), we obtain by using (1.4) that for all $u \in K \cap \partial\Omega_1$,

$$\begin{aligned}\|Tu\| &\leqslant (A + 2\psi^*\psi^+(f^0(\alpha)+\varepsilon)\varrho(m_0))\|u\| \\ &\leqslant (A + 2\psi^* 2\psi^+((\Gamma_0(m_0))^{-1})\varrho(m_0))\|u\| \\ &= A + 2\psi^* 2\psi^+\left(\left(\phi^+\left(\frac{2\psi^*\varrho(m)}{1-A}\right)\right)^{-1}\right)\|u\| = \|u\|.\end{aligned}$$

Now, let $\varepsilon > 0$ be such that $(f_\infty(m_\infty,\theta)-\varepsilon)\Theta(m_\infty,\theta) > 1$. There exists $R_2 > R_1$ such that

$$f(t,(1+t)^2 w,(1+t)z) > (f_\infty(m_\infty,\theta)-\varepsilon)m_\infty(t)\phi(w+z) \tag{5.2}$$

for all $t \in J_\theta$ and $w, z \geqslant 0$ with $|(w,z)| \geqslant R_2$.

Let $\Omega_2 = \{u \in E\colon \|u\|_6 < R_2/\gamma_*\}$, where $\gamma_* = \inf\limits_{t\in J_\theta}\{(\widetilde{\gamma}(t)+\gamma(t))\}$. Hence, for all $u \in K \cap \partial\Omega_2$ and all $\tau \in J_\theta$, we have by Lemma 4.1

$$\frac{u(\tau)}{(1+\tau)^2} + \frac{u'(\tau)}{(1+\tau)} \geqslant (\widetilde{\gamma}(t)+\gamma(t))\|u\|_6 \geqslant \gamma_*\|u\|_6 = R_2.$$

Inserting this in (5.2), we obtain that for $u \in K \cap \partial\Omega_2$ and all $\tau \in J_\theta$, that

$$\begin{aligned}f(\tau,u(\tau),u'(\tau)) &= f\left(\tau,(1+\tau)^2\frac{u(\tau)}{(1+\tau)^2},(1+\tau)\frac{u'(\tau)}{(1+\tau)}\right) \\ &\geqslant (f_\infty(m_\infty,\theta)-\varepsilon)m_\infty(t)\phi\left(\frac{u(\tau)}{(1+\tau)^2}+\frac{u'(\tau)}{(1+\tau)}\right) \\ &\geqslant (f_\infty(m_\infty,\theta)-\varepsilon)m_\infty(t)\phi((\widetilde{\gamma}(t)+\gamma(t))\|u\|_6).\end{aligned}$$

Now, suppose that there are $e \in K \setminus \{0\}$, $t \geqslant 0$, and $u \in K \cap \partial\Omega_2$ such that $u = Tu + te$. Taking in account the above estimate, we obtain by using (1.6) and (1.4) the following contradiction.

$$\begin{aligned}&\|u\|_6 \geqslant \|Tu\|_6 \geqslant \frac{(Tu)'(1/\theta)}{1+(1/\theta)} \geqslant \frac{\theta}{1+\theta}\int_0^{1/\theta}\psi\left(\int_{1/\theta}^{\theta} f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\geqslant \frac{1}{1+\theta}\psi\left(\int_{1/\theta}^{\theta}(f_\infty(m_\infty,\theta)-\varepsilon)m_\infty(\tau)\phi((\widetilde{\gamma}(\tau)+\gamma(\tau))\|u\|_6)\,\mathrm{d}\tau\right) \\ &\geqslant \frac{1}{1+\theta}\psi^-(f_\infty(m_\infty,\theta)-\varepsilon))\psi\left(\phi\left(\|u\|_6\psi^-\left(\int_{1/\theta}^{\theta} m_\infty(\tau)\phi^-((\widetilde{\gamma}(\tau)+\gamma(\tau)))\,\mathrm{d}\tau\right)\right)\right) \\ &= \psi^-((\Theta(m_\infty,\theta))^{-1})\mu(m_\infty,\theta)\|u\|_6 > \|u\|_6.\end{aligned}$$

Therefore, we deduce from Theorem 3.1 that T admits a fixed point $u \in K$ with $R_1 \leqslant \|u\| \leqslant R_2/\gamma_*$ which is by Lemma 4.9 a positive solution to bvp (1.1).

Step 3. Existence in the case when (1.11) holds

Let $\varepsilon > 0$ be such that $(f_0(m_0,\theta) - \varepsilon)\Theta(m_\infty,\theta) > 1$. There exists $\widetilde{R}_1 > 0$ such that

$$f(t,(1+t)^2 w,(1+t)z) > (f_0(m_0,\theta) - \varepsilon)m_0(t)\phi(w+z)$$

for all $w, z \in [0, \widetilde{R}_1]$, and all $t \in J$.

Let $\Omega_1 = \{u \in E\colon \|u\| < \widetilde{R}_1/\gamma_*\}$, where $\gamma_* = \inf\limits_{t\in J_\theta}\{(\widetilde{\gamma}(t) + \gamma(t))\}$. The same calculations as those done in Step 2 show that for all $e \in K\setminus\{0\}$ and all $t \geqslant 0$, such that $u \neq Tu + te$ for all $u \in K \cap \partial\Omega_1$.

Now, let $\varepsilon > 0$ be such that $(f^\infty(m_\infty)+\varepsilon)\Gamma_\infty(m_\infty) < 1$. Then there exists $R_\varepsilon > 0$ such that

$$f(t,(1+t)^2 w,(1+t)z) \leqslant (f^\infty+\varepsilon)m_\infty(t)\phi(w+z)+\omega_{R_\varepsilon}(t)\Psi_{R_\varepsilon}(w,z) \quad \text{for all } w,z>0,$$

where ω_{R_ε} and Ψ_{R_ε} are those given by hypothesis (1.9) for $R = R_\varepsilon$.

Put

$$\begin{aligned}
\Phi_\varepsilon(t) &= \omega_{R_\varepsilon}(t)\Psi_{R_\varepsilon}(R_\varepsilon\widetilde{\gamma}(t), R_\varepsilon\gamma(t)),\\
\overline{\Phi}_\varepsilon &= \sup_{t\geqslant 0}\biggl(\frac{1}{1+t}\int_0^t \psi\biggl(\int_s^\infty \Phi_\varepsilon(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\biggr),\\
\widetilde{R}_2 &= \max\biggl(\frac{\widetilde{R}_1}{\gamma_*}+1, \frac{B+\psi^*\overline{\Phi}_\varepsilon}{1-(2(\psi^*)^2\psi^+(f^\infty(m_\infty)+\varepsilon)\varrho(m_\infty)+A)}\biggr).
\end{aligned}$$

Thus, for all $u \in K\cap\partial\Omega_2$, where $\Omega_2 = \{u\in E\colon \|u\| < \widetilde{R}_2\}$, we have

$$\begin{aligned}
f(\tau,u(\tau),u'(\tau)) &\leqslant (f^\infty(m_\infty)+\varepsilon)m_\infty(\tau)\phi\biggl(\frac{u(\tau)}{(1+\tau)^2}+\frac{u'(\tau)}{(1+\tau)}\biggr)\\
&\quad + \omega_{R_\varepsilon}(\tau)\Psi_{R_\varepsilon}\biggl(\frac{u(\tau)}{(1+\tau)^2},\frac{u'(\tau)}{(1+\tau)}\biggr)\\
&\leqslant (f^\infty(m_\infty)+\varepsilon)m_\infty(\tau)\phi(2\|u\|)+\Phi_\varepsilon(\tau).
\end{aligned}$$

We obtain by using (1.7) and (1.6) that for all $u \in K\cap\partial\Omega_2$,

$$\begin{aligned}
\|T_3u\|_6 &\leqslant \sup_{t\geqslant 0}\frac{1}{1+t}\int_0^t\psi\biggl(\int_s^\infty((f^\infty(m_\infty)+\varepsilon)m_\infty(\tau)\phi(2\|u\|)+\Phi_\varepsilon(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\\
&\leqslant \psi^*\sup_{t\geqslant 0}\biggl(\frac{1}{1+t}\int_0^t\psi\biggl(\int_s^\infty((f^\infty(m_\infty)+\varepsilon)m_\infty(\tau)\phi(2\|u\|)+\Phi_\varepsilon(\tau))\,\mathrm{d}\tau\biggr)\biggr)\,\mathrm{d}s+\psi^*\overline{\Phi}_\varepsilon\\
&\leqslant 2\psi^*\psi^+(f^\infty(\alpha)+\varepsilon)\varrho(m_\infty)\|u\|+\psi^*\overline{\Phi}_\varepsilon.
\end{aligned}$$

Inserting the above estimate in (5.1), we obtain then by using (1.4) that for all $u \in K \cap \partial\Omega_2$

$$\begin{aligned}\|Tu\| &\leqslant (2(\psi^*)^2\psi^+(f^\infty(m_\infty)+\varepsilon)\varrho(m_\infty)+A)\|u\|+B+\psi^*\overline{\Phi}_\varepsilon\\ &\leqslant (2(\psi^*)^2\psi^+((\Gamma_\infty(m_\infty))^{-1})\varrho(m_\infty)+A)\|u\|+B+\psi^*\overline{\Phi}_\varepsilon\\ &= \left(2(\psi^*)^2\psi^+\left(\left(\phi^+\left(\frac{2(\psi^*)^2\varrho(m_\infty)}{1-A}\right)\right)^{-1}\right)\varrho(m_\infty)+A\right)\|u\|+B+\psi^*\overline{\Phi}_\varepsilon\\ &= \|u\|.\end{aligned}$$

We deduce from Theorem 3.1 that T admits a fixed point $u \in K$ with $\widetilde{R}_1/\gamma_* \leqslant \|u\| \leqslant \widetilde{R}_2$ which is, by Lemma 4.9, a positive solution to bvp (1.1).

The proof of the main theorem is complete. □

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