EVENTUALLY SEMISIMPLE WEAK FI-EXTENDING MODULES

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Abstract. In this article, we study modules with the weak FI-extending property. We prove that if M satisfies weak FI-extending, pseudo duo, C_3 properties and $M/\operatorname{Soc} M$ has finite uniform dimension then M decomposes into a direct sum of a semisimple submodule and a submodule of finite uniform dimension. In particular, if M satisfies the weak FI-extending, pseudo duo, C_3 properties and ascending (or descending) chain condition on essential submodules then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (or Artinian, respectively) submodule M_2. Moreover, we show that a nonsingular weak CS (or weak C_{11}^*, or weak FI) module has a direct summand which essentially contains the socle of the module and is a CS (or C_{11}, or FI-extending, respectively) module.

Keywords: CS-module; weak CS-module; uniform dimension; ascending chain on essential submodules; C_{11}-module; FI-extending; weak FI-extending

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1. Introduction

Throughout this paper all rings have identities and modules are unital right modules. Let R be any ring and M a right R-module. Recall that M is called a CS-module (or *extending* module) if every submodule of M is essential in a direct summand of M. Equivalently, every complement in M is a direct summand of M (see [7]). The class of CS-modules contains injective, semisimple and uniform modules (i.e., every nonzero submodule is essential in the module). In particular, the module M has *finite uniform (Goldie) dimension* if M does not contain an infinite direct sum of nonzero submodules. It is well known that a module M has finite uniform dimension if and only if there exist a positive integer n and uniform submodules U_i $(1 \leqslant i \leqslant n)$ of M such that $U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is an essential submodule of M and in this case n is an invariant of the module called the *uniform dimension of* M (see [1], page 294 or [19]).

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Armendariz (see [2], Proposition 1.1) proved that the module M satisfies DCC (the descending chain condition) on essential submodules if and only if $M/(\operatorname{Soc} M)$ is an Artinian module. Goodearl in [8], Proposition 3.6 proved that the module satisfies ACC (the ascending chain condition) on essential submodules if and only if $M/(\operatorname{Soc} M)$ is a Noetherian module. Smith (see [12], Theorem 2.1) proved that the following statements are equivalent for a module M.

(i) M/N has finite uniform dimension for every essential submodule N of M,

(ii) every homomorphic image of $M/(\operatorname{Soc} M)$ has finite uniform dimension.

Camillo and Yousif in [6], Corollary 3 proved that if M is a CS-module and $M/(\operatorname{Soc} M)$ has a finite uniform dimension then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 of M and a submodule M_2 with finite uniform dimension, and in this case M is a direct sum of uniform modules. They deduced that if M is a CS-module then M satisfies ACC (or DCC) on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (or Artinian, respectively) submodule M_2 of M (see [6], Proposition 5).

A module M is called a *weak* CS-module (or WCS) if each semisimple submodule of M is essential in a direct summand of M. Obviously, CS-modules are WCS-modules. It is proved in [11], Corollary 2.7, Theorem 2.8 that the results of [6] mentioned above can be extended to weak CS-modules. A module M is called C_{11}-module if every submodule of M has a complement which is a direct summand of M. Smith and Tercan in [13], Theorem 5.2, Corollary 5.3 extended the results of [6] to modules with C_{11}^{+} (i.e., every direct summand of the module satisfies C_{11} property). A module M is called a *weak* C_{11}-module (or WC_{11}) if each of its semisimple submodules has a complement which is a direct summand of M. Tercan (see [16], Theorem 11, Corollary 12) showed that the aforementioned results of [13] can be extended to modules which have the property that every direct summand satisfies WC_{11}.

Another useful generalization of CS-modules is the FI-extending concept. A module M is called *FI-extending* if every fully invariant submodule (i.e., every submodule such that the image under all endomorphisms is contained in itself) is essential in a direct summand of M (see [3], [4]). A weak version of FI-extending modules is introduced and investigated in [20]. A module is called *weak FI-extending* (or *WFI-extending*) if each of its semisimple fully invariant submodules is essential in a direct summand of M. Tercan and Yaşar in [18] generalized the results of [16], Theorem 11, Corollary 12 to WFI^{+}-extending (i.e., every direct summand of the module satisfies the WFI-extending property) (and also to FI^{+}-extending (i.e., every direct summand of the module satisfies the FI-extending property)) modules with the pseudo duo condition on the class of fully invariant submodules of the module. Recall that a module M is said to have the *pseudo duo property* provided that any semisimple submodule of M has at least one fully invariant (in M) direct summand in its decomposition, i.e., if N is a semisimple submodule of M whenever $N = N_1 \oplus N_2$ then at least one of the N_i $(i = 1, 2)$ is a fully invariant submodule of M (see [18]). Note that the following implications hold for a module M:

$$\begin{array}{ccccc} CS & \Longrightarrow & C_{11} & \Longrightarrow & FI\text{-extending} \\ \Downarrow & & \Downarrow & & \Downarrow \\ WCS & \Longrightarrow & WC_{11} & \Longrightarrow & WFI\text{-extending} \end{array}$$

No other implications can be added to this table, in general. To see why this is the case, we refer to [20]. Notice that it is an open problem whether the FI-extending (and also WFI-extending, WC_{11}, WCS) property is inherited by direct summands or not.

We show that conditions on the direct summands of the mentioned results in [18] can be replaced by the C_3 condition, i.e., if M_1, M_2 are direct summands of M with $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is also a direct summand of M (see [19]). To this end, we arrive at the same results when the module itself is a WFI-extending (or FI-extending) module. Moreover, we provide some special direct summands which enjoy weak versions of the extending property. We think that these results would be helpful to deal with the general framework for the aforementioned open problems. For any unexplained notion or notation, we refer to [1], [5], [19].

2. Weak FI-extending modules

Let R be a ring and M an R-module. Following [6] we call M *eventually semisimple* provided that, for any direct sum

$$M_1 \oplus M_2 \oplus M_3 \oplus \ldots$$

of submodules M_i $(i \geqslant 1)$ of M, there exists a positive integer k such that M_i is semisimple for all $i \geqslant k$. Semisimple modules and modules with finite uniform dimension are eventually semisimple. Camillo and Yousif in [6], Lemma 1 proved that if $M/\mathrm{Soc}(M)$ has finite uniform dimension then M is eventually semisimple. Recall that a module M is called *almost semisimple* if M has an essential socle and every finitely generated semisimple submodule of M is a complement in M. It is obvious that semisimple modules are almost semisimple but the converse is not true in general (see, for example [11]).

Lemma 2.1 ([11], Lemma 2.1). *Let $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 a module with finite uniform dimension. Then the module M is eventually semisimple.*

Lemma 2.2 ([11], Lemma 2.3). *Let M be an eventually semisimple module. Then there exists an almost semisimple complement K in M such that M/K has finite uniform dimension.*

Recall that a module M is *strongly bounded* if and only if every nonzero submodule of M is an essential extension of a fully invariant submodule of M. It is easy to see that if $\operatorname{Soc} M$ is essential in M, then M is strongly bounded. Moreover, if M is strongly bounded, then each semisimple submodule of M is fully invariant in M.

Recall also that for a module M, if X is a homogeneous component of the socle of M then X is a fully invariant submodule of M. It is natural and meaningful for the definition of the WFI-extending notion to consider, whether if X is a semisimple fully invariant in M implies that X is a homogeneous component of $\operatorname{Soc}(M)$. The following example provides a negative answer to this problem and we refer to [5], Example 7.3.13 (ii) for details of its first part.

Example 2.3. (i) Let $\Lambda = \operatorname{End}(\mathbb{Z}(p^\infty))$, where $\mathbb{Z}(p^\infty)$ is the Prufer p-group and p is a prime integer. Let $R = \Lambda \oplus \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$, where the addition is componentwise and the multiplication is defined by

$$(\lambda, m_1, n_1)(\mu, m_2, n_2) = (\lambda\mu, \lambda(m_2) + \mu(m_1), \lambda(n_2) + \mu(n_1))$$

for (λ, m_1, n_1), $(\mu, m_2, n_2) \in R$. Then, it can be seen that

$$R \cong S = \left\{ \begin{bmatrix} \lambda & m & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & n \\ 0 & 0 & 0 & \lambda \end{bmatrix} : \lambda \in \Lambda \text{ and } m, n \in \mathbb{Z}(p^\infty) \right\}$$

with the addition componentwise and the standard matrix multiplication. Observe that Λ is the ring of p-adic integers. Then the ring S is commutative since Λ is commutative. Let us take

$$V = \begin{bmatrix} 0 & \mathbb{Z}/\mathbb{Z}p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{Z}/\mathbb{Z}p \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence V and W are the only minimal ideals of S. Furthermore, $\mathrm{Soc}(S) = V \oplus W$ and $V_S \cong W_S$. Thus $V \oplus W$ is the only homogeneous component of $\mathrm{Soc}(S)$. Now, let $X = V$ (or W). Then X is a semisimple fully invariant submodule of S. However, X is not a homogeneous component of $\mathrm{Soc}(S)$.

(ii) Let F be any field and $V = v_1F \oplus v_2F$ be a vector space over F with a basis $\{v_1, v_2\}$. Let R be the trivial extension of F with V, i.e.,

$$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in F \text{ and } v \in V \right\}.$$

Then R is a commutative ring with $\mathrm{Soc}(R) = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. It can be seen easily that $U = \begin{bmatrix} 0 & v_1F \\ 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & v_2F \\ 0 & 0 \end{bmatrix} = W$. Thus $\mathrm{Soc}(R) = U \oplus W$ is the only homogeneous component of $\mathrm{Soc}(R)$. Now, let $X = U$ (or W). Then X is a semisimple fully invariant submodule of R. However X is not a homogeneous component of $\mathrm{Soc}(R)$.

The next lemma is the WFI-extending with the pseudo duo property version of [17], Lemma 2.3 (see also [11], Lemma 2.5).

Lemma 2.4. *Let M be an eventually semisimple weak FI-extending module with the pseudo duo property. Then every almost semisimple submodule of M is semisimple.*

P r o o f. Let K be an almost semisimple submodule of M. Let $0 \neq x \in K$. Suppose that $\mathrm{Soc}\, xR$ is not finitely generated. Then

$$\mathrm{Soc}\, xR = L_1 \oplus L_2 \oplus \ldots$$

for some infinitely generated submodules L_i $(i \geqslant 1)$ of xR. By hypothesis, there exists a subset $\emptyset \neq \{j_1, j_2, \ldots, j_n, \ldots\} \subseteq \{1, 2, \ldots, n, \ldots\}$ such that L_{j_k} is essential in N_{j_k} where N_{j_k}'s are direct summands of M. Then the sum $N_{j_1} + N_{j_2} + \ldots$ is direct and there exists a positive integer t such that N_{j_t} is semisimple. In this case, $N_{j_t} = L_{j_t}$, so that L_{j_t} is a direct summand of M, and hence also of xR. It follows that L_{j_t} is cyclic, a contradiction. Thus $\mathrm{Soc}\, xR$ is finitely generated. By hypothesis, $\mathrm{Soc}\, xR$ is a complement in xR. But $\mathrm{Soc}\, xR$ is essential in xR, and hence $\mathrm{Soc}\, xR = xR$. It follows that K is semisimple. □

E x a m p l e 2.5. Let R be a principal ideal domain. If R is not a complete discrete valuation ring then there exists an indecomposable torsion-free R-module M of rank 2 (see [10], Theorem 19). For M, $\mathrm{Soc}\, M = 0$ and M has finite uniform dimension, namely 2. Let T be the trivial extension R with M, i.e.,

$$T = \begin{bmatrix} R & M \\ 0 & R \end{bmatrix} = \left\{ \begin{bmatrix} r & m \\ 0 & r \end{bmatrix} : r \in R,\ m \in M \right\}.$$

Then T is a commutative indecomposable ring with respect to usual matrix operations. Since T_T has finite Goldie dimension and $\operatorname{Soc} T_T = 0$, T is eventually semisimple WFI-extending T-module. However, since T is not uniform, T_T is not FI-extending. Further, note that T_T satisfies the C_3 condition.

Corollary 2.6. *Let M be an eventually semisimple FI-extending module with the pseudo duo property. Then every almost semisimple submodule of M is semisimple.*

P r o o f. Immediate by Lemma 2.4. □

Lemmas 2.2 and 2.4 enable us to obtain the following result without the condition that direct summands of the module are WFI-extending. However, we need to use the C_3 condition. To this end, the next result is the extension of [17], Theorem 2.5 (see also [11], Theorem 2.6) with a weaker condition.

Theorem 2.7. *Let M be an eventually semisimple WFI-extending module with the pseudo duo property and C_3. Then $M = M_1 \oplus M_2$ for some semisimple module M_1 and WFI-extending module M_2 with finite uniform dimension.*

P r o o f. Suppose that M is an eventually semisimple WFI-extending module with the pseudo duo property and C_3. By Lemmas 2.2 and 2.4, $M = M_1 \oplus M_2$ for some semisimple module M_1 and M_2 with finite uniform dimension. Let S be a semisimple fully invariant submodule of M_2. Then $M_1 \oplus S$ is a semisimple submodule of M. By hypothesis, at least one of M_1 or S is fully invariant in M. If M_1 is fully invariant in M then M_2 is WFI-extending by [20], Theorem 3.5. Now, assume that S is fully invariant in M. So there exists a direct summand K of M such that S is essential in K. It follows that $M_1 \oplus S$ is essential in $M_1 \oplus K$. By the C_3 condition, $M_1 \oplus K$ is a direct summand of M and by the modular law, $M_1 \oplus K = M_1 \oplus [(M_1 \oplus K) \cap M_2]$, from which we infer that $(M_1 \oplus K) \cap M_2$ is a direct summand of M, and hence also of M_2. Moreover, $S \subseteq (M_1 \oplus K) \cap M_2$. By [1], Proposition 5.20, S is an essential submodule of the direct summand $(M_1 \oplus K) \cap M_2$ of M_2. Thus M_2 is a WFI-extending module. □

Corollary 2.8. *Let M be a WFI- (FI)-extending module with the pseudo duo property and C_3 (or C_2). If $M/(\operatorname{Soc} M)$ has finite uniform dimension then $M = M_1 \oplus M_2$ for some semisimple module M_1 and module M_2 with finite uniform dimension.*

P r o o f. By Theorem 2.7 and [6], Lemma 1. □

Next example makes it clear that the combined conditions in Corollary 2.8 are different from each other.

E x a m p l e 2.9. (i) Let M be the $\mathbb{Z}$-module $\mathbb{Z}$. It is clear that M satisfies all the assumptions of Corollary 2.8 except for the C_2 condition.

(ii) Let R be the trivial extension of $\mathbb{Z}$ with $\mathbb{Z} \oplus \mathbb{Z}$, i.e., $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Then it is easy to see that R_R satisfies all the assumptions of Corollary 2.8 except for the FI-extending condition.

Using Corollary 2.8, we can now have the following result on WFI-extending (and also FI-extending) modules which satisfy ACC (or DCC, respectively) on essential submodules (see [18], Corollary 2.8).

Theorem 2.10. *Let M be a WFI-extending module with the pseudo duo property and C_3. Then M satisfies the ascending (or descending) chain condition on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple module M_1 and Noetherian (or Artinian, respectively) module M_2.*

P r o o f. We provide the proof in the Noetherian case; the proof in the Artinian case is similar. If M is a direct sum of a semisimple module and a Noetherian module, then M satisfies ACC on essential submodules by [6], Lemma 4.

Conversely, suppose that M satisfies ACC on essential submodules. By [6], Lemma 4, $M/\operatorname{Soc} M$ is Noetherian. Now, Corollary 2.8 yields that $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and submodule M_2 with finite uniform dimension. There exists a positive integer k and uniform submodules U_i $(1 \leqslant i \leqslant k)$ of M_2 such that $Y = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ is essential in M_2. Now, M having ACC on essential submodules implies that U_i is Noetherian $(1 \leqslant i \leqslant k)$ and also that M_2/Y is Noetherian. Thus M_2 is Noetherian. $\square$

In a similar vein to Theorem 2.7, we have the next result on FI-extending modules (see Example 2.5).

Theorem 2.11. *Let M be an eventually semisimple FI-extending module with the pseudo duo property and C_3. Then $M = M_1 \oplus M_2$ for some semisimple module M_1 and FI-extending module M_2 with finite uniform dimension.*

P r o o f. By Theorem 2.7, $M = M_1 \oplus M_2$ for some semisimple module M_1 and module M_2 with finite uniform dimension. Let us show that M_2 is FI-extending. Let H be a fully invariant submodule of M_2. Assume $\operatorname{Soc} H = H$. Then $M_1 \oplus H$ is a semisimple submodule of M. By hypothesis, at least one of M_1 or H is fully invariant in M. If M_1 is fully invariant in M then the result follows from [20], Proposition 2.5 and Theorem 3.5. If H is fully invariant in M then the similar argument as in the proof of Theorem 2.7 yields that H is essential in a direct summand of M_2. Thus M_2 is FI-extending.

Now, assume that $\operatorname{Soc} H \neq H$. Observe that H has finite uniform dimension, say k. Then there exist uniform submodules U_i $(1 \leqslant i \leqslant k)$ of H such that $U_1 \oplus U_2 \oplus \ldots \oplus U_k$ is essential in H. Hence $\operatorname{Soc} H = \bigoplus_{i=1}^{k} \operatorname{Soc} U_i$. Since $\operatorname{Soc} H \neq H$, there exists a number j such that $\operatorname{Soc} U_j \neq U_j$ where $1 \leqslant j \leqslant k$. Then $M_1 \oplus \operatorname{Soc} U_j$ is a semisimple submodule of M. By the pseudo duo property, either M_1 or $\operatorname{Soc} U_j$ is fully invariant in M. If M_1 is fully invariant in M then the result follows by [20]. Assume $\operatorname{Soc} U_j$ is fully invariant in M. By FI-extending condition, there exists a direct summand K of M such that $\operatorname{Soc} U_j$ is essential in K. Thus $M_1 \oplus \operatorname{Soc} U_j$ is essential in $M_1 \oplus K$. Therefore $M_1 \oplus K = M_1 \oplus [(M_1 \oplus K) \cap M_2]$ and $(M_1 \oplus K) \cap M_2$ is a direct summand of M and hence also of M_2. It follows that $\operatorname{Soc} U_j$ is essential in $(M_1 \oplus K) \cap M_2$. Since $\bigoplus_{i=1}^{k} U_i$ is essential in H, $\operatorname{Soc} U_j$ is essential in $(M_1 \oplus K) \cap H$. Thus $(M_1 \oplus K) \cap H$ is a uniform module. But this is impossible. Let $l \neq j$ and $0 \neq X = (U_l \oplus U_j) \cap (M_1 \oplus K)$, $0 \neq Y = (U_j \oplus 0) \cap (M_1 \oplus K)$. It is easy to see that X and Y are submodules of $(M_1 \oplus K) \cap H$ such that $X \cap Y = 0$. It follows that $\operatorname{Soc} H = H$. Then the result follows from the first part of the proof. □

The following example illustrates Theorem 2.10 and Theorem 2.11.

Example 2.12. Let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ where p is any prime integer. Then M is an FI-extending (and hence WFI-extending) module. It is straightforward to see that $M_{\mathbb{Z}}$ has the pseudo duo property and C_3 condition. Moreover, since $M_{\mathbb{Z}}$ has finite uniform dimension, $M_{\mathbb{Z}}$ is an eventually semisimple module.

3. Direct summands of weak versions of extending properties

Recall that it is an open problem whether direct summands of an FI-extending module are FI-extending or not. To this end, there are more corresponding open problems for weak CS, weak C_{11} and WFI-extending modules. In the former section, we obtained some certain direct summands enjoying the WFI-extending or FI-extending properties (see Theorems 2.7 and 2.11). In this section, we provide some special direct summands which enjoy the property. For this aim, we deal with nonsingular modules satisfying one of the weak versions of extending properties. We expect the results exhibited in this section would be helpful to the general frame for the aforementioned all related to each other problems.

The next lemma is well known, see for example [14], but its proof is given for the sake of completeness.

Lemma 3.1. *Let N be a submodule of a module M such that N has a unique essential closure K in M. Then K is the sum of all submodules L of M containing N such that N is essential in L.*

Proof. Let H be the sum of submodules L of M such that N is an essential submodule of L. Since N is essential in its closure K, it follows that $K \subseteq H$. Conversely, let L be any submodule of M such that N is an essential submodule of L. Let L' be any closure of L in M. Clearly, L' is a closure of N in M, and so $L' = K$. Thus $L \subseteq K$. It follows that $H \subseteq K$ and hence $H = K$. □

Proposition 3.2. *Let M be a nonsingular WCS R-module. Then there exists a direct summand of M which is CS and essentially contains the socle of M.*

Proof. Let $S = \operatorname{Soc} M$. Then there exists a direct summand D of M such that S is essential in D. Let X be a complement in D. Hence X is a complement in M. By hypothesis, $\operatorname{Soc} X$ is essential in a direct summand D_1 of M. By the nonsingularity assumption, D_1 is the unique essential closure of $\operatorname{Soc} X$ in M. Since $\operatorname{Soc} X = X \cap S$ is essential in X, X is essential in D_1. It follows that $X = D_1$. Hence D is a CS-module. □

Note that the nonsingularity of the module in Proposition 3.2 is not superfluous. For example, let p be any rational prime and M the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$. Then M is a weak CS-module (see [11], Example 1.1). Moreover, $\operatorname{Soc} M$ is essential in only a direct summand of the module itself. However, M is not a CS-module. In fact, the submodule $K = (1 + \mathbb{Z}p, p + \mathbb{Z}p^3)\mathbb{Z}$ is a complement submodule of M which is not a direct summand of M (see [13]).

Our next theorem is based on the class of modules with the following property which is interesting own right. A module M is called WC_{11}^*-module if every direct sum of a semisimple submodule and a direct summand, which has the zero socle, has a complement which is a direct summand of M. It is clear that every WC_{11}^*-module is WC_{11}. Moreover, if $\operatorname{Soc} M = 0$ or it is essential in M or M is indecomposable, then WC_{11}^* and WC_{11} properties coincide.

Theorem 3.3. *Let M be a nonsingular WC_{11}^*-module. Then there exists a direct summand of M which has C_{11} and essentially contains the socle of M.*

Proof. Let $S = \operatorname{Soc} M$. Then there exists a direct summand D' of M such that $D' \cap S = 0$ and $S \oplus D'$ is essential in M. So $M = D \oplus D'$ for some submodule D of M. Since $S \cap D' = 0$, $S = \operatorname{Soc} D \leqslant D$. Thus S is essential in D. Let us show that D is a C_{11}-module. Let $\pi\colon M \to D$ be the canonical projection map and N be a submodule of D. By hypothesis, there exist submodules K, K' of M such that

$M = K \oplus K'$, $(\operatorname{Soc} N \oplus D') \cap K = 0$ and $\operatorname{Soc} N \oplus D' \oplus K$ is essential in M. Since $K \cap D' = 0$, $K \cong \pi(K)$. So $\operatorname{Soc}(\pi(K)) = \pi(K) \cap S$ which is essential in $\pi(K)$. Hence $\operatorname{Soc} K$ is essential in K and $S = \operatorname{Soc} K \oplus \operatorname{Soc} K'$ is essential in $K \oplus \operatorname{Soc} K'$. Thus by Lemma 3.1, $K \oplus \operatorname{Soc} K' \subseteq D$ and so $K \subseteq D$. Now, by the modular law $D = K \oplus (D \cap K')$ and $\operatorname{Soc} N \oplus K = (\operatorname{Soc} N \oplus D' \oplus K) \cap D$ which is essential in D. It is clear that $N \cap K = 0$ and $(N \oplus D' \oplus K) \cap D = N \oplus K$ is essential in D. It follows that D has C_{11}. □

Theorem 3.4. *Let M be a nonsingular WFI-extending module. Then there exists a direct summand of M which is FI-extending and essentially contains the socle of M.*

Proof. Let $S = \operatorname{Soc} M$. Since S is fully invariant in M, there exists a direct summand D of M such that S is essential in D. Now, let X be any fully invariant submodule of D. From [5], Proposition 2.3.3 (iv), D is fully invariant in M. Thus X is fully invariant in M (see [5], Proposition 2.3.3 (ii)). By hypothesis, there exists a direct summand D_1 of M such that $\operatorname{Soc} X$ is essential in D_1. By the nonsingularity asummption, D_1 is the unique essential closure of $\operatorname{Soc} X$ in M. Since $\operatorname{Soc} X = X \cap S$ is essential in X, then $X \subseteq D_1$. It follows that X is essential in D_1.

Assume that $D \neq D + D_1$. Let $d + d_1 \in D + D_1$ be such that $d + d_1 \notin D$ where $d \in D$ and $d_1 \in D_1$. So $d_1 \neq 0$. Hence there exists an essential right ideal L of R such that $d_1 L \subseteq X$. Since D is nonsingular, $0 \neq (d + d_1)L \subseteq D$. Thus D is essential in $D + D_1$. Hence $D = D_1 + D$. Then $D_1 \leqslant D$. Now, by the modular law:

$$D = D \cap (D_1 \oplus D_1') = D_1 \oplus (D \cap D_1')$$

where $M = D_1 \oplus D_1'$ and D_1' is a submodule of M. Therefore D_1 is a direct summand of D. It follows that D is FI-extending. □

Next we collect some examples related to the latter proposition and theorems. So, we make it clear that the nonsingularity with weak CS (WC_{11}, WFI, respectively) does not imply the condition CS (C_{11}, FI, respectively).

Example 3.5. (i) Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Then the right R-module R is not right CS (see [15]). Notice that $Z(R_R) = 0$. Since $\operatorname{Soc}(R_R) = 0$, R_R is a WCS-module.

(ii) Let M be the Specker group, i.e., $M_{\mathbb{Z}} = \mathbb{Z}^{\mathbb{N}}$. By [9], Proposition 1.22, $M_{\mathbb{Z}}$ is nonsingular. Notice that $M_{\mathbb{Z}}$ is not a C_{11}-module from [13]. Now [9], Corollary 1.26 yields that $\operatorname{Soc}(M_{\mathbb{Z}}) = 0$. Hence $M_{\mathbb{Z}}$ is a WC_{11}-module.

(iii) Let D be a simple domain which is not a division ring. Let R be the right R-module where $R = \begin{bmatrix} D & D \oplus D \\ 0 & D \end{bmatrix}$ (see [4]). Then $Z(R_R) = 0$ and also $\operatorname{Soc}(R_R) = 0$.

Thus R_R is a nonsingular WFI-extending module. Since $I = \begin{bmatrix} 0 & 0 \oplus D \\ 0 & 0 \end{bmatrix}$ is a fully invariant submodule of R_R and the nonzero idempotents of R have the form $\begin{bmatrix} 1 & (b,d) \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & (b,d) \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, I is not essential in one of them. It follows that R_R is not FI-extending.

Observe that, for instance, we obtain a direct summand which satisfies weak CS (WC_{11} and WFI-extending, respectively) in former results (see Proposition 3.2, Theorems 3.3 and 3.4). It is clear that we cannot drop the nonsingularity assumption in Proposition 3.2. However, it turns out that whether we can drop the nonsingularity condition in Theorems 3.3 and 3.4 or not is essentially based on the aforementioned open problems. Now we ask: is nonsingularity of the module in Theorems 3.3 and 3.4 superfluous or not?

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