BICYCLIC COMMUTATOR QUOTIENTS WITH ONE NON-ELEMENTARY COMPONENT

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Abstract. For any number field K with non-elementary 3-class group $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$, $e \geqslant 2$, the punctured capitulation type $\varkappa(K)$ of K in its unramified cyclic cubic extensions L_i, $1 \leqslant i \leqslant 4$, is an orbit under the action of $S_3 \times S_3$. By means of Artin's reciprocity law, the arithmetical invariant $\varkappa(K)$ is translated to the punctured transfer kernel type $\varkappa(G_2)$ of the automorphism group $G_2 = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of the second Hilbert 3-class field of K. A classification of finite 3-groups G with low order and bicyclic commutator quotient $G/G' \simeq C_{3^e} \times C_3$, $2 \leqslant e \leqslant 6$, according to the algebraic invariant $\varkappa(G)$, admits conclusions concerning the length of the Hilbert 3-class field tower $\mathrm{F}_3^\infty(K)$ of imaginary quadratic number fields K.

Keywords: Hilbert 3-class field tower; maximal unramified pro-3 extension; unramified cyclic cubic extensions; Galois action; imaginary quadratic fields; bicyclic 3-class group; punctured capitulation types; statistics; pro-3 groups; finite 3-groups; generator rank; relation rank; Schur σ-groups; low index normal subgroups; kernels of Artin transfers; abelian quotient invariants; p-group generation algorithm; descendant trees; antitony principle

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1. Introduction

An indispensable tool for the investigation of the unramified Hilbert 3-class field tower $\mathrm{F}_3^\infty(K)$ of an arbitrary number field $K/\mathbb{Q}$ with bicyclic 3-class group $\mathrm{Cl}_3(K) \simeq C_{3^e} \times C_3$, $e \geqslant 2$, is the *punctured capitulation type* $\varkappa(K) = (\ker(\tau_i))_{1 \leqslant i \leqslant 4}$ of K in its four unramified cyclic cubic extensions L_1, L_2, L_3; L_4. The puncture at L_4 is motivated by the special role of L_4 as common subfield of all four unramified

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extensions of degree nine of K. Here we denote by $\tau_i\colon\ \mathrm{Cl}_3(K) \to \mathrm{Cl}_3(L_i)$, $\mathfrak{a}\mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_{L_i})\mathcal{P}_{L_i}$, the *transfers* (extension homomorphisms) of 3-classes of K into L_i for $1 \leqslant i \leqslant 4$. We expand these ideas exemplarily for $e = 2$. With minor modifications, however, they may be adopted for any $e \geqslant 3$.

By means of Artin's reciprocity law (see [2], [3]), we translate the arithmetical invariant $\varkappa(K)$ to the *punctured transfer kernel type* $\varkappa(G_2)$ of the automorphism group $G_2 = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of the second Hilbert 3-class field of K. Based on the lattice of normal subgroups between a pro-3 group G with $G/G' \simeq C_9 \times C_3$ and its commutator subgroup G' in Section 2.1, we define the group theoretic *Artin transfers* $T_i : G/G' \to H_i/H_i'$ from G to maximal subgroups H_i with $(G : H_i) = 3$, corresponding to the arithmetical transfers τ_i, in Section 2.2.

We explain in Section 2.3 why only the *orbit under the action of* $S_3 \times S_3$ of the punctured transfer kernel type $\varkappa(G) = (\ker(T_i))_{1 \leqslant i \leqslant 4}$ is an *invariant* of G, but not an individual orbit representative.

We conclude these preliminaries in Section 2.4 with an overview of all 52 combinatorially possible orbits, emphasizing those which can actually be realized as $\varkappa(G)$ by a 3-group G.

In Section 3 we devote our attention to a collection of algebraic invariants of the smallest metabelian 3-groups G with commutator quotient $G/G' \simeq C_9 \times C_3$ and coclass $2 \leqslant \mathrm{cc}(G) \leqslant 3$, in order to establish the required inventory of Galois groups $\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ for imaginary quadratic fields K in Sections 6 and 8, supplemented by $e \in \{3, 4\}$ in Section 4 and periodic Schur σ-groups in Sections 7 and 9.

In Section 5.1, we provide evidence of the fact that the set of non-metabelian groups G whose second derived quotient G/G'' is isomorphic to an assigned metabelian group M is contained in the same tree as M itself, i.e., the *covers* $\mathrm{cov}(M)$ (see [24]) are separated by descendant trees. In Section 5.2 we collect information on the relation rank of G and the Galois action of the absolute group $\mathrm{Gal}(K/\mathbb{Q})$ on G, which admits the determination of the length $l_3(K)$ of the 3-class field tower of K in Section 5.

2. Kernels of Artin transfers and abelian quotient invariants

2.1. Low index subgroups. Every two-generated pro-3 group $G = \langle x, y\rangle$ with $G/G' \simeq (9, 3)$, such that $x^9 \in G'$ and $y^3 \in G'$, possesses the following self-conjugate intermediate groups $G' \leqslant J_i, H_i \leqslant G$ between the commutator subgroup G' and G, as shown in Figure 1:

▷ *first layer*: four *maximal* normal subgroups H_i of index $(G : H_i) = 3$,

$$H_1 = \langle x, G'\rangle, \quad H_2 = \langle xy, G'\rangle, \quad H_3 = \langle xy^2, G'\rangle, \quad H_4 = \langle x^3, y, G'\rangle,$$

▷ *second layer*: four *second maximal* normal subgroups J_i of index $(G : J_i) = 9$,

$$J_1 = \langle y, G' \rangle, \quad J_2 = \langle x^3 y, G' \rangle, \quad J_3 = \langle x^3 y^2, G' \rangle, \quad J_4 = \langle x^3, G' \rangle.$$

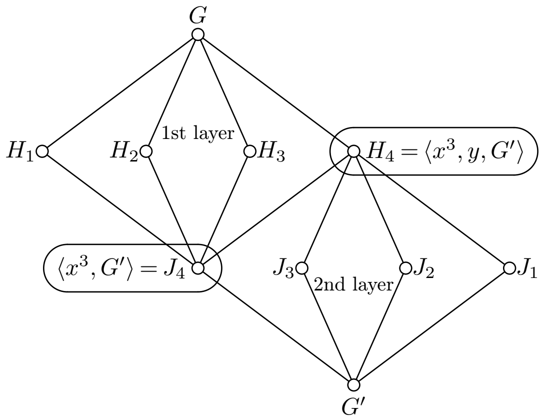


Figure 1. Low index subgroups of a pro-3 group G with G/G' of type $(9, 3)$.

For both layers of subgroups, we use the subscript 4 to indicate the *distinguished maximal subgroup* $H_4 = \prod_{i=1}^{4} J_i$, for which the quotient $H_4/G' = \langle x^3, y \rangle$ is bicyclic of type $(3, 3)$, whereas H_i/G' is cyclic of order 9 for $1 \leqslant i \leqslant 3$, and to emphasize the *distinguished second maximal subgroup*, $J_4 = \bigcap_{i=1}^{4} H_i$, which coincides with the *Frattini subgroup* $\Phi(G) = G^3 G'$ of G, whereas J_i is contained in H_4 only, for $1 \leqslant i \leqslant 3$.

2.2. Artin transfers. We characterize a finite 3-group G by the usual invariants, order $|G| = 3^m$, logarithmic order $\mathrm{lo}(G) = m$, nilpotency class $\mathrm{Cl}(G) = c$, coclass $\mathrm{cc}(G) = m - c$, and derived length $\mathrm{dl}(G) = l$. Additionally we use advanced invariants associated with certain homomorphisms, the *Artin transfers* (see [3]) from G to its maximal subgroups H_i, $1 \leqslant i \leqslant 4$,

$$T_i \colon G/G' \to H_i/H_i', \quad g \mapsto \begin{cases} g^3 & \text{if } g \in G \setminus H_i \ (\textit{outer transfer}), \\ g^{\mathrm{S}_3(h)} & \text{if } g \in H_i \ (\textit{inner transfer}), \end{cases}$$

where $\mathrm{S}_3(h) = 1 + h + h^2 \in \mathbb{Z}[G]$, with an arbitrary element $h \in G \setminus H_i$, denotes the third *trace* element (*Spur*) in the group ring of G, acting as symbolic exponent. It turns out that, for fixed derived length $\mathrm{dl}(G) = 2$, a metabelian group G is occasionally determined uniquely by its *transfer kernel type* (TKT), $\varkappa(G) = (\ker(T_i))_{1 \leqslant i \leqslant 4}$, in conjunction with its *abelian quotient invariants* (AQI), $\alpha(G) = (H_i/H_i')_{1 \leqslant i \leqslant 4}$. The *Artin pattern* of G is the pair $\mathrm{AP}(G) = (\alpha, \varkappa)$. Here, we restrict the TKT and AQI to the first layer.

2.3. Orbits of punctured transfer kernel types. Although the capitulation over a few imaginary quadratic fields K of type $(9,3)$ has been investigated in [34], [14], [18], [5] already, an invariant characterization of the possible TKTs $\varkappa(G_2)$ of their second 3-class groups $G_2 = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ was missing up to now. An adequate model, motivated by Figure 1, is therefore established in the sequel, for the first time.

Definition 1. There are five possibilities for the kernel of the transfer T_i for each $1 \leqslant i \leqslant 4$. Either $\ker(T_i) = J_k/G' \simeq C_3$ for some $1 \leqslant k \leqslant 4$, and we denote the *one-dimensional transfer kernel* by the singlet $\varkappa(i) = k$ (see [19], Section 2.2, page 475), or $\ker(T_i) = H_4/G' \simeq C_3 \times C_3$, and we denote the *two-dimensional transfer kernel* by the singlet $\varkappa(i) = 0$. Due to the distinguished role of the subscript 4, we combine the singlets in the following way to form a multiplet

$$\varkappa = ((\varkappa(1), \varkappa(2), \varkappa(3)); \varkappa(4)) \in [0,4]^3 \times [0,4],$$

which we call the *punctured transfer kernel type* (pTKT) of the group $G = \langle x, y\rangle$ with respect to the selected generators x, y. In order to be independent of the choice of generators and of the arrangement of the subgroups H_1, H_2, H_3 and J_1, J_2, J_3, we define the $(S_3 \times S_3)$-*orbit*

$$\varkappa^{S_3 \times S_3} = \{\tilde{\sigma} \circ \varkappa \circ \hat{\tau} \colon\ \sigma, \tau \in S_3\}$$

of $\varkappa$ under the operation of $S_3 \times S_3$ as an isomorphism invariant $\varkappa(G)$ of G. Here, $\tilde{\sigma}$ denotes the extension of σ from $[1,3]$ to $[0,4]$ which fixes 0 and 4, and $\hat{\tau}$ denotes the extension of τ from $[1,3]$ to $[1,4]$ which fixes 4. (The broader context of this definition is explained in [23].)

Throughout this work, we adhere to the convention that the subscript 4 is distinguished and invariant under any permutation of the other subscripts (τ for the domain and σ for the codomain).

Definition 2. Two further *isomorphism invariants* of G are defined by the number of distinguished transfer kernels $\mu = \mu(G) = \#\{1 \leqslant i \leqslant 4\colon\ \varkappa(i) = 4\}$, and the number of two-dimensional transfer kernels $\nu = \nu(G) = \#\{1 \leqslant i \leqslant 4\colon\ \varkappa(i) = 0\}$.

2.4. Combinatorially possible punctured transfer kernel types. In this section, we arrange all combinatorially possible $(S_3 \times S_3)$-orbits of the 5^4 punctured quartets $\varkappa \in [0,4]^3 \times [0,4]$ by increasing invariant $0 \leqslant \mu \leqslant 4$ and cardinality of the image. Table 1 shows the punctured quartets with invariant $\nu = 0$, and Table 2 those with invariant $1 \leqslant \nu \leqslant 4$, as possible pTKTs $\varkappa(G)$ of 3-groups G with G/G' of type $(9,3)$, or punctured capitulation types $\varkappa(K)$ of number fields K with 3-class group $\mathrm{Cl}_3(K)$ of type $(9,3)$, according to Artin's reciprocity law (see [19], Section 2.3, pages 476–478). The orbits are divided into sections (Sec.), denoted by letters and identified by ordinal numbers (Nr.). Each orbit contains a canonical representative.

Sec.	Nr.	Repres. of orbit $\varkappa$	Occupation numbers $o(\varkappa)$	μ	Taussky type κ	Charact. property	Cardinality of orbit $\lvert\varkappa^{S_3\times S_3}\rvert$	Realizing 3-group G
A	1	$(111;1)$	(04000)	0	$(BBBA)$	constant	3	$\langle 3^4,6\rangle$
B	2	$(111;2)$	(03100)	0	$(BBBA)$	nearly	6	$\langle 3^8,1682\vert 1685\rangle$
B	3	$(112;1)$	(03100)	0	$(BBBA)$	constant	18	
C	4	$(112;2)$	(02200)	0	$(BBBA)$		18	$\langle 3^8,1683\vert 1687\rangle$
D	5	$(112;3)$	(02110)	0	$(BBBA)$		18	$\langle 3^8,1684\vert 1686\rangle$
D	6	$(123;1)$	(02110)	0	$(BBBA)$		18	$\langle 3^8,1744\vert 1782\rangle$
B	7	$(111;4)$	(03001)	1	$(BBBA)$	nearly	3	$\langle 3^6,16\vert 19\rangle$
B	8	$(114;1)$	(03001)	1	$(BBAA)$	constant	9	
D	9	$(112;4)$	(02101)	1	$(BBBA)$		18	exists
D	10	$(114;2)$	(02101)	1	$(BBAA)$		18	$\langle 3^8,1689\vert 1690\rangle$
D	11	$(124;1)$	(02101)	1	$(BBAA)$		36	$\langle 3^6,14\vert 15\rangle$
E	12	$(123;4)$	(01111)	1	$(BBBA)$	per-	6	$\langle 3^6,17\vert 20\rangle$
E	13	$(124;3)$	(01111)	1	$(BBAA)$	mutation	18	
C	14	$(114;4)$	(02002)	2	$(BBAA)$		9	
C	15	$(144;1)$	(02002)	2	$(BAAA)$		9	
D	16	$(124;4)$	(01102)	2	$(BBAA)$		18	exists
D	17	$(144;2)$	(01102)	2	$(BAAA)$		18	exists
B	18	$(144;4)$	(01003)	3	$(BAAA)$	nearly	9	exists
B	19	$(444;1)$	(01003)	3	$(AAAA)$	constant	3	exists
A	20	$(444;4)$	(00004)	4	$(AAAA)$	constant	1	$\langle 3^4,4\rangle,\langle 3^5,22\rangle,\langle 3^6,12\rangle$
						Total number:	256	

Table 1. 20 $(S_3\times S_3)$-orbits of $\varkappa\in[1,4]^4$ with $\nu=0$.

Table 1 gives a coarse classification into sections A to E, an identification by ordinal numbers 1 to 20, and a set-theoretic characterization. Table 2 gives a coarse classification into sections a to e, an identification by ordinal numbers 1 to 32, and a set-theoretic characterization.

We denote by $o(\varkappa)=(\#\varkappa^{-1}\{i\})_{0\leqslant i\leqslant 4}$ the family of *occupation numbers* of the selected orbit representative $\varkappa$ and by κ the quartet of *Taussky's coarse capitulation types* A and B (see [36]) associated with $\varkappa$ (that is, $\kappa(i)=A$ if the meet $H_i\cap\ker(T_i)>1$ is nontrivial, and $\kappa(i)=B$ otherwise).

Sec.	Nr.	Repres. of orbit $\varkappa$	Occupation numbers $o(\varkappa)$	μ	Taussky type κ	Charact. property	Cardinality of orbit $\lvert\varkappa^{S_3\times S_3}\rvert$	Realizing 3-group G
a	1	$(000;0)$	(40000)	0	$(AAAA)$	constant	1	$\langle 3^4,3\rangle,\langle 3^5,15\vert 17\rangle$
b	2	$(000;1)$	(31000)	0	$(AAAA)$	nearly	3	$\langle 3^5,16\rangle$
b	3	$(001;0)$	(31000)	0	$(AABA)$	constant	9	$\langle 3^5,19\vert 20\rangle$
c	4	$(001;1)$	(22000)	0	$(AABA)$		9	
c	5	$(011;0)$	(22000)	0	$(ABBA)$		9	
d	6	$(001;2)$	(21100)	0	$(AABA)$		18	
d	7	$(012;0)$	(21100)	0	$(ABBA)$		18	
b	8	$(011;1)$	(13000)	0	$(ABBA)$	nearly	9	
b	9	$(111;0)$	(13000)	0	$(BBBA)$	constant	3	
d	10	$(011;2)$	(12100)	0	$(ABBA)$		18	$\langle 3^6,13\rangle$
d	11	$(012;1)$	(12100)	0	$(ABBA)$		36	
d	12	$(112;0)$	(12100)	0	$(BBBA)$		18	
e	13	$(012;3)$	(11110)	0	$(ABBA)$	per-	18	
e	14	$(123;0)$	(11110)	0	$(BBBA)$	mutation	6	$\langle 3^6,18\vert 21\rangle$
b	15	$(000;4)$	(30001)	1	$(AAAA)$	nearly	1	$\langle 3^5,13\vert 14\rangle,\langle 3^6,9\rangle$
b	16	$(004;0)$	(30001)	1	$(AAAA)$	constant	3	$\langle 3^5,18\rangle$
d	17	$(001;4)$	(21001)	1	$(AABA)$		9	exists
d	18	$(004;1)$	(21001)	1	$(AAAA)$		9	
d	19	$(014;0)$	(21001)	1	$(ABAA)$		18	
d	20	$(011;4)$	(12001)	1	$(ABBA)$		9	
d	21	$(014;1)$	(12001)	1	$(ABAA)$		18	
d	22	$(114;0)$	(12001)	1	$(BBAA)$		9	
e	23	$(012;4)$	(11101)	1	$(ABBA)$	per-	18	
e	24	$(014;2)$	(11101)	1	$(ABAA)$	muta-	36	
e	25	$(124;0)$	(11101)	1	$(BBAA)$	tion	18	
c	26	$(004;4)$	(20002)	2	$(AAAA)$		3	
c	27	$(044;0)$	(20002)	2	$(AAAA)$		3	$\langle 3^6,11\rangle$
d	28	$(014;4)$	(11002)	2	$(ABAA)$		18	
d	29	$(044;1)$	(11002)	2	$(AAAA)$		9	
d	30	$(144;0)$	(11002)	2	$(BAAA)$		9	exists
b	31	$(044;4)$	(10003)	3	$(AAAA)$	nearly	3	$\langle 3^6,10\rangle$
b	32	$(444;0)$	(10003)	3	$(AAAA)$	constant	1	
Total number: $625-256=369$								

Table 2. 32 $(S_3\times S_3)$-orbits of $\varkappa\in[0,4]^4\setminus[1,4]^4$ with $1\leqslant\nu\leqslant 4$.

If an orbit $\varkappa^{S_3 \times S_3}$ can be realized as pTKT $\varkappa(G)$, then a suitable 3-group G is given by its identifier in the SmallGroups library (see [6]). In contrast to [19], Tables 6–7, pages 492–493, we are unable to mark an orbit as "impossible" when no realization as pTKT has been known until now, since currently, as opposed to $G/G' \simeq (3,3)$ (see [31]), we do not have parametrized power-commutator presentations of all metabelian 3-groups G with $G/G' \simeq (9,3)$.

ord	id	α	$\varkappa$	TKT	n	d_2	Action
27	2	$(2,2,2;11)$	$(000;0)$	a.1	1	3	$\langle 12,4\rangle$
81	3	$(21,21,21;111)$	$(000;0)$	a.1	3	4	$\langle 12,4\rangle$
81	4	$(21,21,21;21)$	$(444;4)$	A.20	2	3	$\langle 6,1\rangle$
81	6	$(3,3,3;21)$	$(111;1)$	A.1	0	2	$\langle 6,1\rangle$
243	13	$(21,21,21;1111)$	$(000;4)$	b.15	1	4	$\langle 12,4\rangle$
243	14	$(21,21,21;211)$	$(000;4)$	b.15	1	4	$\langle 12,4\rangle$
243	15	$(21,21,21;211)$	$(000;0)$	a.1	1	4	$\langle 12,4\rangle$
243	16	$(21,21,21;211)$	$(000;3)$	b.2	0	3	$\langle 6,2\rangle$
243	17	$(21,21,211;111)$	$(000;0)$	a.1	1	4	$\langle 4,2\rangle$
243	18	$(21,21,211;111)$	$(004;0)$	b.16	0	3	$\langle 2,1\rangle$
243	19	$(21,21,31;111)$	$(001;0)$	b.3	0	3	$\langle 4,2\rangle$
243	20	$(21,21,31;111)$	$(001;0)$	b.3	0	3	$\langle 4,2\rangle$
243	22	$(21,21,21;21)$	$(444;4)$	A.20	0	2	$\langle 2,1\rangle$
729	9	$(211,211,211;1111)$	$(000;4)$	b.15	3	5	$\langle 12,4\rangle$
729	10	$(211,211,211;211)$	$(044;4)$	b.31	2	4	$\langle 4,2\rangle$
729	11	$(211,211,211;211)$	$(044;0)$	c.27	2	4	$\langle 4,2\rangle$
729	12	$(211,211,211;1111)$	$(444;4)$	A.20	2	4	$\langle 6,2\rangle$
729	13	$(211,31,31;211)$	$(011;3)$	d.10	1	3	$\langle 2,1\rangle$
729	14	$(211,31,31;211)$	$(423;3)$	D.11	0	2	$\langle 2,1\rangle$
729	15	$(211,31,31;211)$	$(432;3)$	D.11	0	2	$\langle 2,1\rangle$
729	16	$(31,31,31;1111)$	$(111;4)$	B.7	1	3	$\langle 12,4\rangle$
729	17	$(31,31,31;211)$	$(123;4)$	E.12	1	3	$\langle 12,4\rangle$
729	18	$(31,31,31;211)$	$(132;0)$	e.14	1	3	$\langle 12,4\rangle$
729	19	$(31,31,31;1111)$	$(111;4)$	B.7	1	3	$\langle 12,4\rangle$
729	20	$(31,31,31;211)$	$(132;4)$	E.12	1	3	$\langle 12,4\rangle$
729	21	$(31,31,31;211)$	$(123;0)$	e.14	1	3	$\langle 12,4\rangle$

Table 3. Finite 3-groups G with $G/G' \simeq C_9 \times C_3$ and low order.

3. Finite 3-groups with commutator quotient $(9, 3)$

In Table 3 we collect invariants of crucial metabelian 3-groups G with $G/G' \simeq C_9 \times C_3$ and low order $27 \leqslant \#G \leqslant 729$. The punctured AQI and TKT form the Artin pattern $(\alpha, \varkappa)$ of G, which is used in the strategy of pattern recognition via Artin transfers (see [26]) in order to identify the isomorphism class of the Galois group $G_2 = \mathrm{Gal}(\mathrm{F}_3^2(K)/K)$ of the second Hilbert 3-class field of a number field K by the capitulation kernels $\varkappa(K)$ and the abelian type invariants $\alpha(K)$ of the unramified cyclic cubic extensions L_i, $1 \leqslant i \leqslant 4$, of a number field with $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$. The *nuclear rank* n specifies the position of G in a descendant tree, deciding whether G is a terminal leaf with $n = 0$ or a root with further descendants if $n \geqslant 1$. The *relation rank* d_2 of G frequently admits an estimate of the length of the Hilbert 3-class field tower $\mathrm{F}_3^\infty(K)$ of a number field K. Finally, the *action* of the absolute Galois group $\mathrm{Gal}(K/\mathbb{Q})$ on the Frattini quotient $Q = G/\Phi(G)$ decides whether G is admissible as $\mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ for the field K. Generally, the isomorphism class of a group G is determined by its name in the SmallGroups database (see [6]) which has the shape $\langle \mathrm{ord}, \mathrm{id} \rangle$ containing the order and a numerical identifier in angle brackets. Table 3 is illuminated by Figure 2.

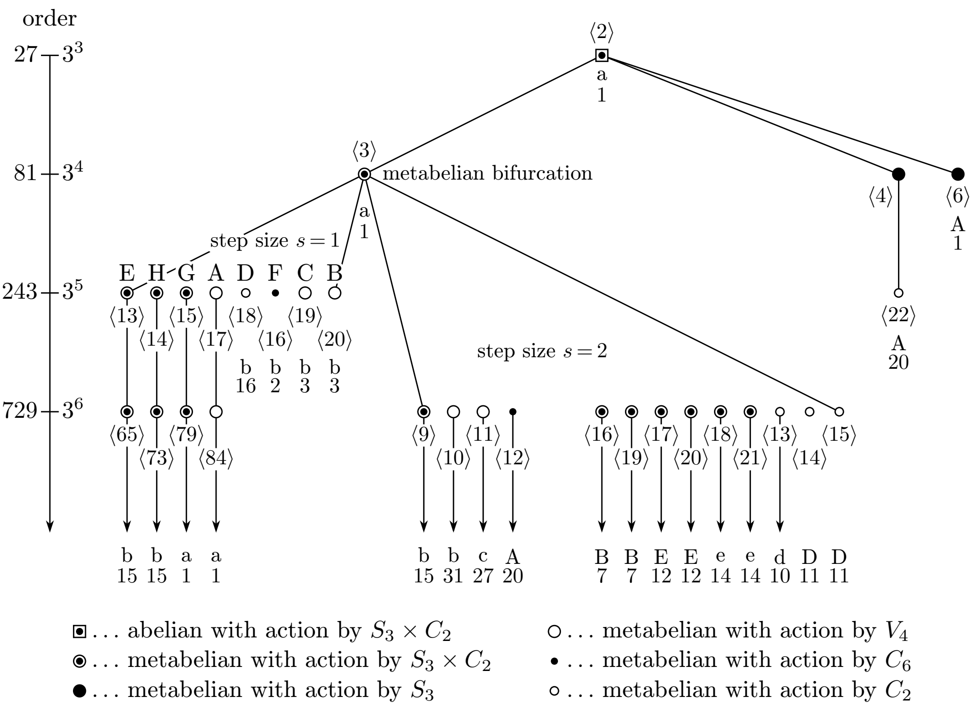


Figure 2. Finite 3-groups G with commutator quotient $G/G' \simeq C_9 \times C_3$.

By a 3-group *of type* $(9,3)$ we understand a finite group G with derived quotient $G/G' \simeq C_9 \times C_3$. Such groups of the second maximal class, that is of coclass $\mathrm{cc}(G) = 2$, were called *CF-groups* by Ascione et al. (see [4], Section 7, pages 272–274). Ascione denoted those of nilpotency class $\mathrm{Cl}(G) = 3$ by capital letters $\mathrm{A}, \dots, \mathrm{H}$ as in Figure 2. However, most of our 3-class tower groups $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ arise as descendants of step size $s = 2$ of the group $\langle 81, 3\rangle$ with remarkable metabelian bifurcation to coclass $\mathrm{cc} = 3$, in the sense of [21].

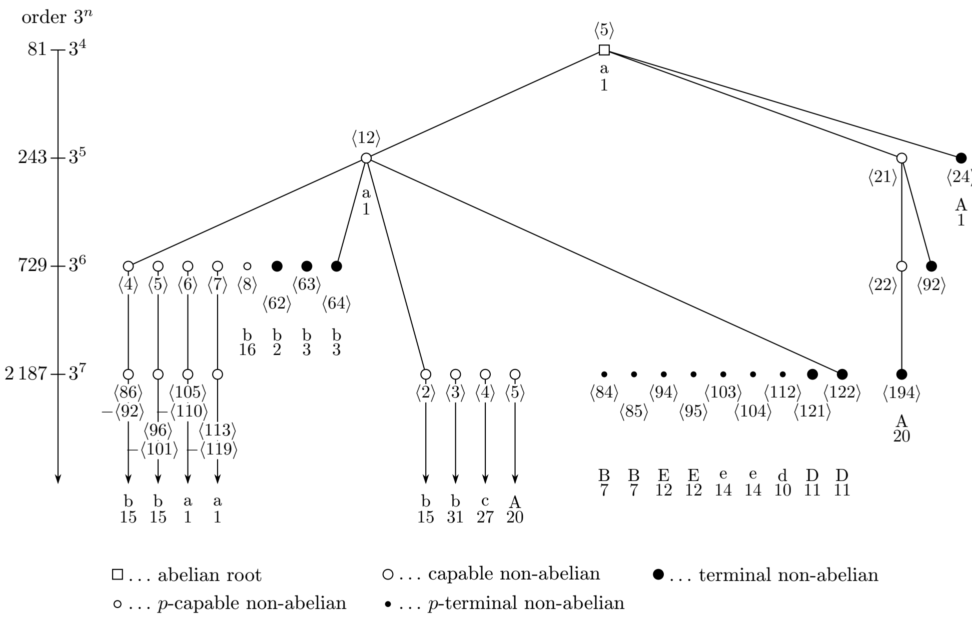


Figure 3. Finite 3-groups G with commutator quotient $G/G' \simeq (27,3)$.

4. Finite 3-groups with commutator quotient $(27,3)$ or $(81,3)$

In Figures 3, and 4, we give identifiers of finite 3-groups with commutator quotient $G/G' \simeq (27,3)$, and $G/G' \simeq (81,3)$, respectively. Directed edges lead from descendants D to parents $\pi(D) = D/\gamma_c(D)$, which usually differ from p-parents $\pi_p(D) = D/P_{c_p-1}(D)$. Up to minor modifications, the structure of the descendant trees in Figures 3 and 4 is the same as in Figure 2. On the left hand side, there are the *CF-groups* (cyclic factors) G for which the factors $\gamma_i(G)/\gamma_{i+1}(G) \simeq C_3$, $i \geqslant 3$, are always cyclic. They are higher analogues in branches $\Phi_s(b)$, $b \geqslant 1$, of Ascione $\mathrm{A}, \dots, \mathrm{H}$ (see [4]) in stems $\Phi_s(0)$ of isoclinism classes. On the right hand side, there are the *non-CF groups* G, which are called *BCF-groups* (bicyclic or cyclic factors) by Nebelung (see [31]), since $\gamma_3(G)/\gamma_4(G) \simeq C_3 \times C_3$ is bicyclic.

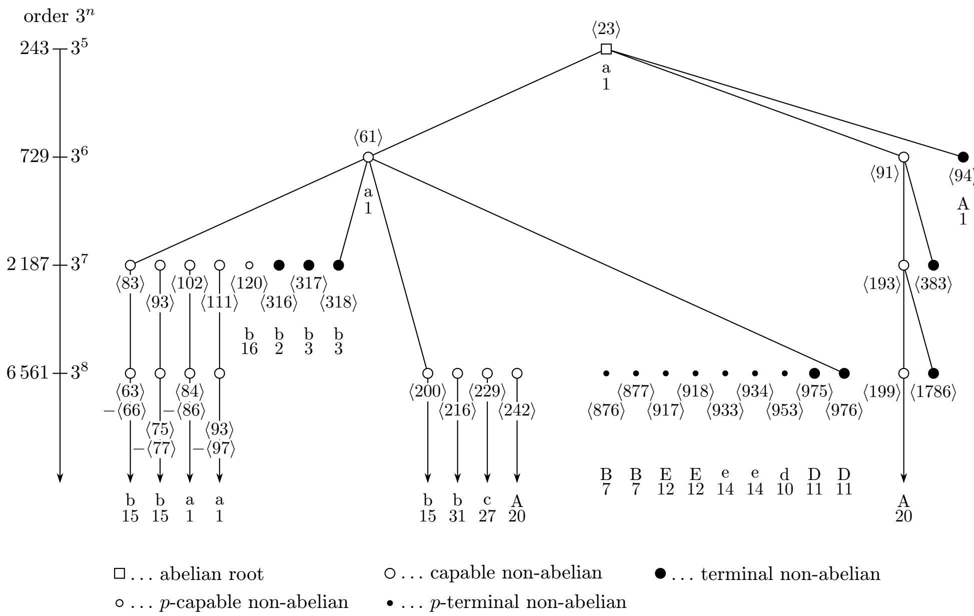


Figure 4. Finite 3-groups G with commutator quotient $G/G' \simeq (81,3)$.

5. Length of the Hilbert 3-class field tower

5.1. Separation of covers by descendant trees. In the following propositions, let P and D be finite 3-groups with isomorphic commutator quotients $(9,3)$, and let the parent P be a quotient of the descendant D by a normal subgroup contained in the commutator subgroup D'.

Proposition 1. *The components of $\alpha(P)$ are quotients of the corresponding components of $\alpha(D)$, which is exactly the meaning of the partial order relation $\alpha(P) \leqslant \alpha(D)$.*

Proof. The statement is part of the theorem on the antitony $\alpha(P) \leqslant \alpha(D)$ and $\varkappa(P) \geqslant \varkappa(D)$ of the components of the Artin pattern $(\alpha, \varkappa)$ with respect to (parent, descendant)-pairs (P, D), where P is a quotient of D, which we proved in [22], Sections 5.1–5.4, pages 78–87. □

Corollary 1. *None of the metabelian descendants of $\langle 243, j\rangle$ with $13 \leqslant j \leqslant 20$ can be the metabelianization of any non-metabelian descendant of $\langle 729, i\rangle$ with $13 \leqslant i \leqslant 21$.*

P r o o f. According to Table 3, the $\alpha(N)$ of non-metabelian descendants N of the groups $\langle 729, i\rangle$ with $13 \leqslant i \leqslant 21$ have at least two components (31). Since the $\alpha(M)$ of metabelian descendants M of the groups $\langle 243, j\rangle$ with $13 \leqslant j \leqslant 20$ have at least two components (21), none of the metabelianizations N/N'' (each of which has AQI coinciding with $\alpha(N)$) can be isomorphic to one of the groups M. □

Proposition 2. *The ranks of the components of $\alpha(D)$ cannot be smaller than the ranks of the corresponding components of $\alpha(P)$.*

P r o o f. This is an immediate consequence of Proposition 1. □

Corollary 2. *None of the metabelian descendants of $\langle 243, j\rangle$ with $13 \leqslant j \leqslant 20$ can be the metabelianization of any non-metabelian descendant of $\langle 729, i\rangle$ with $9 \leqslant i \leqslant 12$.*

P r o o f. According to Table 3, the four components of $\alpha(P)$ have at least rank three for the groups $P = \langle 729, i\rangle$ with $9 \leqslant i \leqslant 12$. By the antitony principle, this is also true for $\alpha(D)$ of any non-metabelian descendant D of one of these four roots P. However, metabelian descendants M of the groups $\langle 243, j\rangle$ with $13 \leqslant j \leqslant 20$ have $\alpha(M)$ with at least two components of rank two. Since each of the metabelianizations D/D'' has AQI coinciding with $\alpha(D)$, none of them can be isomorphic to one of the groups M. □

5.2. Relation rank and Galois action. Constraints arise from two issues, bounds for the relation rank of the tower group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$, and the Galois action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Cl}_3(K) \simeq G/G'$. By $\langle o, i\rangle$ we denote the groups in the SmallGroups database of Magma (see [17]). In tree diagrams, the order o is given on a scale, and we abbreviate the identifiers by $\langle i\rangle$.

Theorem 1. *For a number field K with 3-class rank $\varrho = 2$, in particular for $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$, the Galois group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ of the 3-class field tower must satisfy the following conditions.*

(1) *The relation rank d_2 of G must be bounded by $2 \leqslant d_2 \leqslant 2 + r + \theta$, where $r = r_1 + r_2 - 1$ denotes the torsion free Dirichlet unit rank of the field K with signature (r_1, r_2), and $\theta = 1$ if K contains the primitive third roots of unity, $\theta = 0$ otherwise.*
(2) *The automorphism group $\mathrm{Aut}(Q)$ of the Frattini quotient $Q = G/\Phi(G)$ must contain a subgroup isomorphic to $\mathrm{Gal}(K/\mathbb{Q})$.*

P r o o f. According to the Burnside basis theorem, the generator rank d_1 of G coincides with the generator rank of the Frattini quotient $Q = G/\Phi(G) = G/(G' \cdot G^3)$, or the derived quotient $G/G' \simeq \mathrm{Cl}_3(K)$, that is, the 3-class rank $\varrho = 2$ of K.

(1) According to the Shafarevich theorem (see [20], Theorem 5.1, page 28), the relation rank d_2 of G is bounded by $d_1 \leqslant d_2 \leqslant d_1 + r + \theta$. Together with the generator rank $d_1 = \varrho = 2$ this gives the bounds $2 \leqslant d_2 \leqslant 2 + r + \theta$. The original [35] contains a fatal misprint.

(2) The absolute Galois group $\mathrm{Gal}(K/\mathbb{Q})$ of K acts on the 3-class group $\mathrm{Cl}_3(K) \simeq G/G'$ and thus also on the Frattini quotient $Q = G/\Phi(G) = G/(G' \cdot G^3)$, whence $\mathrm{Aut}(Q)$ contains a subgroup isomorphic to $\mathrm{Gal}(K/\mathbb{Q})$. $\square$

By the same proof as for item (2) of Theorem 1, with $G/G' \simeq \mathrm{Cl}_3(K)$ replaced by

$$G_k/G_k' \simeq \mathrm{Gal}(\mathrm{F}_3^k(K)/K) \;/\; \mathrm{Gal}(\mathrm{F}_3^k(K)/\mathrm{F}_3^1(K)) \simeq \mathrm{Gal}(\mathrm{F}_3^1(K)/K) \simeq \mathrm{Cl}_3(K),$$

we obtain the same requirement for the Galois action on G_k (but not for the relation rank of G_k):

Corollary 3. *Let k be a positive integer and denote by $G_k = \mathrm{Gal}(\mathrm{F}_3^k(K)/K)$ the Galois group of the kth Hilbert 3-class field $\mathrm{F}_3^k(K)$ of K. The automorphism group $\mathrm{Aut}(Q)$ of the Frattini quotient $Q = G_k/\Phi(G_k)$ must contain a subgroup isomorphic to $\mathrm{Gal}(K/\mathbb{Q})$.*

6. Imaginary quadratic fields with 3-class group $(9, 3)$

An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ has signature $(r_1, r_2) = (0, 1)$, and torsionfree Dirichlet unit rank $r = r_1 + r_2 - 1 = 0$. If its class number is bigger than one, then it does not contain primitive third roots of unity, i.e., $\theta = 0$. Shafarevich bounds for the relation rank of $G_\infty = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ are given by $2 = d_1 \leqslant d_2 \leqslant d_1 + r + \theta = 2$, i.e., Schur σ-groups are mandatory. Among the 875 imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $-10^6 < d < 0$ and $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$, the pTKTs and second 3-class groups are distributed as in Table 4.

Experiment 1. Among imaginary quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminants $d < 0$ and 3-class group $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$, a dominant proportion of 46.4% has a metabelian 3-class field tower with length $l_3(K) = 2$ and Schur σ-group (see [16], [1], [9]) $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq$ either $\langle 729, 14\rangle$ or $\langle 729, 15\rangle$ having generator- and relator-inverting action by C_2 and punctured transfer kernel type D.11, $\varkappa(G) \sim (124; 1)$. Even in the case of equidistribution among the two candidate groups, the proportion is dominant with 23.2% for each. (Computed with [12], [17].)

#	%	$\lvert d_0\rvert$	pTKT	AQI	$\mathrm{Gal}(\mathrm{F}_3^2(K)/K)$	$l_3(K)$
406	46.4	3 299	D.11		$\langle 729, 14\vert 15\rangle$	2
75	8.6	5 703	E.12		$\in \mathcal{T}\langle 729, 17\vert 20\rangle$	$\geqslant 3$
64	7.3	54 695	B.7		$\in \mathcal{T}\langle 729, 16\vert 19\rangle$	$\geqslant 3$
20	2.3	289 704	A.20		$\in \mathcal{T}\langle 729, 12\rangle$	$\geqslant 3$
46	5.3	11 651	D.10		$\langle 6561, 1689\vert 1690\rangle$	3
20	2.3	17 723	D.5	heterocyclic	$\langle 6561, 1686\rangle$	3
40	4.6	31 983	D.6		$\langle 6561, 1744\vert 1782\rangle$	3
14	1.6	35 331	C.4	heterocyclic	$\langle 6561, 1687\rangle$	3
30	3.4	42 567	C.4	homocyclic	$\langle 6561, 1683\rangle$	3
15	1.7	116 419	D.5	homocyclic	$\langle 6561, 1684\rangle$	3

Table 4. Length $l_3(K)$ of 3-class field towers of imaginary quadratic fields K.

We give proofs for the dominant situation of 2 two-stage towers (nearly 50%) and for the significant contribution of 8 towers with precisely three stages (more than 18.8%).

Theorem 2 (Two-stage tower). *Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with fundamental discriminant $d < 0$, 3-class group $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$ and punctured capitulation type* D.11, $\varkappa(K) \sim (124; 1)$.

(1) *The Galois group G_2 of the second Hilbert 3-class field $\mathrm{F}_3^2(K)$ of K is one of the two unique metabelian Schur σ-groups $\mathrm{Gal}(\mathrm{F}_3^2(K)/K) \simeq \langle 729, 14\rangle$ or $\langle 729, 15\rangle$ (see Figure 2).*

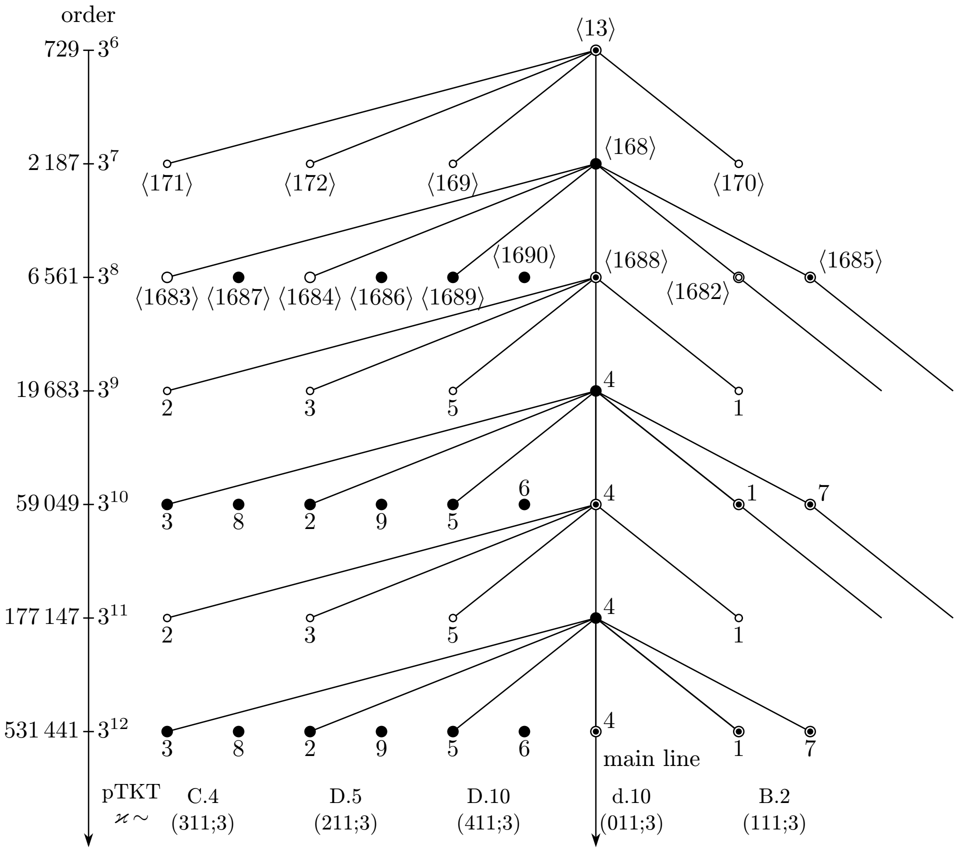
(2) *The abelian type invariants of the 3-class groups $\mathrm{Cl}_3(L_i)$ of the four unramified cyclic cubic extensions L_i, $1 \leqslant i \leqslant 4$, of K are given by $\alpha(K) \sim (31, 31, 211; 211)$.*

(3) *The 3-class field tower of K stops at the second stage, that is, $\mathrm{F}_3^2(K) = \mathrm{F}_3^\infty(K)$ is the maximal unramified pro-3 extension of K.*

Proof. The groups $\langle 729, 14\rangle$ or $\langle 729, 15\rangle$ are unique with Artin pattern $\alpha(K) = (31, 31, 211; 211)$, $\varkappa(K) \sim (124; 1)$, of type D.11. □

In the next theorem, a tower with precisely three stages is warranted, independently of the AQI. For the unambiguous identification of G_2 and G_∞, however, a specification of the AQI is mandatory. This is a generalization of the results in [10].

Theorem 3 (Three-stage towers). *Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with fundamental discriminant $d < 0$, 3-class group $\mathrm{Cl}_3(K) \simeq C_9 \times C_3$ and punctured capitulation type either* D.10, $\varkappa(K) \sim (411; 3)$ *or* C.4, $\varkappa(K) \sim (311; 3)$ *or* D.5, $\varkappa(K) \sim (211; 3)$ *or* D.6, $\varkappa(K) \sim (123; 1)$.



◎… GI action by C_2, homocyclic AQI

○… GI and RI action by C_2, homocyclic AQI

⊙… GI action by C_2

●… GI and RI action by C_2

∘… non-σ

Figure 5. Metabelian skeleton of the coclass tree $\mathcal{T}_3\langle 729, 13\rangle$.

(1) *The* 3*-class field tower of* K *stops at the third stage, that is,* $\mathrm{F}_3^3(K) = \mathrm{F}_3^\infty(K)$ *is the maximal unramified pro-*3 *extension of* K*, for all four assigned pTKTs.*

(2) *Let the abelian type invariants of the* 3*-class groups* $\mathrm{Cl}_3(L_i)$ *of the four unramified cyclic cubic extensions* L_i*,* $1 \leqslant i \leqslant 4$*, of* K *be denoted by* $\alpha(K)$*. The Galois group* G_2 *of the second Hilbert* 3*-class field* $\mathrm{F}_3^2(K)$ *of* K *and the Galois group* G_∞ *of the Hilbert* 3*-class field tower* $\mathrm{F}_3^3(K) = \mathrm{F}_3^\infty(K)$ *of* K *are given by*

   (a) $G_2 \simeq \langle 6561, 1683\rangle$ *(see Figure* 5*) and* $G_\infty \simeq \langle 2187, 168\rangle - \#2; 2$ *if* $\varkappa(K) \sim (311; 3)$ *and* $\alpha(K) \sim (222, 31, 31; 211)$ *(homocyclic type* C.4*),*

   (b) $G_2 \simeq \langle 6561, 1684\rangle$ *(see Figure* 5*) and* $G_\infty \simeq \langle 2187, 168\rangle - \#2; 3$ *if* $\varkappa(K) \sim (211; 3)$ *and* $\alpha(K) \sim (222, 31, 31; 211)$ *(homocyclic type* D.5*),*

(c) $G_2 \simeq \langle 6561, 1686\rangle$ *(see Figure 5) and* $G_\infty \simeq \langle 2187, 168\rangle - \#2; 5$ *if* $\varkappa(K) \sim (211; 3)$ *and* $\alpha(K) \sim (321, 31, 31; 211)$ *(heterocyclic type* D.5*),*

(d) $G_2 \simeq \langle 6561, 1687\rangle$ *(see Figure 5) and* $G_\infty \simeq \langle 2187, 168\rangle - \#2; 6$ *if* $\varkappa(K) \sim (311; 3)$ *and* $\alpha(K) \sim (321, 31, 31; 211)$ *(heterocyclic type* C.4*),*

(e) $G_2 \simeq \langle 6561, 1689\rangle$ *or* $G_2 \simeq \langle 6561, 1690\rangle$ *(see Figure 5) if* $\varkappa(K) \sim (411; 3)$ *and* $\alpha(K) \sim (321, 31, 31; 211)$ *(type* D.10*),*

(f) $G_2 \simeq \langle 6561, 1782\rangle$ *(see Figure 6) or* $G_2 \simeq \langle 6561, 1744\rangle$ *if* $\varkappa(K) \sim (123; 1)$ *and* $\alpha(K) \sim (31, 31, 31; 321)$ *(type* D.6*).*

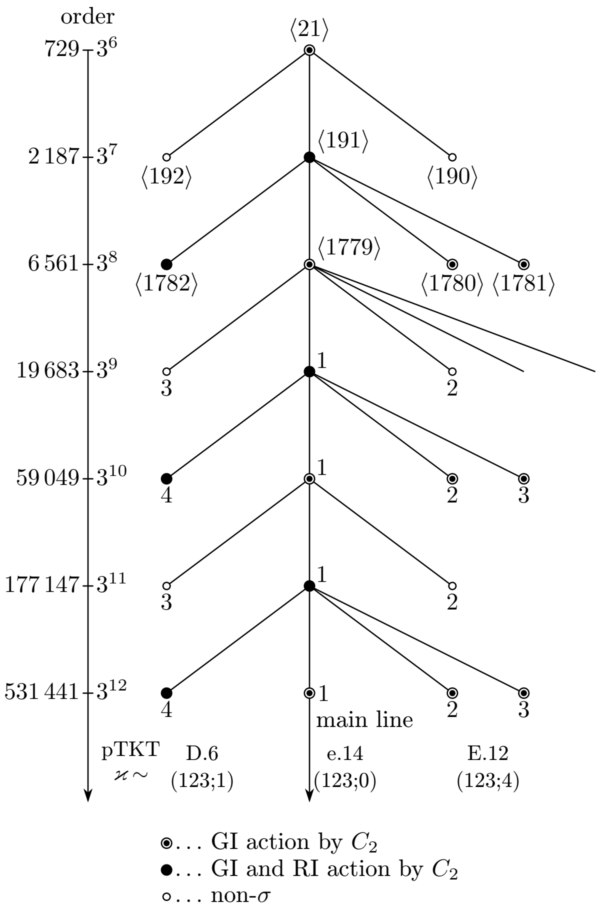


Figure 6. Metabelian skeleton of the coclass tree $\mathcal{T}_3\langle 729, 21\rangle$.

In each case, G_∞ *is a non-metabelian Schur* σ*-group with* $\mathrm{ord} = 19\,683$ *and* $\mathrm{dl} = 3$.

Proof. See Theorem 8. The notation with relative identifiers is due to [13], and [17]. □

7. Metabelian Schur σ-groups G with $G/G' \simeq (3^e,3)$, $e \geqslant 3$

According to Table 4, nearly one half of the imaginary quadratic fields $K=\mathbb{Q}(\sqrt{d})$ with 3-class group $\mathrm{Cl}_3(K) \simeq (9,3)$ possesses a metabelian 3-class field tower with automorphism group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K) \simeq \mathrm{SmallGroup}(729,i)$, where $i \in \{14,15\}$. More precisely and more generally, this trend continues for bigger commutator quotients $G/G' \simeq (3^e,3)$ and absolute discriminants $|d|$ below a given upper bound B: 406 among 875, that is 46.4% with $G \simeq \langle 729, 14|15\}$, $G/G' \simeq (9,3)$ when $B=10^6$, 433 among 930, that is 46.56% with $G \simeq \langle 2187, 121|122\}$, $G/G' \simeq (27,3)$ when $B = 3\cdot 10^6$, 999 among 2174, that is 45.95% with $G \simeq \langle 6561, 975|976\}$, $G/G' \simeq (81,3)$ when $B = 20\cdot 10^6$.

This high proportion of metabelian Schur σ-groups suggested to search for a general theoretical statement concerning metabelian Schur σ-groups G (with derived length $\mathrm{dl}(G)=2$). Indeed, we succeeded in finding another periodicity, underpinned with two infinite limit groups by Newman, and justified rigorously by a parametrized pc-presentation. See also [30].

Theorem 4. *For all integers $e \geqslant 3$, the unique pair of metabelian Schur σ-groups G with commutator quotient $G/G' \simeq (3^e,3) \hat{=} (e1)$ is given by the periodic sequence*

$$\mathrm{SmallGroup}(729,8)(-\#1;1)^{e-3} - \#1;i, \quad i \in \{2,3\}. \tag{7.1}$$

Corollary 4. *The order of these groups is $\#G = 3^{4+e}$, and their Artin pattern $(\varkappa,\alpha)$ is given by constant punctured transfer kernel type* D.11, $\varkappa \sim (124;1)$, *and increasing abelian quotient invariants, $\alpha \sim ((e+1)1,(e+1)1,e11;e11)$.*

Theorem 5. *For all integers $e \geqslant 3$, the unique pair of metabelian Schur σ-groups G with commutator quotient $G/G' \simeq (3^e,3) \hat{=} (e1)$ is alternatively given by $G \simeq (L_{11}/P_e(L_{11})) - \#1;i$, $i \in \{2,3\}$, where the infinite limit group L_{11} is given by the finite presentation*

$$L_{11} = \langle a,t,u\colon\ [t,a]=u,\ [u,a]=t^3,\ t^3=[u,t],\ u^3=1\rangle. \tag{7.2}$$

Theorem 6. *For all integers $e \geqslant 3$, the unique pair of metabelian Schur σ-groups G with commutator quotient $G/G' \simeq (3^e,3) \hat{=} (e1)$ is alternatively given by $G \simeq L_{16}/\langle a^{\mp 3^e} \cdot t^3 \cdot [t,a,t]\rangle$, where the infinite limit group L_{16} is given by the finite presentation*

$$L_{16} = \langle a,t\colon\ [t,a,a]=t^3,\ [t,a]^3=1,\ [t,a,t,t]=1\rangle. \tag{7.3}$$

Additionally, the p-parent of the next pair (with $G/G' \simeq (3^{e+1},3)$) is given by $L_{16}/\langle t^3\cdot[t,a,t]\rangle$.

Theorem 7. *For all integers $e \geqslant 3$, the unique pair of metabelian Schur σ-groups G with commutator quotient $G/G' \simeq (3^e, 3) \hat{=} (e1)$ is alternatively given by the parametrized presentation*

(7.4)

$$G = \langle x, y, s_2, s_3, t_3, w\colon\ s_2 = [y, x],\ s_3 = [s_2, x],\ t_3 = [s_2, y],\ x^{3^e} = w,\ y^3 = s_3,\ R\rangle,$$

where R denotes either the relation $t_3 = s_3 \cdot w$ or $t_3 = s_3 \cdot w^2$.

Proof. Theorems 4–6 are proved by means of formula (7.4) in Theorem 7. □

In particular, for $e = 3$ and $e = 4$ we have the pairs

$$\mathrm{SmallGroup}(2187, 121) \simeq \langle x, y, s_2, s_3, t_3, w\colon\ x^{27} = w,\ y^3 = s_3,\ t_3 = s_3 \cdot w\rangle,$$
$$\mathrm{SmallGroup}(2187, 122) \simeq \langle x, y, s_2, s_3, t_3, w\colon\ x^{27} = w,\ y^3 = s_3,\ t_3 = s_3 \cdot w^2\rangle;$$

and

$$\mathrm{SmallGroup}(6561, 975) \simeq \langle x, y, s_2, s_3, t_3, w\colon\ x^{81} = w,\ y^3 = s_3,\ t_3 = s_3 \cdot w\rangle,$$
$$\mathrm{SmallGroup}(6561, 976) \simeq \langle x, y, s_2, s_3, t_3, w\colon\ x^{81} = w,\ y^3 = s_3,\ t_3 = s_3 \cdot w^2\rangle.$$

The smallest pair, for $e = 2$, however, is exceptional:

$$\mathrm{SmallGroup}(729, 14) \simeq \langle x, y, s_2, s_3, t_3\colon\ x^9 = s_3,\ y^3 = t_3\rangle,$$
$$\mathrm{SmallGroup}(729, 15) \simeq \langle x, y, s_2, s_3, t_3\colon\ x^9 = s_3,\ y^3 = t_3^2\rangle.$$

8. Imaginary quadratic fields with bicyclic 3-class group of order > 27

When we pass from finite 3-groups G with simplest non-elementary bicyclic commutator quotient G/G' of type $(9, 3)$ to situations with order > 27, that is, the cases $(27, 3)$, $(81, 3)$, $(243, 3)$, etc., then the construction of the groups by means of the p-group generation algorithm (see [32], [33]) becomes increasingly difficult, since the algorithm uses the definition of (parent, descendant)-pairs $(\pi_p(D), D)$ with $\pi_p(D) = D/P_k(D)$ in terms of the lower exponent-p central series $(P_i(D))_{0\leqslant i\leqslant k}$, whereas the parent $\pi(D) = D/\gamma_c(D)$ on the coclass tree of D may have nuclear rank $n = 0$.

In order to emphasize the broad scope of our current investigations, let us summarize the characteristic properties of all coclass trees with metabelian mainline of type d.10, independently of the arbitrary commutator quotient $(9, 3)$, $(27, 3)$, $(81, 3)$, etc. For fixed commutator quotient, this tree is unique. The infinite main line consists of metabelian vertices with punctured transfer kernel type d.10, $\varkappa \sim (011; 3)$, infinitely capable. The metabelian vertices of depth one with respect to the main line have either

type B.2, $\varkappa \sim (111;3)$, finitely capable for fixed coclass, infinitely for all step sizes, or
type D.5, $\varkappa \sim (211;3)$, terminal, or
type C.4, $\varkappa \sim (311;3)$, terminal, or
type D.10, $\varkappa \sim (411;3)$, terminal.

Type B.2 gives rise to brushwood descendants (if multifurcation is admitted, then the derived length is probably unbounded), which prohibits straightforward statements about the tower length (except that it must be at least equal to three). However, we believe that the types D.5, C.4 and D.10 of imaginary quadratic fields are always associated with a tower of precise length three.

When we assign a fixed punctured transfer target type α (i.e., the abelian quotient invariants of maximal subgroups), then our belief can be proven rigorously for commutator quotient $(9,3)$ (Theorem 8, June 2013), $(27,3)$ (Theorem 9, 16 July 2021), $(81,3)$ (Theorem 10, 20 July 2021), $(243,3)$ (Theorem 11, 28 July 2021), $(729,3)$ and higher (Theorems 12 and 13, 30 July 2021).

In the following theorems, a tower with precisely three stages is warranted, independently of the AQI. For the unambiguous identification of the metabelianization $M = G/G''$ and the group G, however, a specification of the AQI is mandatory.

Theorem 8. *An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with 3-class group of type $(9,3)$ and one of the following six kinds of Artin pattern $\mathrm{AP}(K) = (\varkappa, \alpha)$ has a 3-class field tower of precise length $l_3(K) = 3$:*

type C.4, $\varkappa \sim (311;3)$, $\alpha \sim (222,31,31;211)$ *with homocyclic* 1*st component or*
type D.5, $\varkappa \sim (211;3)$, $\alpha \sim (222,31,31;211)$ *with homocyclic* 1*st component or*
type D.5, $\varkappa \sim (211;3)$, $\alpha \sim (321,31,31;211)$ *with heterocyclic* 1*st component or*
type C.4, $\varkappa \sim (311;3)$, $\alpha \sim (321,31,31;211)$ *with heterocyclic* 1*st component or*
type D.10, $\varkappa \sim (411;3)$, $\alpha \sim (321,31,31;211)$ *with heterocyclic* 1*st component or*
type D.6, $\varkappa \sim (123;1)$, $\alpha \sim (31,31,31;321)$ *with heterocyclic* 4*th component.*

The tower group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ is of order 3^9, and its metabelianization $M = G/G''$ is of order 3^8.

P r o o f. In each case, G is a non-metabelian Schur σ-group with $\mathrm{ord} = 19\,683$ and $\mathrm{dl} = 3$.

(1) $M \simeq \langle 6561, 1683\rangle$ and $G \simeq \langle 2187, 168\rangle - \#2;2$ if $\varkappa(K) \sim (311;3)$ and $\alpha(K) \sim (222,31,31;211)$ (homocyclic),
(2) $M \simeq \langle 6561, 1684\rangle$ and $G \simeq \langle 2187, 168\rangle - \#2;3$ if $\varkappa(K) \sim (211;3)$ and $\alpha(K) \sim (222,31,31;211)$ (homocyclic),
(3) $M \simeq \langle 6561, 1686\rangle$ and $G \simeq \langle 2187, 168\rangle - \#2;5$ if $\varkappa(K) \sim (211;3)$ and $\alpha(K) \sim (321,31,31;211)$ (heterocyclic),

(4) $M \simeq \langle 6561, 1687\rangle$ and $G \simeq \langle 2187, 168\rangle - \#2; 6$ if $\varkappa(K) \sim (311; 3)$ and $\alpha(K) \sim (321, 31, 31; 211)$ (heterocyclic),
(5) $M \simeq \langle 6561, 1689\rangle$, $G \simeq \langle 2187, 168\rangle - \#2; 8$ or $M \simeq \langle 6561, 1690\rangle$, $G \simeq \langle 2187, 168\rangle - \#2; 9$ if $\varkappa(K) \sim (411; 3)$ and $\alpha(K) \sim (321, 31, 31; 211)$,
(6) $M \simeq \langle 6561, 1744\rangle$, $G \simeq \langle 2187, 181\rangle - \#2; 4$ or $M \simeq \langle 6561, 1782\rangle$, $G \simeq \langle 2187, 191\rangle - \#2; 4$ if $\varkappa(K) \sim (123; 1)$ and $\alpha(K) \sim (31, 31, 31; 321)$.

□

Example 1. Concrete realizations of Artin patterns in Theorem 8:
type C.4, homocyclic, for $d = -42\,567$,
type D.5, homocyclic, for $d = -116\,419$,
type D.5, heterocyclic, for $d = -17\,723$,
type C.4, heterocyclic, for $d = -35\,331$,
type D.10 for $d = -11\,651$,
type D.6 for $d = -31\,983$.

Theorem 9. *An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with 3-class group of type $(27, 3)$ and one of the following four kinds of Artin pattern $\mathrm{AP}(K) = (\varkappa, \alpha)$ has a 3-class field tower of precise length $l_3(K) = 3$:*

type D.10, $\varkappa \sim (411; 3)$, $\alpha \sim (322, 41, 41; 311)$ with homocyclic 1st component or
type D.5, $\varkappa \sim (211; 3)$, $\alpha \sim (421, 41, 41; 311)$ with heterocyclic 1st component or
type C.4, $\varkappa \sim (311; 3)$, $\alpha \sim (421, 41, 41; 311)$ with heterocyclic 1st component or
type D.6, $\varkappa \sim (123; 1)$, $\alpha \sim (41, 41, 41; 322)$ with homocyclic 4th component.

The tower group $G = \mathrm{Gal}(\mathrm{F}_3^{\infty}(K)/K)$ is of order 3^{10}, and its metabelianization $M = G/G''$ is of order 3^9.

Proof. It suffices to give the isomorphism classes of G and M. The unique metabelian 3-group with commutator quotient $(27, 3)$, order 3^8, and punctured transfer kernel type d.10, $\varkappa \sim (011; 3)$, is given by the fork $B := \mathrm{SmallGroup}(6561, 98)$. It has nuclear rank $n(B) = 2$ and thus causes a bifurcation which uniquely determines G (and thus of course also $M = G/G''$), for all given Artin patterns, with two solutions (Schur σ-candidates) each:

$G \simeq B - \#2; 2$ or 3, $M \simeq B - \#1; 4$ or 5, for $\varkappa \sim (411; 3)$, $\alpha \sim (322, 41, 41; 311)$,
$G \simeq B - \#2; 6$ or 8, $M \simeq B - \#1; 8$ or 10, for $\varkappa \sim (211; 3)$, $\alpha \sim (421, 41, 41; 311)$,
$G \simeq B - \#2; 5$ or 9, $M \simeq B - \#1; 7$ or 11, for $\varkappa \sim (311; 3)$, $\alpha \sim (421, 41, 41; 311)$.

There exist two metabelian 3-groups with commutator quotient $(27, 3)$, order 3^8, and punctured transfer kernel type e.14, $\varkappa \sim (123; 0)$, namely the forks $B_1 := \mathrm{SmallGroup}(6561, 88)$ and $B_2 := \mathrm{SmallGroup}(6561, 91)$. They have nuclear rank $n(B_i) = 2$. Their bifurcation determines two Schur σ-candidates for G: $G \simeq B_i - \#2; 4$, and $M \simeq B_i - \#1; 4$ with $1 \leqslant i \leqslant 2$, for $\varkappa \sim (123; 1)$, $\alpha \sim (41, 41, 41; 322)$. □

E x a m p l e 2. Concrete realizations of Artin patterns in Theorem 9:

type D.10 for $d = -110\,059$,
type D.5 for $d = -382\,232$,
type C.4 for $d = -41\,631$,
type D.6 for $d = -155\,224$.

Theorem 10. *An imaginary quadratic field* $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, *with* 3*-class group of type* $(81, 3)$ *and one of the following four kinds of Artin pattern* $\mathrm{AP}(K) = (\varkappa, \alpha)$ *has a* 3*-class field tower of precise length* $l_3(K) = 3$*:*

type D.10, $\varkappa \sim (411; 3)$, $\alpha \sim (422, 51, 51; 411)$ *with homocyclic* 1*st component or*
type D.5, $\varkappa \sim (211; 3)$, $\alpha \sim (521, 51, 51; 411)$ *with heterocyclic* 1*st component or*
type C.4, $\varkappa \sim (311; 3)$, $\alpha \sim (521, 51, 51; 411)$ *with heterocyclic* 1*st component or*
type D.6, $\varkappa \sim (123; 1)$, $\alpha \sim (51, 51, 51; 422)$ *with homocyclic* 4*th component.*

The tower group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ *is of order* 3^{11}*, and its metabelianization* $M = G/G''$ *is of order* 3^{10}*.*

P r o o f. For the leading three Artin patterns, we start with $A_1 :=$ SmallGroup $(6561, 93)$, a metabelian 3-group with coclass $\mathrm{cc}(A_1) = 4$ and Artin pattern $\varkappa \sim (000; 0)$, $\alpha \sim (421, 41, 41; 311)$. It has nuclear rank $n(A_1) = 2$. The metabelianizations M of the 3-class field tower groups G are given by

$A_1 - \#2; 7$ or 8 for type D.10, $\varkappa \sim (411; 3)$, $\alpha \sim (422, 51, 51; 411)$,
$A_1 - \#2; 11$ or 13 for type D.5, $\varkappa \sim (211; 3)$, $\alpha \sim (521, 51, 51; 411)$,
$A_1 - \#2; 10$ or 14 for type C.4, $\varkappa \sim (311; 3)$, $\alpha \sim (521, 51, 51; 411)$.

They all have nuclear rank $n(M) = 1$ and a unique terminal descendant $M - \#1; 1$, which is exactly the Schur σ-group G.

For the trailing Artin pattern we start with $A_2 :=$ SmallGroup$(6561, 85)$, a metabelian 3-group with coclass $\mathrm{cc}(A_2) = 4$ and Artin pattern $\varkappa \sim (000; 0)$, $\alpha \sim (41, 41, 41; 322)$. It has nuclear rank $n(A_2) = 2$. The metabelianizations M of the 3-class field tower groups G are given by

$A_2 - \#2; 8$ or 12 for type D.6, $\varkappa \sim (123; 1)$, $\alpha \sim (51, 51, 51; 422)$.

They have nuclear rank $n(M) = 1$ and a unique terminal descendant $M - \#1; 1$, which is exactly the Schur σ-group G. □

E x a m p l e 3. Concrete realizations of Artin patterns in Theorem 10:

type D.10 for $d = -469\,283$,
type D.5 for $d = -584\,411$,
type C.4 for $d = -617\,363$,
type D.6 for $d = -548\,939$.

R e m a r k 1. The common parent, with respect to the usual lower central, of the leading 12 groups G and M is $B = A_1 - \#1;2$ with $n(B) = 1$. But B is useless for the construction process, since it has only 3 terminal descendants with Artin pattern of type d.10, $\varkappa \sim (011;3)$, $\alpha \sim (421, 51, 51; 411)$.

Theorem 11. *An imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$, $d < 0$, with 3-class group of type $(243, 3)$ and one of the following four kinds of Artin pattern $\mathrm{AP}(K) = (\varkappa, \alpha)$ has a 3-class field tower of precise length $l_3(K) = 3$:*

type D.10, $\varkappa \sim (411;3)$, $\alpha \sim (522, 61, 61; 511)$ *with homocyclic* 1*st component or*
type D.5, $\varkappa \sim (211;3)$, $\alpha \sim (621, 61, 61; 511)$ *with heterocyclic* 1*st component or*
type C.4, $\varkappa \sim (311;3)$, $\alpha \sim (621, 61, 61; 511)$ *with heterocyclic* 1*st component or*
type D.6, $\varkappa \sim (123;1)$, $\alpha \sim (61, 61, 61; 522)$ *with homocyclic* 4*th component.*

The tower group $G = \mathrm{Gal}(\mathrm{F}_3^\infty(K)/K)$ is of order 3^{12}, and its metabelianization $M = G/G''$ is of order 3^{11}.

P r o o f. First, we start with $A_2 := \mathrm{SmallGroup}(6561, 93) - \#2;2$, a metabelian 3-group with coclass $\mathrm{cc}(A_2) = 5$ and Artin pattern $\varkappa \sim (400;0)$, $\alpha \sim (522, 51, 51; 411)$. It has nuclear rank $n(A_2) = 1$. The metabelianizations M of the 3-class field tower groups G are given by $A_2 - \#1;2$ or 3 for type D.10, $\varkappa \sim (411;3)$, $\alpha \sim (522, 61, 61; 511)$.

Then, we start with $A_4 := \mathrm{SmallGroup}(6561, 93) - \#2;4$, a metabelian 3-group with coclass $\mathrm{cc}(A_4) = 5$ and Artin pattern $\varkappa \sim (000;0)$, $\alpha \sim (521, 51, 51; 411)$. It has nuclear rank $n(A_4) = 1$. The metabelianizations M of the 3-class field tower groups G are given by $A_4 - \#1;2$ or 3 for type D.5, $\varkappa \sim (211;3)$, $\alpha \sim (621, 61, 61; 511)$.

Next, we start with $A_5 := \mathrm{SmallGroup}(6561, 93) - \#2;5$, a metabelian 3-group with coclass $\mathrm{cc}(A_5) = 5$ and Artin pattern $\varkappa \sim (000;0)$, $\alpha \sim (521, 51, 51; 411)$. It has nuclear rank $n(A_5) = 1$. The metabelianizations M of the 3-class field tower groups G are given by $A_5 - \#1;2$ or 3 for type C.4, $\varkappa \sim (311;3)$, $\alpha \sim (621, 61, 61; 511)$.

Finally, we start with $A_0 := \mathrm{SmallGroup}(6561, 85) - \#2;4$, a metabelian 3-group with coclass $\mathrm{cc}(A_0) = 5$ and Artin pattern $\varkappa \sim (000;0)$, $\alpha \sim (51, 51, 51; 422)$. It has nuclear rank $n(A_0) = 1$. The metabelianizations M of the 3-class field tower groups G are given by $A_0 - \#1;2$ or 3 for type D.6, $\varkappa \sim (123;1)$, $\alpha \sim (61, 61, 61; 522)$.

All these metabelian groups M have nuclear rank $n(M) = 1$ and a unique terminal descendant $M - \#1;1$, which is exactly the Schur σ-group G. □

E x a m p l e 4. Concrete realizations of Artin patterns in Theorem 11:

type D.10 for $d = -2\,115\,951$,
type D.5 for $d = -2\,105\,871$,
type C.4 for $d = -5\,687\,591$,
type D.6 for $d = -5\,368\,119$.

E x a m p l e 5. Concrete realizations of Artin patterns for $\mathrm{Cl}_3(\mathbb{Q}(\sqrt{d})) \simeq (729, 3)$ in Theorem 12:

type D.10 for $d = -21\,658\,691$,
type D.5 for $d = -8\,421\,559$,
type C.4 for $d = -16\,554\,479$,
type D.6 for $d = -6\,720\,503$.

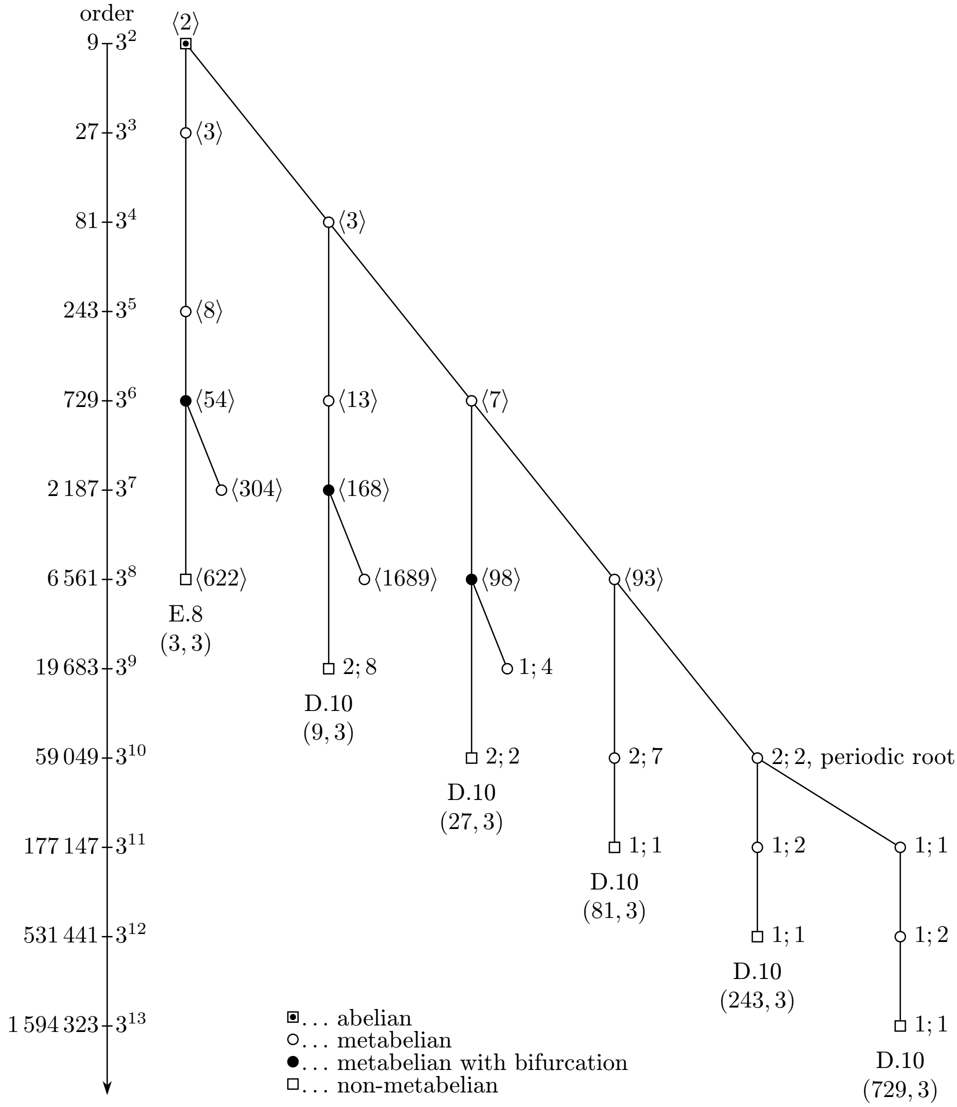


Figure 7. Schur σ-groups G with commutator quotient $G/G' \simeq (3^e, 3)$, $1 \leqslant e \leqslant 6$.

9. Periodicity and limits for $G/G' \simeq (3^e, 3)$, $e \geqslant 5$

In Figure 7, all directed edges lead from descendants D to p-parents $\pi_p(D) = D/P_{c_p-1}(D)$, rather than to parents $\pi(D) = D/\gamma_c(D)$. The figure admits actual descendant construction.

Figure 7 shows that the construction process for the eight non-metabelian Schur σ-groups G with order $\#G = 3^{7+e}$ and punctured transfer kernel types D.10, C.4, D.5 and D.6, becomes increasingly difficult for the commutator quotients $G/G' \simeq (27,3)$, $(81,3)$, $(243,3)$. For the commutator quotient $G/G' \simeq (729,3)$, however, an unexpected tranquilization occurs, and the construction process becomes settled with a simple step size one periodicity.

Theorem 12. *The four pairs of Schur σ-groups G with derived length $\mathrm{dl}(G) = 3$, commutator quotient $G/G' \simeq (3^e,3) \hat{=} (e1)$, $e \geqslant 5$, punctured transfer kernel types* D.10, D.5, C.4, D.6, *and order $\#G = 3^{7+e}$ are given by the following bottom up construction process.*

- ▷ *For type* D.10, $\varkappa \sim (411;3)$, $\alpha \sim (e22,(e+1)1,(e+1)1;e11)$, *let* $A_2 := \mathrm{SmallGroup}(6561,93) - \#2;2$. *Then*

$$G \simeq A_2(-\#1;1)^{e-5} - \#1;i - \#1;1, \quad i \in \{2,3\}. \tag{9.1}$$

- ▷ *For type* D.5, $\varkappa \sim (211;3)$, $\alpha \sim ((e+1)21,(e+1)1,(e+1)1;e11)$, *let* $A_4 := \mathrm{SmallGroup}(6561,93) - \#2;4$. *Then*

$$G \simeq A_4(-\#1;1)^{e-5} - \#1;i - \#1;1, \quad i \in \{2,3\}. \tag{9.2}$$

- ▷ *For type* C.4, $\varkappa \sim (311;3)$, $\alpha \sim ((e+1)21,(e+1)1,(e+1)1;e11)$, *let* $A_5 := \mathrm{SmallGroup}(6561,93) - \#2;5$. *Then*

$$G \simeq A_5(-\#1;1)^{e-5} - \#1;i - \#1;1, \quad i \in \{2,3\}. \tag{9.3}$$

- ▷ *For type* D.6, $\varkappa \sim (123;1)$, $\alpha \sim ((e+1)1,(e+1)1,(e+1)1;e22)$, *let* $A_0 := \mathrm{SmallGroup}(6561,85) - \#2;4$. *Then*

$$G \simeq A_0(-\#1;1)^{e-5} - \#1;i - \#1;1, \quad i \in \{2,3\}. \tag{9.4}$$

For all types D.10, C.4, D.5 and D.6, Newman has found infinite limit groups whose quotients give rise to the Schur σ-groups G of order $\#G \geqslant 3^{12}$, that is, $e \geqslant 5$.

Theorem 13. *The four pairs of Schur σ-groups G with derived length $\mathrm{dl}(G) = 3$, commutator quotient $G/G' \simeq (3^e,3) \hat{=} (e1)$, $e \geqslant 5$, punctured transfer kernel types* D.10, D.5, C.4, D.6, *and order $\#G = 3^{7+e}$ are alternatively given by the following top down construction process.*

▷ *For type* D.10, $\varkappa \sim (411;3)$, $\alpha \sim (e22,(e+1)1,(e+1)1;e11)$, *let the infinite limit group be given by the finite presentation*

$$L_{10} = \langle a,t,u\colon\ [t,a]=u,\ [u,a]=[u,t],\ [u,t,u]=1,\\ t^3=[u,t,t,t],\ u^3=[u,t,t]^2\cdot[u,t,t,t]\rangle. \tag{9.5}$$

Then $G \simeq (L_{10}/P_e(L_{10})) - \#1;i - \#1;1$, $i \in \{2,3\}$.

▷ *For type* D.5, $\varkappa \sim (211;3)$, $\alpha \sim ((e+1)21,(e+1)1,(e+1)1;e11)$, *let the infinite limit group be given by the finite presentation*

$$L_5 = \langle a,t,u\colon\ [t,a]=u,\ [u,a]=t^3\cdot[u,t],\ t^3=[u,t,t,t]^{-1}\rangle. \tag{9.6}$$

Then $G \simeq (L_5/P_e(L_5)) - \#1;i - \#1;1$, $i \in \{2,3\}$.

▷ *For type* C.4, $\varkappa \sim (311;3)$, $\alpha \sim ((e+1)21,(e+1)1,(e+1)1;e11)$, *let the infinite limit group be given by the finite presentation*

$$L_4 = \langle a,t,u\colon\ [t,a]=u,\ [u,a]=[u,t]\cdot[u,t,t,t]^{-1},\ [u,t,u]=1,\\ t^3=[u,t,t,t],\ u^3=[u,t,t]^2\cdot[u,t,t,t],\ [u,t]^3=[u,t,t,t]^2\rangle. \tag{9.7}$$

Then $G \simeq (L_4/P_e(L_4)) - \#1;i - \#1;1$, $i \in \{2,3\}$.

▷ *For type* D.6, $\varkappa \sim (123;1)$, $\alpha \sim ((e+1)1,(e+1)1,(e+1)1;e22)$, *let the infinite limit group be given by the finite presentation*

$$L_6 = \langle a,t,u\colon\ [t,a]=u,\ [u,a]=t^6\cdot u^6,\ [u,t]=t^9,\ u^9=1,\ [u,t]^3=1\rangle. \tag{9.8}$$

Then $G \simeq (L_6/P_e(L_6)) - \#1;i - \#1;1$, $i \in \{2,3\}$.

Proof. The proof consists of the construction of successive descendants of A_2, A_4, A_5, A_0 in the way indicated in Theorem 12 by means of the p-group generation algorithm (see [15]) by Newman (see [32]) and O'Brien (see [33]), which is implemented in the computational algebra system Magma (see [7], [8], [17]), and verifying isomorphism to the descendants of quotients of limit groups as claimed in Theorem 13. For a proof with parametrized pc-presentations see [30]. □

10. Deterministic laws for CF- and BCF-groups

The directed edges of the graphs in Figure 3 and 4 are not suitable for the actual construction of the vertices, since they lead from descendants D to parents $P = D/\gamma_c(D)$. Examplarily, we illustrate some hidden directed edges from descendants D to p-parents (ancestors) $A = D/P_{c_p-1}(D)$.

Theorem 14. *For each exponent $e \geqslant 3$, we assume that $a+b = e+1$ with integers $0 \leqslant a - b \leqslant 1$. Then*

- ▷ *there are exactly two* BCF*-groups D with punctured transfer kernel type* e.14 *and Artin pattern $\varkappa \sim (123; 0)$, $\alpha \sim ((e+1)1, (e+1)1, (e+1)1; ab1)$; their common p-parent is the* CF*-group $A = D/P_e(D)$ with type* a.1 *and Artin pattern $\varkappa \sim (000; 0)$, $\alpha \sim (e1, e1, e1; ab1)$;*
- ▷ *there is a unique* BCF*-group D with punctured transfer kernel type* d.10 *and Artin pattern $\varkappa \sim (110; 2)$, $\alpha \sim ((e+1)1, (e+1)1, e11; e11)$; its p-parent is the* CF*-group $A = D/P_e(D)$ with type* a.1 *and Artin pattern $\varkappa \sim (000; 0)$, $\alpha \sim (e1, e1, e11; (e-1)11)$;*
- ▷ *there are exactly two* BCF*-groups D with punctured transfer kernel type* D.11 *and Artin pattern $\varkappa \sim (124; 2)$, $\alpha \sim ((e+1)1, (e+1)1, e11; e11)$; their common p-parent is the* CF*-group $A = D/P_e(D)$ with type* b.16*, Artin pattern $\varkappa \sim (004; 0)$, $\alpha \sim (e1, e1, e11; (e-1)11)$.*

The p-class of the BCF*-groups D is always $c_p(D) = e+1$ and their order is $\#D = 3^{4+e}$. The commutator quotient of all groups D and A is $(3^e, 3)$.*

The groups of the last item are the metabelian Schur σ-groups in Section 7.

Proof. We generally denote some crucial commutators by $s_2 = [y, x]$, $s_3 = [s_2, x]$, $t_3 = [s_2, y]$.

- ▷ $D = \langle x, y \colon\ x^{3^e} = w,\ y^3 = s_3^2,\ t_3 = w \text{ or } w^2\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y \colon\ x^{3^e} = 1,\ y^3 = s_3^2,\ t_3 = 1\rangle$.
- ▷ $D = \langle x, y \colon\ x^{3^e} = w,\ y^3 = 1,\ t_3 = s_3 w\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y \colon\ x^{3^e} = 1,\ y^3 = 1,\ s_3 = t_3\rangle$.
- ▷ $D = \langle x, y \colon\ x^{3^e} = w,\ y^3 = s_3,\ t_3 = s_3 w \text{ or } s_3 w^2\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y \colon\ x^{3^e} = 1,\ y^3 = s_3 = t_3\rangle$.

□

In particular, we have some relations between groups with SmallGroup identifiers for $e \in \{3, 4\}$.

Corollary 5. *The p-class of the* BCF*-groups $D \simeq \langle 2187, i\rangle$ with $i \in \{103, 104, 112, 121, 122\}$ and punctured transfer kernel types* e.14, d.10, D.11 *is $c_p = 4$, and their p-parents are the* CF*-groups $A = D/P_3(D) \simeq \langle 729, j\rangle$ with $j \in \{6, 7, 8\}$ and types* a.1 *and* b.16.

Proof. Using Theorem 14 for $e = 3$, we obtain:

- ▷ For $D = \langle 2187, 103|104\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 6\rangle$.
- ▷ For $D = \langle 2187, 112\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 7\rangle$.
- ▷ For $D = \langle 2187, 121|122\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 8\rangle$. □

Corollary 6. *The p-class of the* BCF-*groups* $D \simeq \langle 6561, i\rangle$ *with* $i \in \{933, 934, 953, 975, 076\}$ *and punctured transfer kernel types* e.14, d.10, D.11 *is* $c_p = 5$, *and their p-parents are the* CF-*groups* $A = D/P_4(D) \simeq \langle 2187, j\rangle$ *with* $j \in \{102, 111, 120\}$ *and types* a.1, b.16.

Proof. Using Theorem 14 for $e = 4$, we get:

▷ For $D = \langle 6561, 933|934\rangle$: $P_4(D) \simeq C_3$, and thus $A = D/P_4(D) = \langle 2187, 102\rangle$.

▷ For $D = \langle 6561, 953\rangle$: $P_4(D) \simeq C_3$, and thus $A = D/P_4(D) = \langle 2187, 111\rangle$.

▷ For $D = \langle 6561, 975|976\rangle$: $P_4(D) \simeq C_3$, and thus $A = D/P_4(D) = \langle 2187, 120\rangle$. □

There exist many other similar deterministic laws for groups which were not in the focus of the present paper, in contrast to those of Theorem 14. See the following appendix.

11. Group theoretic appendix

Arithmetical evaluation of the groups investigated in this appendix is difficult. The exposition is purely group theoretical. It supplements further interesting deterministic laws for CF-groups and BCF-groups (with moderate, or elevated, rank distribution).

Definition 3. By the *rank distribution* of a pro-3 group G with bicyclic commutator quotient G/G' we understand the punctured quartet

$$\varrho(G) \sim (\mathrm{rank}_3(H_i/H_i'))_{(G:H_i)=3}.$$

Theorem 15. *For each exponent* $e \geqslant 3$ *we assume that* $a+b = e+1$ *with integers* $0 \leqslant a-b \leqslant 1$. *Then:*

▷ *there are exactly two* BCF-*groups* D *with punctured transfer kernel type* B.7 *and Artin pattern* $\varkappa \sim (111; 4)$, $\alpha \sim ((e+1)1, (e+1)1, (e+1)1; (e-1)111)$*; their common p-parent is the* CF-*group* $A = D/P_e(D)$ *with type* b.15 *and Artin pattern* $\varkappa \sim (000; 4)$, $\alpha \sim (e1, e1, e1; (e-1)111)$*;*

▷ *there are exactly two* BCF-*groups* D *with punctured transfer kernel type* E.12 *and Artin pattern* $\varkappa \sim (123; 4)$, $\alpha \sim ((e+1)1, (e+1)1, (e+1)1; ab1)$*; their common p-parent is the* CF-*group* $A = D/P_e(D)$ *with type* b.15, *Artin pattern* $\varkappa \sim (000; 4)$, $\alpha \sim (e1, e1, e1; ab1)$.

These two cases together with the cases in Theorem 14 *are summarized in Table* 5. *The p-class of the* BCF-*groups* D *is always* $c_p(D) = e+1$ *and their order is* $\#D = 3^{4+e}$. *The commutator quotient of all groups* D *and* A *is* $(3^e, 3)$, *and they are tied together by the rank distribution* ϱ.

			BCF-group D	
#	pTKT	$\varkappa$	α	ϱ
2	B.7	$111;4$	$(e+1)1,(e+1)1,(e+1)1;(e-1)111$	$2,2,2;4$
2	E.12	$123;4$	$(e+1)1,(e+1)1,(e+1)1;ab1$	$2,2,2;3$
2	e.14	$123;0$	$(e+1)1,(e+1)1,(e+1)1;ab1$	$2,2,2;3$
1	d.10	$110;2$	$(e+1)1,(e+1)1,e11;e11$	$2,2,3;3$
2	D.11	$124;2$	$(e+1)1,(e+1)1,e11;e11$	$2,2,3;3$

	CF-group $A = D/P_e(D)$	
pTKT	$\varkappa$	α
b.15	$000;4$	$e1,e1,e1;(e-1)111$
b.15	$000;4$	$e1,e1,e1;ab1$
a.1	$000;0$	$e1,e1,e1;ab1$
a.1	$000;0$	$e1,e1,e11;(e-1)11$
b.16	$004;0$	$e1,e1,e11;(e-1)11$

Table 5. BCF-groups with moderate rank distribution ϱ.

Proof. We only have to justify the statements in the first two rows. Everything else has been proved in Theorem 14.

▷ $D = \langle x, y\colon\ x^{3^e} = w,\ y^3 = 1,\ t_3 = w$ or $w^2\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = 1,\ t_3 = 1\rangle$.

▷ $D = \langle x, y\colon\ x^{3^e} = w,\ y^3 = s_3,\ t_3 = w$, or $w^2\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = s_3,\ t_3 = 1\rangle$. □

Example 6. By specialization of Theorem 15 to $e = 3$, we obtain:

▷ For $D = \langle 2187, 84|85\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 4\rangle$.

▷ For $D = \langle 2187, 94|95\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 5\rangle$.

By specialization of Theorem 15 to $e = 4$, we get:

▷ For $D = \langle 6561, 876|877\rangle$: $P_4(D) \simeq C_3$, and thus $A = D/P_4(D) = \langle 2187, 83\rangle$.

▷ For $D = \langle 6561, 917|918\rangle$: $P_4(D) \simeq C_3$, and thus $A = D/P_4(D) = \langle 2187, 93\rangle$.

We call the construction method of BCF-groups with moderate rank distribution an *endo-genetic propagation* since the descendant D and the p-parent A share a common commutator quotient. The deeper reasons will be illuminated in [29].

Now we come to the exo-genetic propagation of CF-groups and BCF-groups with elevated rank distribution. For the latter, see also [27].

Theorem 16. *For each exponent $e \geqslant 3$ we assume that $a+b = e+1$ with integers $0 \leqslant a-b \leqslant 1$ and $c+d = e$ with integers $0 \leqslant c-d \leqslant 1$. Then Table 6 shows five* CF-*groups D with commutator quotient $(3^{e+1}, 3)$ and their p-parents A with commutator quotient $(3^e, 3)$. The p-class of the* CF-*groups D is always $c_p(D) = e+1$ and their* *order is $\#D = 3^{4+e}$. D and A are tied together by the punctured transfer kernel type $\varkappa$ and the rank distribution ϱ, but the second component α of the Artin pattern $(\varkappa, \alpha)$ is distinct.*

CF-group D				CF-group $A = D/P_e(D)$
α	ϱ	pTKT	$\varkappa$	α
$(e+1)1, (e+1)1, (e+1)1; e111$	$2,2,2;4$	b.15	$000;4$	$e1, e1, e1; (e-1)111$
$(e+1)1, (e+1)1, (e+1)1; ab1$	$2,2,2;3$	b.15	$000;4$	$e1, e1, e1; cd1$
$(e+1)1, (e+1)1, (e+1)1; ab1$	$2,2,2;3$	a.1	$000;0$	$e1, e1, e1; cd1$
$(e+1)1, (e+1)1, (e+1)11; e11$	$2,2,3;3$	a.1	$000;0$	$e1, e1, e11; (e-1)11$
$(e+1)1, (e+1)1, (e+1)11; e11$	$2,2,3;3$	b.16	$004;0$	$e1, e1, e11; (e-1)11$

Table 6. CF-groups and their rank distribution ϱ.

Proof. As before, we denote the main commutators by $s_2 = [y, x]$, $s_3 = [s_2, x]$, $t_3 = [s_2, y]$.

- $D = \langle x, y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w, y^3 = 1,\ t_3 = 1\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = 1,\ t_3 = 1\rangle$.
- $D = \langle x, y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w,\ y^3 = s_3, t_3 = 1\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = s_3,\ t_3 = 1\rangle$.
- $D = \langle x, y\colon\ x^{3^{e+1}} = 1, x^{3^e} = w,\ y^3 = s_3^2,\ t_3 = 1\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = s_3^2,\ t_3 = 1\rangle$.
- $D = \langle x, y\colon\ x^{3^{e+1}} = 1, x^{3^e} = w, y^3 = 1, t_3 = s_3\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = 1,\ t_3 = s_3\rangle$.
- $D = \langle x, y\colon\ x^{3^{e+1}} = 1, x^{3^e} = w,\ y^3 = s_3,\ t_3 = s_3\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x, y\colon\ x^{3^e} = 1,\ y^3 = s_3,\ t_3 = s_3\rangle$. □

Example 7. Using Theorem 16 for $e = 3$, we obtain:

- For $D = \langle 2187, 83\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 4\rangle$.
- For $D = \langle 2187, 93\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 5\rangle$.
- For $D = \langle 2187, 102\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 6\rangle$.
- For $D = \langle 2187, 111\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 7\rangle$.
- For $D = \langle 2187, 120\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 729, 8\rangle$.

Theorem 17. *For each exponent $e \geqslant 3$ we assume that $a + b = e + 2$ with integers $0 \leqslant a - b \leqslant 1$ and $c + d = e + 1$ with integers $0 \leqslant c - d \leqslant 1$. Then Table 7 shows four* BCF*-groups D with commutator quotient $(3^{e+1}, 3)$ and their p-parents A with commutator quotient $(3^e, 3)$. The p-class of the* BCF*-groups D is always $c_p(D) = e + 1$ and their order is $\#D = 3^{5+e}$. D and A are tied together by the punctured transfer kernel type $\varkappa$ and the rank distribution ϱ, but the second component α of the Artin pattern $(\varkappa, \alpha)$ is distinct.*

BCF-group D				BCF-group $A = D/P_e(D)$
α	ϱ	pTKT	$\varkappa$	α
$(e+1)11, (e+1)11, (e+1)11; e111$	$3,3,3;4$	b.15	$000;4$	$e11, e11, e11; (e-1)111$
$(e+1)11, (e+1)11, (e+1)11; ab1$	$3,3,3;3$	b.31	$044;4$	$e11, e11, e11; cd1$
$(e+1)11, (e+1)11, (e+1)11; ab1$	$3,3,3;3$	c.27	$044;0$	$e11, e11, e11; cd1$
$(e+1)11, (e+1)11, (e+1)11; e111$	$3,3,3;4$	A.20	$444;4$	$e11, e11, e11; (e-1)111$

Table 7. BCF-groups with elevated rank distribution ϱ.

Proof. Again, we put $s_2 = [y,x]$, $s_3 = [s_2,x]$, $t_3 = [s_2,y]$.

- ▷ $D = \langle x,y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w,\ y^3 = 1\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x,y\colon\ x^{3^e} = 1,\ y^3 = 1\rangle$.
- ▷ $D = \langle x,y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w,\ y^3 = s_3\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x,y\colon\ x^{3^e} = 1,\ y^3 = s_3\rangle$.
- ▷ $D = \langle x,y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w,\ y^3 = s_3^2\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x,y\colon\ x^{3^e} = 1,\ y^3 = s_3^2\rangle$.
- ▷ $D = \langle x,y\colon\ x^{3^{e+1}} = 1,\ x^{3^e} = w,\ y^3 = t_3\rangle$, $P_e(D) = \langle w\rangle \simeq C_3$, and thus $A = D/P_e(D) = \langle x,y\colon\ x^{3^e} = 1,\ y^3 = t_3\rangle$. □

Example 8. Using Theorem 17 for $e = 3$, we obtain:

- ▷ For $D = \langle 6561, 200\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 2187, 2\rangle$.
- ▷ For $D = \langle 6561, 216\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 2187, 3\rangle$.
- ▷ For $D = \langle 6561, 229\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 2187, 4\rangle$.
- ▷ For $D = \langle 6561, 242\rangle$: $P_3(D) \simeq C_3$, and thus $A = D/P_3(D) = \langle 2187, 5\rangle$.

12. Conclusion

Since the relevant 3-groups G in this paper are realized as Galois groups $G \simeq \mathrm{Gal}(\mathrm{F}_3^k(K)/K)$ of iterated unramified Hilbert 3-class fields ($2 \leqslant k \leqslant \infty$) of imaginary quadratic fields K, they must be σ-groups. Therefore, only every other branch of a descendant tree consists of admissible vertices. We say the groups on the first admissible branch are in the ground state and the groups on higher admissible branches are in excited states.

Using this terminology, we can easily point out the entirely different nature of the infinite limit groups in the articles [11], [25] and those in Sections 7 and 9 of the present paper.

- ▷ The limit groups in [11] and [25] admit the construction of all excited states with fixed commutator quotient ($(3,3)$ or $(9,3)$ in [11], Theorem 4.1, only $(3,3)$ in [25], Theorem 3.5).
- ▷ The limit groups in the present paper admit the construction of the ground state with varying commutator quotient ($(3^e,3)$ with $e \geqslant 5$ in Theorem 13).

Our theory in Section 7 is complete since groups with punctured transfer kernel type D.11 can be called sporadic (outside of coclass trees) and exist only in the ground state, namely as metabelian Schur σ-groups G. Since these groups are of class $\mathrm{Cl}(G) = 3$, the periodicity in Theorems 4–7 sets in with $e = 3$. With the aid of Theorem 14, the metabelian Schur σ-groups are embedded into more general deterministic laws, which express a remarkable relationship between BCF-groups (with moderate rank distribution) and CF-groups.

The theory in Section 9 is not complete, since Theorems 12 and 13 only deal with the ground state of groups with punctured transfer kernel types D.10, C.4, D.5, D.6, which are of class $\mathrm{Cl}(G) = 5$, whence periodicity sets in with $e = 5$. But these groups are periodic vertices of coclass trees.

In theorems concerning the first excited state of groups with punctured transfer kernel types D.10, C.4, D.5, D.6, which are of class $\mathrm{Cl}(G) = 7$, periodicity sets in with $e = 7$, see [28], and so on.

Figure 7 illuminates the broad range of non-metabelian Schur σ-groups G with growing commutator quotients G/G', beginning with $(3, 3)$ in [25] and $(9, 3)$ in Theorem 8, where descendants are still constructed with (usual) parents in coclass trees, over the increasingly irregular cases $(27, 3)$ in Theorem 9 and $(81, 3)$ in Theorem 10, up to the new periodicity for $(3^e, 3)$ with $e \geqslant 5$, which starts in Theorem 11 and continues in Theorem 12, where descendants are constructed with p-parents, the coclass trees begin to hide, and the bifurcations degenerate to simple descendant relations.

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