

FIXED POINT THEOREMS FOR HYBRID PAIR  
OF WEAK COMPATIBLE MAPPINGS  
IN PARTIAL METRIC SPACES

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*Abstract.* The notions of compatible mappings play a crucial role in metrical fixed point theory. Partial metric spaces are a generalization of the notion of a metric space in the sense that distance of a point from itself is not necessarily zero. In this paper, we prove coincidence and fixed point theorems for a pair of single-valued and multi-valued weak compatible mappings on a complete partial metric space. Our main results generalize, in particular, the results of Kaneko and Sessa (1989), Pathak (1995) and Kessy, Kumar and Kakiko (2017). Examples that illustrate the generality of our results are also provided.

*Keywords:* partial metric space; weak compatible mapping; hybrid pair of mapping

*MSC 2020:* 47H10, 54H25

## 1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle has been generalized in different dimensions. One of the concepts is using commuting maps, Sessa [22] introduced the concept of weakly commuting maps. Jungck [12] made more generalized commuting and weakly commuting maps called compatible maps. Further, Murthy et al. [18], Pathak and Khan [21], Al-Thagafi and Shahzad [2], Bouhadjera and Djoud [8], Abbas and Rhoades [1] gave another generalizations of noncommuting mappings without continuity in generalized metric spaces. Kaneko and Sessa [13] extended the concept of compatible mappings due to Jungck [12] to include multi-valued mappings  $F$  as well as single-valued mappings  $f$ . They followed the works of Kubiak [15] and Nadler [19] and proved coincidence and fixed point theorems for a hybrid pair of compatible mappings. Pathak [20] extended the concept of compatible hybrid mappings to  $f$ -weak

compatible hybrid mappings and proved a coincidence theorem, an extension of the results of Kaneko and Sessa [13], for the mappings satisfying the following contractive condition:

$$(1.1) \quad H(Fx, Fy) \leq h \max \left\{ d(fx, fy), d(fx, Fx), d(fy, Fy), \right. \\ \left. \frac{1}{2}[d(fx, Fy) + d(fy, Fx)] \right\}$$

for all  $x, y$  in a complete metric space  $(X, d)$ , where  $0 \leq h < 1$ .

On the other hand, Mathews [16] introduced the distance notion which he named partial metric while studying denotational semantics in data flow networks, and generalized Banach contraction principle to partial metric spaces. Matthews also in [17] investigated rigorously the topological aspects for partial metric spaces. Inspired by results in [16], Ćirić et al. [10] proved some common fixed point theorems for generalized contractions on partial metric spaces. Altun and Romaguera [3] introduced the notion of 0-Cauchy sequence, 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

In 2012 Aydi et al. [5] introduced and studied the notion of partial Hausdorff metric and used it to obtain the Nadler's fixed point theorem for multivalued contraction mappings [19] in partial metric spaces. This important notion of multivalued contraction has been introduced on other distance spaces as well, for example, metric like spaces and existence of fixed point for generalized multivalued contractions have been investigated [6], [7]. In [11], Haghi showed vividly that some partial metric fixed point results can be obtained from their corresponding results in metric spaces. Recently, there has been several studies on possible generalizations of the existing metric fixed point results to partial metric spaces. We refer the reader to Vetro and Vetro [24], where they have proved the coincidence point and common fixed point theorems for two self-mappings satisfying generalized contractive conditions, defined by implicit relations in the setting of partial metric space. This paper forms a part of the studies for metric fixed point results for a hybrid pair of weak compatible mappings.

In this paper we therefore establish the coincidence and fixed point theorems for a hybrid pair of weak compatible mappings in partial metric spaces that satisfy the following condition:

$$(1.2) \quad H_p(Fx, Fy) \leq h \max \left\{ p(fx, fy), p(fx, Fx), p(fy, Fy), \right. \\ \left. \frac{1}{2}[p(fx, Fy) + p(fy, Fx)] \right\}$$

for all  $x, y$  in a complete partial metric space  $(X, p)$ , where  $0 \leq h < 1$ .

In the sequel, we will require the following definitions and preliminary results.

**Definition 1.1.** Let  $X$  be a nonempty set. Let  $F: X \rightarrow 2^X$ , where  $2^X$  denotes the collection of all nonempty subsets of  $X$ , be a multi-valued mapping and  $f: X \rightarrow X$  be a single-valued mapping. Then a point  $s$  in  $X$  is called a fixed point of the mappings  $F$  and  $f$  if  $s = fs \in Fs$ . A point  $z \in X$  is called a coincidence point of  $F$  and  $f$  if  $fz \in Fz$ .

**Definition 1.2** ([17]). Let  $X$  be a nonempty set. A partial metric space is a pair  $(X, p)$ , where  $p$  is a function  $p: X \times X \rightarrow [0, \infty)$ , called the partial metric, such that for all  $x, y, z \in X$  it satisfies the following properties:

- (P1)  $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$ ;
- (P2)  $p(x, x) \leq p(x, y)$ ;
- (P3)  $p(x, y) = p(y, x)$ ;
- (P4)  $p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$ .

Clearly, if  $p(x, y) = 0$ , then by (P1), (P2) and (P3),  $x = y$ . But the converse is in general not true.

An example of partial metric space is the pair  $([0, \infty), p)$ , where  $p(x, y) = \max\{x, y\}$  for all  $x, y \in [0, \infty)$ . More examples may be found in [16], [9].

Each partial metric  $p$  on  $X$  generates a  $T_0$  topology  $\tau_p$  on  $X$ , whose basis is the collection of all open  $p$ -balls  $\{B_p(x, \varepsilon): x \in X, \varepsilon > 0\}$ , where  $B_p(x, \varepsilon) = \{y \in X: p(x, y) < p(x, x) + \varepsilon\}$  for all  $x \in X$  and a real number  $\varepsilon$ .

Let  $(X, p)$  be a partial metric space. Let  $G$  be a nonempty subset of  $X$ . It is well known [4] that  $x \in X$  is a point of closure of  $G$ , denoted  $x \in \overline{G}$ , if and only if  $p(x, G) = p(x, x)$ . Also, the set  $G$  is said to be closed in  $(X, p)$  if and only if  $G = \overline{G}$ .

**Definition 1.3** ([17]). (i) A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  is said to converge to some  $x \in X$  if  $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$ .

(ii) A sequence  $\{x_n\}$  in a partial metric space  $(X, p)$  is a Cauchy sequence if  $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$  exists and is finite.

(iii) A partial metric space  $(X, p)$  is said to be complete if every Cauchy sequence  $\{x_n\}$  in  $X$  converges with respect to the topology  $\tau_p$  to a point  $x \in X$  such that  $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$ .

**Lemma 1.4** ([17]). Let  $(X, p)$  be a partial metric space. Then the mapping  $p^s: X \times X \rightarrow [0, \infty)$  given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all  $x, y \in X$  defines a metric on  $X$ .

Thus, a partial metric on a nonempty set  $X$  induces a metric  $d$  on  $X$ , where  $d = p^s$ .

**Lemma 1.5** ([9]).

- (i) A sequence  $\{x_n\}$  is a Cauchy sequence in a partial metric space  $(X, p)$  if and only if it is a Cauchy sequence in the metric space  $(X, p^s)$ .
- (ii) A partial metric space  $(X, p)$  is complete if and only if the metric space  $(X, p^s)$  is complete.

In 2012, Aydi et al. [5] extended Nadler's [19] multivalued concept in a metric space to a multivalued partial metric space by introducing partial Hausdorff metric as follows.

Let  $(X, p)$  be a partial metric space. Let  $CB^p(X)$  be the family of all nonempty, closed and bounded subsets of the partial metric space  $(X, p)$ , induced by the partial metric  $p$ . Note that the closeness is taken from  $(X, \tau_p)$  ( $\tau_p$  is a topology induced by  $p$ ) and the boundedness is given as follows:  $A$  is a bounded subset in  $(X, p)$  if there exists  $x_0 \in X$  and  $M \geq 0$  such that for all  $a \in A$  we have  $a \in B_p(x_0, M)$ , that is,  $p(x_0, a) < p(a, a) + M$ .

**Definition 1.6** ([5]). Let  $(X, p)$  be a partial metric space and  $CB^p(X)$  denote the collection of all nonempty bounded and closed subsets of  $X$ . For  $A, B \in CB^p(X)$ , define

$$H_p(A, B) = \max \{ \delta_p(A, B), \delta_p(B, A) \},$$

where  $p(x, A) = \inf \{ p(x, a) : a \in A \}$  and  $\delta_p(A, B) = \sup \{ p(a, B) : a \in A \}$ . Then the mapping  $H_p$  is a partial metric, called the partial Hausdorff metric, on  $CB^p(X)$  induced by the partial metric  $p$ .

It is immediate to check that  $p(x, A) = 0 \Rightarrow p^s(x, A) = 0$ , where  $p^s(x, A) = \inf \{ p^s(x, a), a \in A \}$ .

Now, we shall give some properties of mapping  $\delta_p: CB^p(X) \times CB^p(X) \rightarrow [0, \infty)$ .

**Proposition 1.7** ([5]). Let  $(X, p)$  be a partial metric space. For any  $A, B, C \in CB^p(X)$  we have the following:

- (i)  $\delta_p(A, A) = \sup \{ p(a, a), a \in A \}$ ;
- (ii)  $\delta_p(A, A) \leq \delta_p(A, B)$ ;
- (iii)  $\delta_p(A, B) = 0$  implies that  $A \subseteq B$ ;
- (iv)  $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$ .

**Proposition 1.8** ([5]). Let  $(X, p)$  be a partial metric space. For any  $A, B, C \in CB^p(X)$  we have the following:

(h1)  $H_p(A, A) \leq H_p(A, B)$ ;

(h2)  $H_p(A, B) = H_p(B, A)$ ;

(h3)  $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$ .

**Corollary 1.9** ([5]). Let  $(X, p)$  be a partial metric space. For  $A, B, \in CB^p(X)$ , the following holds:  $H_p(A, B) = 0$  implies that  $A = B$ .

By Proposition 1.8 and Corollary 1.9, we say that the mapping  $H_p: CB^p(X) \times CB^p(X) \rightarrow [0, \infty)$  is a partial Hausdorff metric induced by  $p$ .

**Lemma 1.10** ([5]). Let  $(X, p)$  be a partial metric space. Let  $A, B \in CB^p(X)$  and  $q > 1$ . Then for any  $a \in A$  there exists  $b \in B$  that depends on  $a$  such that the following holds:

$$p(a, b) \leq qH_p(A, B).$$

**Definition 1.11** ([13]). The mappings  $f: X \rightarrow X$  and  $F: X \rightarrow CB(X)$  are compatible if  $fFx \in CB(X)$  for all  $x \in X$  and  $H(Ffx_n, fFx_n) \rightarrow 0$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $Fx_n \rightarrow N \in CB(X)$  and  $fx_n \rightarrow s \in N$ .

**Definition 1.12** ([20]). Let  $(X, d)$  be a metric space and  $F: X \rightarrow CB(X)$  and  $f: X \rightarrow X$  be mappings. The mappings  $F$  and  $f$  are  $f$ -weak compatible if  $fFx \in CB(X)$  for all  $x \in X$  and the following limits exist and satisfy:

(i)  $\lim_{n \rightarrow \infty} H(fFx_n, Ffx_n) \leq \lim_{n \rightarrow \infty} H(Ffx_n, Fx_n)$ , and

(ii)  $\lim_{n \rightarrow \infty} D(fFx_n, fx_n) \leq \lim_{n \rightarrow \infty} H(Ffx_n, Fx_n)$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $Fx_n \rightarrow N \in CB(X)$  and  $fx_n \rightarrow s \in N$ .

Example 1 of Pathak [20] shows that the compatible mappings  $F$  and  $f$  are  $f$ -weak compatible mappings. But the converse may not hold.

**Lemma 1.13** ([20]). Let  $F: X \rightarrow CB(X)$  and  $f: X \rightarrow X$  be  $f$ -weak compatible mappings. If  $fz \in Fz$  for some  $z \in X$ , then  $fFz = Ffz$ .

We present an extension of the notion of weak compatibility of hybrid pair of mappings of Pathak [20] on metric spaces in partial metric spaces.

**Definition 1.14.** Let  $(X, p)$  be a partial metric space. Let  $F: X \rightarrow CB^p(X)$  and  $f: X \rightarrow X$  be mappings. The mappings  $F$  and  $f$  are said to be  $f$ -weak compatible if  $fFx \in CB^p(X)$  for all  $x \in X$  and the following limits exist and satisfy:

(i)  $\lim_{n \rightarrow \infty} H_p(fFx_n, Ffx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n)$ , and

(ii)  $\lim_{n \rightarrow \infty} p(fFx_n, fx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n)$  whenever  $\{x_n\}$  is a sequence in  $X$  such that  $Fx_n \rightarrow N \in CB^p(X)$  and  $fx_n \rightarrow s \in N$ .

Clearly, the class of weak compatible mappings on a partial metric space include that of compatible mappings (cf. [14]).

**Example 1.15.** Let  $X = [0, \infty]$  be endowed with the Euclidean partial metric defined by  $p(x, y) = |x - y|$ . Let  $f: X \rightarrow X$  and  $T: X \rightarrow CB^p(X)$  be mappings defined by

$$fx = x^2 + 1, \quad Fx = \left[0, \frac{2}{3}(2x + 1)\right] \quad \text{if } x \geq 0,$$

and  $x_n = 1 + 1/(n + 1)$  is a sequence in  $X$  such that  $x_n \rightarrow 1$ .  $F$  and  $f$  are clearly continuous:  $F(X) = f(X) = X$ . Thus, we have

$$\lim_{n \rightarrow \infty} Fx_n \rightarrow [0, 2], \quad \lim_{n \rightarrow \infty} fx_n \rightarrow 2 \in [0, 2].$$

Now, we compute the metrics involving  $F$ ,  $f$  and  $x_n$  as follows:

$$\begin{aligned} Ffx_n &= \left[1, \left(\frac{2}{3}(2x_n + 1)\right)^2 + 1\right] = \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right], \\ fFx_n &= \left[0, \frac{2}{3}(2(x_n^2 + 1) + 1)\right] = \left[0, \frac{4x_n^2 + 6}{3}\right], \\ Fx_n &= \left[0, \frac{2}{3}(2x_n + 1)\right] = \left[0, \frac{4x_n + 2}{3}\right], \\ fx_n &= x_n^2 + 1. \end{aligned}$$

Applying Definition 1.11, we get

$$\begin{aligned} (1.3) \quad H_p(fFx_n, Ffx_n) &= H_p\left(\left[0, \frac{4x_n^2 + 6}{3}\right], \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right]\right) \\ &\leq \max \left\{ \delta_p\left[\left[0, \frac{4x_n^2 + 6}{3}\right], \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right]\right], \right. \\ &\quad \left. \delta_p\left(\left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right], \left[0, \frac{4x_n^2 + 6}{3}\right]\right) \right\}, \end{aligned}$$

$$\begin{aligned} (1.4) \quad \delta_p\left(\left[0, \frac{4x_n^2 + 6}{3}\right], \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right]\right) &= \max \left\{ p\left(0, \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right]\right), \right. \\ &\quad \left. p\left(\frac{4x_n^2 + 6}{3}, \left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right]\right) \right\} \\ &\leq \max \left\{ 1, \frac{4x_n^2 + 16x_n - 5}{9} \right\} = \frac{4x_n^2 + 16x_n - 5}{9}, \end{aligned}$$

$$\begin{aligned} (1.5) \quad \delta_p\left(\left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right], \left[0, \frac{4x_n^2 + 6}{3}\right]\right) &= \max \left\{ p\left(1, \left[0, \frac{4x_n^2 + 6}{3}\right]\right), p\left(\frac{16x_n^2 + 16x_n + 13}{9}, \left[0, \frac{4x_n^2 + 6}{3}\right]\right) \right\} \\ &\leq \max \left\{ 1, \frac{4x_n^2 + 16x_n - 5}{9} \right\} = \frac{4x_n^2 + 16x_n - 5}{9}. \end{aligned}$$

Using (1.4) and (1.5) in (1.3) we obtain

$$H_p(fFx_n, Ffx_n) = \frac{4x_n^2 + 16x_n - 5}{9},$$

$$\lim_{n \rightarrow \infty} H_p(fFx_n, Ffx_n) = \lim_{n \rightarrow \infty} \frac{4x_n^2 + 16x_n - 5}{9} = \frac{15}{9}.$$

Similarly, we calculate

$$H_p(Ffx_n, Fx_n) = H_p\left(\left[1, \frac{16x_n^2 + 16x_n + 13}{9}\right], \left[0, \frac{4x_n + 2}{3}\right]\right),$$

$$H_p(Ffx_n, Fx_n) = \frac{16x_n^2 + 4x_n + 7}{9},$$

$$\lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n) = \lim_{n \rightarrow \infty} \frac{16x_n^2 + 4x_n + 7}{9} = \frac{27}{9} = 3,$$

$$\lim_{n \rightarrow \infty} H_p(fFx_n, Ffx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n).$$

Next, we have

$$H_p(fFx_n, fx_n) = H_p\left(\left[0, \frac{4x_n^2 + 6}{3}\right], x_n^2 + 1\right)$$

$$\leq \min\left\{p(0, x_n^2 + 1), p\left(\frac{4x_n^2 + 6}{3}, x_n^2 + 1\right)\right\}$$

$$\leq \min\left\{x_n^2 + 1, \frac{x_n^2 + 3}{3}\right\}$$

$$= \frac{x_n^2 + 3}{3},$$

$$\lim_{n \rightarrow \infty} H_p(fFx_n, fx_n) = \lim_{n \rightarrow \infty} \frac{x_n^2 + 3}{3} = \frac{4}{3},$$

$$\lim_{n \rightarrow \infty} H_p(fFx_n, fx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n).$$

Thus, the mappings  $f$  and  $F$  are  $f$ -weak compatible mappings.

We present an extension of Lemma 1.13 in partial metric spaces.

**Lemma 1.16.** *Let  $(X, p)$  be a partial metric space. Let  $F: X \rightarrow CB^p(X)$  and  $f: X \rightarrow X$  be  $f$ -weak compatible mappings. If  $fz \in Fz$  for some  $z \in X$ , then  $fFz = Ffz$ .*

*Proof.* Let  $x_n = z$  for each  $n$ . Then  $fx_n \rightarrow fz$  and  $Fx_n \rightarrow N = Fz$ . Assume that  $fz \in Fz$ . Then by compatibility of the mappings  $f$  and  $F$  (see Definition 1.14) the following hold:

$$(1.6) \quad H_p(fFz, Ffz) \leq H_p(Ffz, Fz)$$

and

$$p(fFz, fz) = p(fFz, fz) \leq H_p(Ffz, Fz).$$

Now, we consider  $H_p(Ffz, Fz)$ . Using (1.2) we have the following:

$$\begin{aligned} & H_p(Ffz, Fz) \\ & \leq h \max \left\{ p(ffz, fz), p(fz, Ffz), p(fz, Fz), \frac{1}{2}[p(ffz, Fz) + p(fz, Ffz)] \right\} \\ & \leq h \max \{ H_p(Ffz, Fz), H_p(Ffz, Fz), p(fz, fz), H_p(Ffz, Fz) \}. \end{aligned}$$

Therefore  $H_p(fFz, Fz) \leq hH_p(Ffz, Fz)$ . This holds only when  $H_p(Ffz, Fz) = 0$ , which implies  $Ffz = Fz$ . So by (1.6) we have  $Ffz = Ffz$ .  $\square$

Pathak in [20] established the following coincidence point result.

**Theorem 1.17.** *Let  $(X, d)$  be a complete metric space,  $f: X \rightarrow X$  and  $F: X \rightarrow CB(X)$  be  $f$ -weak compatible continuous mappings such that  $F(X) \subseteq f(X)$  and satisfying condition (1.1). Then there exists a point  $s \in X$  such that  $fs \in Ts$ .*

In this paper, we state and prove a coincidence theorem for a pair of hybrid mappings in partial metric spaces. We generalize Theorem 1.17 to complete partial metric spaces and obtain fixed point theorems for the mappings.

## 2. MAIN RESULTS

Now, we are ready to present our main results.

**Theorem 2.1.** *Let  $(X, p)$  be a complete partial metric space. Let  $f: X \rightarrow X$  and  $F: X \rightarrow CB^p(X)$  be  $f$ -weak compatible continuous mappings such that  $F(X) \subseteq f(X)$  and satisfying condition (1.2). Then there exists a point  $s \in X$  such that  $fs \in Fs$ .*

*Proof.* Let  $x_0 \in X$  be arbitrary. Since  $F(X) \subseteq f(X)$ , we can find  $x_1 \in X$  such that  $fx_1 \in Fx_0$ . By the definition of  $H_p$  (see Definition 1.6), and (1.2) for  $h = 0$ , we have

$$p(fx_1, Fx_1) \leq H_p(Fx_0, Fx_1) = 0$$

from which by Definition 1.2 we have

$$p(fx_1, fx_1) = p(fx_1, Fx_1).$$

Thus,  $fx_1$  is contained in  $Fx_1$  since  $Fx_1$  is closed.

Assume  $0 < h < 1$ . Let us define  $q = 1/\sqrt{h}$ . So  $q > 1$ . By Lemma 1.10, there exists a point  $z_1 \in Fx_1$  such that  $p(z_1, fx_1) \leq qH_p(Fx_1, Fx_0)$ . This inequality may be in reverse direction if  $q > 0$  or  $q \leq 1$ . Since  $F(X) \subseteq f(X)$ , we can find  $x_2 \in X$  such that  $z_1 = fx_2 \in Fx_1$ .



In general, after selecting  $x_n$ , we can choose  $x_{n+1} \in X$  and set  $z_n = fx_{n+1} \in Fx_n$  satisfying  $p(z_n, fx_n) = p(fx_{n+1}, fx_n) \leq qH_p(Fx_n, Fx_{n-1})$  for each  $n \geq 1$ .

By (1.2), we have the following:

$$\begin{aligned}
 (2.1) \quad p(fx_n, fx_{n+1}) &\leq qH_p(Fx_{n-1}, Fx_n) \\
 &\leq qh \max \left\{ p(fx_{n-1}, fx_n), p(fx_{n-1}, Fx_{n-1}), p(fx_n, Fx_n), \right. \\
 &\quad \left. \frac{1}{2}[p(fx_{n-1}, Fx_n) + p(fx_n, Fx_{n-1})] \right\} \\
 &\leq \sqrt{h} \max \left\{ p(fx_{n-1}, fx_n), p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1}), \right. \\
 &\quad \left. \frac{1}{2}[p(fx_{n-1}, fx_{n+1}) + p(fx_n, fx_n)] \right\} \\
 &\leq \sqrt{h} \max \left\{ p(fx_n, fx_{n-1}), p(fx_n, fx_{n+1}), \right. \\
 &\quad \left. \frac{1}{2}[p(fx_n, fx_{n-1}) + p(fx_n, fx_{n+1})] \right\} \\
 &\leq \sqrt{h} \max \{ p(fx_n, fx_{n-1}), p(fx_n, fx_{n+1}) \}, \\
 (2.2) \quad p(fx_n, fx_{n+1}) &\leq \sqrt{h} p(fx_n, fx_{n-1}) \quad \forall n \geq 2.
 \end{aligned}$$

By mathematical induction, we obtain

$$(2.3) \quad p(fx_n, fx_{n+1}) \leq (\sqrt{h})^{n-1} p(fx_2, fx_1) \quad \forall n \in \mathbb{N}.$$

By (2.3) and the triangle inequality property (see Definition 1.2 (P4)), for any  $m \in \mathbb{N}$  we have

$$\begin{aligned}
 p(fx_n, fx_{n+m}) &\leq p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2}) \\
 &\quad + \dots + p(fx_{n+m-2}, fx_{n+m-1}) + p(fx_{n+m-1}, fx_{n+m}) \\
 &\leq [(\sqrt{h})^{n-1} + (\sqrt{h})^n + \dots + (\sqrt{h})^{n+m-3} + (\sqrt{h})^{n+m-2}] p(x_2, x_1) \\
 &\leq \frac{(\sqrt{h})^{n-1}}{1 - \sqrt{h}} p(x_2, x_1) \rightarrow 0 \quad \text{as } n \rightarrow \infty \text{ since } 0 < h < 1.
 \end{aligned}$$

By Lemma 1.4, for any  $m \in \mathbb{N}$  we have  $p^s(fx_n, fx_{n+m}) \leq 2p(fx_n, fx_{n+m}) \rightarrow 0$  as  $n \rightarrow \infty$ . This yields that  $\{fx_n\}$  is a Cauchy sequence with respect to  $p^s$  and hence convergent in a complete metric space  $(X, p^s)$ . Therefore, by Lemma 1.5, there exists some  $t \in X$  such that

$$(2.4) \quad p(t, t) = \lim_{n \rightarrow \infty} p(fx_n, t) = \lim_{n, m \rightarrow \infty} p(fx_n, fx_m).$$

From (2.1) and (2.2) we have

$$\begin{aligned} qH_p(Fx_n, Fx_{n-1}) &\leq \sqrt{h}p(fx_{n-1}, fx_n), \\ H_p(Fx_n, Fx_{n-1}) &\leq hp(fx_{n-1}, fx_n) \quad \text{for } n \geq 2. \end{aligned}$$

Since  $\{fx_n\}$  is a Cauchy sequence, this implies that  $\{Fx_n\}$  is a Cauchy sequence. Hence,  $\{Fx_n\}$  is convergent in a complete partial metric space  $(CB^p(X), H_p)$ . Now, let  $Fx_n \rightarrow N \in CB^p(X)$ . Then we have the following:

$$\begin{aligned} p(s, N) &\leq p(s, fx_n) + p(fx_n, N) - p(fx_n, fx_n) \\ &\leq p(s, fx_n) + H_p(Fx_{n-1}, N) - H_p(Fx_{n-1}, Fx_{n-1}) \\ &\leq p(s, s) \quad \text{as } n \rightarrow \infty \text{ by (2.4)} \\ &= p(s, s). \end{aligned}$$

Therefore  $s \in N$ , since  $N$  is closed.

We now show that  $s \in X$  is a coincidence point of the mappings  $f$  and  $F$ . Since  $f$  and  $F$  are  $f$ -weak compatible mappings, the following hold:

$$\lim_{n \rightarrow \infty} H_p(fFx_n, Ffx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n),$$

and  $\lim_{n \rightarrow \infty} p(fFx_n, fx_n) \leq \lim_{n \rightarrow \infty} H_p(Ffx_n, Fx_n)$ . Therefore by the continuity of the mappings  $f$  and  $F$  we have

$$(2.5) \quad H_p(fN, Fs) \leq H_p(Fs, N)$$

and

$$(2.6) \quad p(fs, s) \leq H_p(Fs, N) \text{ since } p(fs, s) \leq p(fN, s).$$

We consider  $p(fs, Fs)$ :

$$\begin{aligned} p(fs, Fs) &\leq p(fs, ffx_{n+1}) + p(ffx_{n+1}, Fs) - p(ffx_{n+1}, ffx_{n+1}) \\ &\leq p(fs, ffx_{n+1}) + H_p(fFx_n, Fs) - p(ffx_{n+1}, ffx_{n+1}) \\ &\leq p(fs, ffx_{n+1}) + H_p(fFx_n, Ffx_n) + H_p(Ffx_n, Fs) \\ &\quad - p(ffx_{n+1}, ffx_{n+1}) - H_p(Fx_n, Fx_n) \\ &\leq H_p(fN, Fs) \quad \text{as } n \rightarrow \infty \\ &\leq H_p(Fs, N) \quad \text{by (2.5)}. \end{aligned}$$

We now consider  $H_p(Fx_n, Fs)$ :

Using (1.2) we have

$$\begin{aligned}
 H_p(Fx_n, Fs) &\leq h \max \left\{ p(fx_n, fs), p(fx_n, Fx_n), p(fs, Fs), \right. \\
 &\quad \left. \frac{1}{2}[p(fx_n, Fs) + p(fs, Fx_n)] \right\} \\
 &\leq h \max \left\{ p(fx_n, fs), p(fx_n, Fx_n), p(fs, Fs), \right. \\
 &\quad \left. \frac{1}{2}[p(fx_n, Fs) + p(fs, fx_n) + p(fx_n, Fx_n)] \right\} \\
 &\leq h \max \left\{ p(s, fs), p(s, N), p(fs, Fs), \right. \\
 &\quad \left. \frac{1}{2}[p(s, Fs) + p(s, fs) + p(s, N)] \right\} \quad \text{as } n \rightarrow \infty \\
 &\leq h \max \left\{ H_p(Fs, N), p(s, s), H_p(Fs, N), \right. \\
 &\quad \left. \frac{1}{2}[H_p(Fs, N) + H_p(Fs, N) + p(s, s)] \right\} \quad \text{by (2.5) and (2.6).}
 \end{aligned}$$

Therefore,  $H_p(N, Fs) \leq h[H_p(N, Fs) + p(s, s)]$ , which implies  $H_p(N, Fs) = 0$  since  $0 < h < 1$  and  $p(s, s) \leq H_p(N, Fs)$ . Thus,  $p(fs, Fs) = 0$ , which implies  $p(fs, fs) = p(fs, Fs)$ . Whence  $fs \in Fs$ , as desired.  $\square$

Now, we will provide an example to support the results proved in Theorem 2.1.

**Example 2.2.** Let  $X, f$  and  $F$  be as defined in Example 1.1. Clearly  $F(X) = f(X) = X$ . Now, for  $x > y$  we have

$$\begin{aligned}
 H_p(Fx, Fy) &= \max\{\delta_p(Fx, Fy), \delta_p(Fy, Fx)\}, \\
 H_p\left(\left[0, \frac{4x+2}{3}\right], \left[0, \frac{4y+2}{3}\right]\right) &= \max\left\{\delta_p\left(\left[0, \frac{4x+2}{3}\right], \left[0, \frac{4y+2}{3}\right]\right), \right. \\
 &\quad \left. \delta_p\left(\left[0, \frac{4y+2}{3}\right], \left[0, \frac{4x+2}{3}\right]\right)\right\}, \\
 \delta_p\left(\left[0, \frac{4x+2}{3}\right], \left[0, \frac{4y+2}{3}\right]\right) &\leq \max\left\{p\left(0, \left[0, \frac{4y+2}{3}\right]\right), p\left(\frac{4x+2}{3}, \left[0, \frac{4y+2}{3}\right]\right)\right\} \\
 &= \frac{4(x-y)}{3}, \\
 \delta_p\left(\left[0, \frac{4y+2}{3}\right], \left[0, \frac{4x+2}{3}\right]\right) &\leq \max\left\{p\left(0, \left[0, \frac{4x+2}{3}\right]\right), p\left(\frac{4y+2}{3}, \left[0, \frac{4x+2}{3}\right]\right)\right\} \\
 &= \frac{4(y-x)}{3}, \\
 H_p(Fx, Fy) &= \max\left\{\frac{4(x-y)}{3}, \frac{4(y-x)}{3}\right\} = \frac{4(x-y)}{3}, \\
 p(fx, fy) &= p\{x^2 + 1, y^2 + 1\} = x^2 - y^2, \\
 p(fx, Fx) &= p\left\{x^2 + 1, \left[0, \frac{4x+2}{3}\right]\right\} = \frac{3x^2 - 4x + 1}{3},
 \end{aligned}$$

$$\begin{aligned}
p(fy, Fy) &= p\left\{y^2 + 1, \left[0, \frac{4y + 2}{3}\right]\right\} = \frac{3y^2 - 4y + 1}{3}, \\
p(fx, Fy) &= p\left\{x^2 + 1, \left[0, \frac{4y + 2}{3}\right]\right\} = \frac{3x^2 - 4y + 1}{3}, \\
p(fy, Fx) &= p\left\{y^2 + 1, \left[0, \frac{4x + 2}{3}\right]\right\} = \frac{3y^2 - 4x + 1}{3}.
\end{aligned}$$

Using contraction inequality we have

$$\begin{aligned}
H_p(Fx, Fy) &\leq \max\left\{x^2 - y^2, \frac{3x^2 - 4x + 1}{3}, \frac{3y^2 - 4y + 1}{3}, \right. \\
&\quad \left. \frac{1}{2}\left[\frac{3x^2 - 4y + 1}{3} + \frac{3y^2 - 4x + 1}{3}\right]\right\}, \\
\frac{4(x - y)}{3} &\leq h(x^2 - y^2),
\end{aligned}$$

which is true for  $0 < h < 1$ .

Notice that our example satisfies all conditions of equation (2.1). Thus, all conditions of Theorem 2.1 are satisfied.

**Remark 2.3.** Even in the case when  $p$  in Example 2.2 defines a metric, Theorem 1.17 does not apply since the mapping  $f$  is discontinuous.

**Corollary 2.4.** *Let  $(X, p)$  be a complete partial metric space. Let  $F: X \rightarrow CB^p(X)$  and  $f: X \rightarrow X$  be continuous mappings satisfying  $H_p(Fx, Fy) \leq hp(fx, fy)$  for all  $x, y \in X$ , where  $0 \leq h < 1$  and  $Ffx = fFx$ . If the mappings  $f, F$  are such that  $F(X) \subseteq f(X)$ , then the mappings have a coincidence point.*

**Remark 2.5.** Let  $(X, p)$  be a partial metric space. Denote by  $PB^p(X)$  the collection of all nonempty bounded subsets  $A$  of  $X$  such that for each  $x \in X$  there exists a point  $y \in A$  with  $p(x, y) = p(x, A)$ . Let  $F: X \rightarrow PB^p(X)$  be a mapping. Then the iterative process  $z_n$  in the above proof of Theorem 2.1 can be simplified to the iteration scheme of Smithson (see [23]), where  $Fx$  is compact and therefore contained in  $PB^p(X)$ . This can be done as follows: after selecting  $x_n$ , let  $x_{n+1} \in X$  be such that  $z_n = fx_{n+1} \in Fx_n$  and  $p(fx_n, z_n) = p(fx_n, Fx_n)$ . Clearly  $PB^p(X) \subseteq CB^p(X)$ . Therefore we have the following corollary to Theorem 2.1.

**Corollary 2.6.** *Let  $(X, p)$  be a complete partial metric space. Let  $f: X \rightarrow X$  and  $F: X \rightarrow PB^p(X)$  be continuous mappings such that  $fFx \in PB^p(X)$  for all  $x \in X$ . If the mappings  $f$  and  $F$  are such that  $F(X) \subseteq f(X)$ ,  $H_p(Ffx, fFx) \leq p(fx, Fx)$  for all  $x \in X$ , and satisfy condition (1.2), where  $0 \leq h < 1$ , then the mappings  $f$  and  $F$  have a common coincidence point.*

We present a fixed point theorem by imposing appropriate restrictions to the mappings  $f$  and  $F$  as they are defined in Theorem 2.1.

**Theorem 2.7.** Let  $(X, p)$  be a complete partial metric space. Let  $f: X \rightarrow X$  and  $F: X \rightarrow CB^p(X)$  be compatible continuous mappings such that  $F(X) \subseteq f(X)$  and satisfying condition (1.2). Furthermore, if for each  $x \in X$  either  $fx \neq f^2x$  implies  $fx \notin Fx$  or  $fx \in Fx$  implies that  $f^n x \rightarrow y$  for some  $y \in X$ , then there exists a point  $s \in X$  such that  $s = fs \in Fs$ .

*Proof.* By Theorem 2.1, there exists a point  $s \in X$  such that  $fs \in Fs$ . Assume that  $fx \neq f^2x$  implies  $fx \notin Fx$  for each  $x \in X$ . Now, suppose  $fx \in Fx$  for each  $x \in X$ . Then by continuity of  $f$  and Lemma 1.16 we have  $f^2s \in fFs = Ffs$ . Thus, we have  $fs = f^2s \in Ffs$ , i.e.,  $f(s)$  is a common fixed point for the mappings  $f$  and  $F$ .

Assume that for each  $x \in X$ ,  $fx \in Fx$  implies that  $f^n x \rightarrow y$  for some  $y \in X$ . By the continuity of  $f$  we have  $fy = y$ . We now show that  $y$  is also a fixed point for  $F$ . By Lemma 1.16,  $f^n s \in Ff^{n-1}s$  for each natural number  $n$ , and the continuity of  $F$  we have

$$\begin{aligned} p(y, Fy) &\leq p(y, f^n s) + p(f^n s, Fy) - p(f^n s, f^n s) \\ &\leq p(y, f^n s) + H_p(Ff^{n-1}s, Fy) - H_p(Ff^{n-1}s, Ff^{n-1}s) \leq p(y, y) \text{ as } n \rightarrow \infty. \end{aligned}$$

So we have  $p(y, y) = p(y, Fy)$ , which implies that  $y \in Fy$  since  $Fy$  is closed. Therefore,  $y = fy \in Fy$ . □

**Remark 2.8.** Vetro and Vetro in [24] used implicit contractive type conditions to prove the results while in this paper, we used a contraction condition for hybrid pair of weak compatible mappings in partial metric spaces. Thus, the approach of the contraction condition as well as the mappings are different from the approach of Vetro and Vetro (see [24]).

**Remark 2.9.** Altun and Romaguera in [3] used  $w$ -distance property to determine fixed point for 0-weakly contractive mapping on partial metric space. In this paper, Theorem 2.1 is proved using  $f$ -weak compatible continuous mappings for complete partial metric space while Theorem 2.7 is proved for compatible mappings in complete partial metric space. Thus, the approach for the two results is different from each other.

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