SOME APPLICATIONS OF SUBORDINATION THEOREMS ASSOCIATED WITH FRACTIONAL q-CALCULUS OPERATOR

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Abstract. Using the operator $\mathfrak{D}^m_{q,\varrho}(\lambda, l)$, we introduce the subclasses $\mathfrak{Y}^{*m}_{q,\varrho}(l,\lambda,\gamma)$ and $\mathfrak{K}^{*m}_{q,\varrho}(l,\lambda,\gamma)$ of normalized analytic functions. Among the results investigated for each of these function classes, we derive some subordination results involving the Hadamard product of the associated functions. The interesting consequences of some of these subordination results are also discussed. Also, we derive integral means results for these classes.

Keywords: analytic function; subordination principle; subordinating factor sequence; Hadamard product; q-difference operator; fractional q-calculus operator

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1. Introduction and preliminaries

Let $\mathcal{A}$ represent the class of functions analytic in $\mathbb{D} = \{z\colon z \in \mathbb{C} \text{ and } |z| < 1\}$ satisfying the normalized condition $f'(0) - 1 = f(0) = 0$. Each $f \in \mathcal{A}$ has the following Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa, \quad z \in \mathbb{D}. \tag{1.1}$$

Definition 1.1. Let $g(z) \in \mathcal{A}$ be defined by

$$g(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa,$$

and $f(z)$ be given by (1.1), the convolution or Hadamard product $(f \star g)$ is defined by

$$(f \star g)(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa = (g \star f)(z).$$

Note that $(f \star g)$ is analytic and univalent in the open disc $\mathbb{D}$.

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In the recent years, practical applications of q-calculus (quantum calculus) in the fields of q-difference equation, optimal control, q-transform analysis and number theory are an efficient area of research. Jackson (see [13], [14]) was the first to successfully develop q-integral and q-derivative in a systematic way and later geometrical interpretation of the q-analysis has been recognized through studies of quantum groups.

Fractional calculus appears more and more frequently for the modelling of relevant systems in several fields of applied sciences. Fractional q-calculus is the q-extension of ordinary fractional calculus. Researchers have claimed to construct and investigated several classes of analytic and bi-univalent functions and their interesting results are extremely numerous to discuss.

We initially present various definitions and notations in q-calculus which are useful to interpret the subject of this paper.

Definition 1.2. For $q \in (0,1)$, the q-number n is given by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & n \in \mathbb{C}, \\ \sum_{\kappa=0}^{n-1} q^\kappa = 1 + q + q^2 + \ldots + q^{n-1}, & n \in \mathbb{N} = \{1, 2, \ldots\}, \\ 0, & n = 0 \end{cases} \tag{1.2}$$

and

$$\lim_{q \to 1^-} [n]_q = n.$$

Definition 1.3 ([24]). For $\nu \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the q-shifted factorial is defined by

$$(\nu; q)_0 = 1, \quad (\nu; q)_n = \prod_{\kappa=0}^{n-1} (1 - \nu q^\kappa), \tag{1.3}$$

and in terms of the basic (or q-) gamma function

$$(q^\nu; q)_n = \frac{(1-q)^n \Gamma_q(\nu + n)}{\Gamma_q(\nu)}, \quad n \in \mathbb{N}_0,$$

where the q-gamma function is defined by

$$\Gamma_q(z) = \frac{(1-q)^{1-z} (q; q)_\infty}{(q^z; q)_\infty}, \quad |q| < 1.$$

We note that

$$(\nu; q)_\infty = \prod_{\kappa=0}^{\infty} (1 - \nu q^\kappa), \quad |q| < 1.$$

For the q-gamma function $\Gamma_q(z)$, it is known that (see [10])

$$\Gamma_q(z+1) = [z]_q \Gamma_q(z),$$

where $[z]_q$ denotes by (1.2). It is also known, in terms of the classical gamma function $\Gamma(z)$, that

$$\lim_{q\to 1^-} \Gamma_q(z) = \Gamma(z).$$

Definition 1.4. Jackson in [13] defined the q-derivative of a function $f(z)$ of the form (1.1) as

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} = 1 + \sum_{\kappa=2}^{\infty} [\kappa]_q a_\kappa z^{\kappa-1}, \quad z \neq 0, \tag{1.4}$$

where $[\kappa]_q$ given by (1.2) and

$$\lim_{q\to 1^-} D_q f(z) = f'(z).$$

A q-analog of the class of starlike functions was first introduced by Ismail et al. in [12] by means of the q-difference operator $D_q f(z)$, $f(z) \in \mathcal{A}$ and $0 < q < 1$.

Definition 1.5. Jackson in [13] introduced the q-integral of a function $f(z)$ of the form (1.1) as

$$\int_0^z f(t)\, \mathrm{d}_q t = z(1-q) \sum_{\kappa=0}^{\infty} q^\kappa f(zq^\kappa),$$

provided that the series converges and

$$\lim_{q\to 1^-} \int_0^z f(t)\, \mathrm{d}_q t = \int_0^z f(t)\, \mathrm{d}t,$$

where $\int_0^z f(t)\, \mathrm{d}t$ is the ordinary integral.

Definition 1.6 (Fractional q-integral operator, see [19], page 57, Definition 1)**.** The fractional q-integral operator $J^{\varrho}_{q,z}$ of a function $f(z)$ of order ϱ is defined by (see also [1], page 257, Definition 1.1)

$$J^{\varrho}_{q,z} f(z) = D^{-\varrho}_{q,z} f(z) = \frac{1}{\Gamma_q(\varrho)} \int_0^z (z - qt)_{\varrho-1} f(t)\, \mathrm{d}_q t, \quad \varrho > 0, \tag{1.5}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z-qt)_{\varrho-1}$ is given by

$$(z-qt)_{\varrho-1} = z^{\varrho-1} \prod_{\kappa=0}^{\infty} \Big(\frac{1-(qt/z)q^\kappa}{1-(qt/z)q^{\varrho+\kappa-1}} \Big) = z^{\varrho-1} {}_1\Psi_0 \Big[q^{-\varrho+1}; -; q; \frac{tq^\varrho}{z} \Big].$$

The series ${}_1\Psi_0[\varrho;-;q,z]$ is single valued when $|\arg(z)| < \pi$ and $|z| < 1$ (for details see [10], pages 104–106). Therefore, the function $(z-qt)_{\varrho-1}$ in (1.5) is single valued when $|\arg(-q^{\varrho}t/z)| < \pi$, $|q^{\varrho}t/z| < 1$ and $|\arg(z)| < \pi$.

Definition 1.7 (Fractional q-derivative operator, see [19], page 58, Definition 2)**.** The fractional q-derivative operator $D^{\varrho}_{q,z}$ of a function $f(z)$ of order ϱ is defined by (see also [1], page 257, Definition 1.2)

$$D^{\varrho}_{q,z}f(z) = D_{q,z}J^{1-\varrho}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\varrho)}D_q\int_0^z (z-qt)_{-\varrho}f(t)\,\mathrm{d}_qt, \quad 0 \leqslant \varrho < 1,$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\varrho}$ is removed as in Definition 1.6.

Definition 1.8 (Extended fractional q-derivative operator, see [19], page 58, Definition 3)**.** The fractional q-derivative operator $D^{\varrho}_{q,z}$ of a function $f(z)$ of order ϱ is defined by

$$D^{\varrho}_{q,z}f(z) = D^{n}_{q,z}J^{n-\varrho}_{q,z}f(z)$$

where $n-1 \leqslant \varrho < n$, $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$.

Purohit and Raina (see [19], page 59) with $n=1$ defined a fractional q-differ-integral operator $\Upsilon^{\varrho}_{q,z}f(z)\colon \mathcal{A} \to \mathcal{A}$ as

$$\begin{aligned}\Omega^{\varrho}_{q,z}f(z) &= \frac{\Gamma_q(2-\varrho)}{\Gamma_q(2)}z^{\varrho-1}D^{\varrho}_{q,z}f(z)\\ &= 1+\sum_{\kappa=2}^{\infty}\frac{\Gamma_q(2-\varrho)\Gamma_q(\kappa+1)}{\Gamma_q(2)\Gamma_q(\kappa+1-\varrho)}a_\kappa z^{\kappa-1}, \quad \varrho < 2;\ 0<q<1;\ z \in \mathbb{D}.\end{aligned}$$

We note that the function $\Upsilon^{\varrho}_{q,z}f(z)$ is defined by

$$\Upsilon^{\varrho}_{q,z}f(z) = z\Omega^{\varrho}_{q,z}f(z) \tag{1.6}$$

and $\Upsilon^{0}_{q,z}f(z) = f(z)$.

Now we define a linear multiplier fractional q-differ-integral operator $\mathfrak{D}^{m}_{q,\varrho}(\lambda,l)$ as

$$\begin{aligned}&\mathfrak{D}^{0}_{q,\varrho}(\lambda,l)f(z) = f(z),\\ &\mathfrak{D}^{1}_{q,\varrho}(\lambda,l)f(z) = \Big(1-\frac{\lambda}{l+1}\Big)\Upsilon^{\varrho}_{q,z}f(z)+\frac{\lambda}{l+1}zD_q(\Upsilon^{\varrho}_{q,z}f(z)) = \mathfrak{D}_{q,\varrho}(\lambda,l)f(z)\\ &\qquad = z+\sum_{\kappa=2}^{\infty}\Big(\frac{\Gamma_q(2-\varrho)\Gamma_q(\kappa+1)}{\Gamma_q(2)\Gamma_q(\kappa+1-\varrho)}\frac{l+1+\lambda([\kappa]_q-1)}{l+1}\Big)a_\kappa z^\kappa,\\ &\qquad\ \vdots\end{aligned}$$

and

$$\mathfrak{D}^{m}_{q,\varrho}(\lambda,l)f(z) = \mathfrak{D}_{q,\varrho}(\lambda,l)(\mathfrak{D}^{m-1}_{q,\varrho}(\lambda,l)f(z)), \tag{1.7}$$

where $\lambda \geqslant 0$, $l > -1$, $m \in \mathbb{N}_0$, $\varrho < 2$ and $0 < q < 1$. It follows from (1.1) and (1.7) that

$$\mathfrak{D}^m_{q,\varrho}(\lambda, l)f(z) = z + \sum_{\kappa=2}^{\infty} \Theta^{m,\lambda,l}_{q,\varrho}(\kappa) a_\kappa z^\kappa \tag{1.8}$$

where

$$\Theta^{m,\lambda,l}_{q,\varrho}(\kappa) = \Big(\frac{\Gamma_q(2-\varrho)\Gamma_q(\kappa+1)}{\Gamma_q(2)\Gamma_q(\kappa+1-\varrho)} \frac{l+1+\lambda([\kappa]_q - 1)}{l+1}\Big)^m.$$

By virtue of (1.6) and (1.8), $\mathfrak{D}^m_{q,\varrho}(\lambda, l)f(z)$ can be written in terms of convolution as

$$\mathfrak{D}^m_{q,\varrho}(\lambda, l)f(z) = \underbrace{[(\Upsilon^\varrho_{q,z}(z) \star \mathfrak{G}^q_{l,\lambda}(z)) \star \ldots \star (\Upsilon^\varrho_{q,z}(z) \star \mathfrak{G}^q_{l,\lambda}(z))]}_{m\text{-times}} \star f(z)$$

where

$$\mathfrak{G}^q_{l,\lambda}(z) = \frac{z - (1 - \lambda/(1+l))qz^2}{(1-z)(1-qz)}.$$

Remark 1.1. Note that the operator $\mathfrak{D}^m_{q,\varrho}(\lambda, l)$ generalizes several previously studied familiar operators, and we mention some of the interesting particular cases as

(i) For $l = 0$ we obtain the operator $\mathcal{D}^{\varrho,m}_{\lambda,q}$ studied by Abelman et al. (see [1]);

(ii) For $l = 0$ and $\varrho = 0$ we obtain the operator $D^m_{\lambda,q}$ studied by Aouf et al. (see [4]);

(iii) For $l = 0$, $\varrho = 0$ and $\lambda = 1$ we obtain the operator $\mathcal{S}^m_q$ studied by Govindaraj and Sivasubramanian (see [11]);

(iv) For $q \to 1^-$ we obtain the operator $D^{\varrho,m}_{l,\lambda}$ studied by El-Ashwah et al. with $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 2 - \varrho$ (see [9]);

(v) For $q \to 1^-$ and $\varrho = 0$ we obtain the operator $D^m_{l,\lambda}$ studied by Cătas (see [6]);

(vi) For $q \to 1^-$ and $l = 0$ we obtain the operator $D^{\varrho,m}_\lambda$ studied by Al-Oboudi and Al-Amoudi (see [3]);

(vii) For $q \to 1^-$, $\varrho = 0$ and $\lambda = 1$ we obtain the operator I^m_l, $l \geqslant 0$, studied by Cho and Srivastava (see [7]);

(viii) For $q \to 1^-$, $\varrho = 0$ and $l = 0$ we obtain the operator D^m_λ studied by Al-Oboudi (see [2]);

(ix) For $q \to 1^-$, $\varrho = 0$, $\lambda = 1$ and $l = 0$ we obtain the operator D^m studied by Sălăgean (see [20]);

(x) For $q \to 1^-$, $l = \lambda = 0$ and $m = 1$ we obtain the operator D_ϱ studied by Owa and Srivastava (see [18]).

Making use of the linear multiplier fractional q-differ-integral operator given by (1.8), we introduce the subclass $\mathfrak{Y}^m_{q,\varrho}(l, \lambda, \gamma)$ of q-starlike functions of order γ in $\mathbb{D}$ and the subclass $\mathfrak{K}^m_{q,\varrho}(l, \lambda, \gamma)$ of q-convex functions of order γ in $\mathbb{D}$ as

$$\mathrm{Re}\Big(\frac{zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda, l)f(z))}{\mathfrak{D}^m_{q,\varrho}(\lambda, l)f(z)}\Big) < \gamma, \quad 0 < q < 1,\ \gamma > 1,\ 0 \leqslant \varrho < 2, \tag{1.9}$$

or, equivalently

$$\Big|\Big(\frac{zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))}{\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)}-1\Big)\Big(\frac{zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))}{\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)}-(2\gamma-1)\Big)^{-1}\Big|<1$$

and

$$\operatorname{Re}\Big(\frac{D_q(zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)))}{D_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))}\Big)<\gamma,\quad 0<q<1,\ \gamma>1,\ 0\leqslant\varrho<2, \tag{1.10}$$

or, equivalently

$$\Big|\Big(\frac{D_q(zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)))}{D_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))}-1\Big)\Big(\frac{D_q(zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)))}{D_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))}-(2\gamma-1)\Big)^{-1}\Big|<1,$$

respectively, where $\lambda\geqslant 0$, $l>-1$, $m\in\mathbb{N}_0$ and $f(z)\in\mathcal{A}$. From (1.9) and (1.10), it follows that $\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z)\in\mathfrak{K}^m_{q,\varrho}(l,\lambda,\gamma)\Leftrightarrow zD_q(\mathfrak{D}^m_{q,\varrho}(\lambda,l)f(z))\in\mathfrak{Y}^m_{q,\varrho}(l,\lambda,\gamma)$.

We note that:

(i) $\mathfrak{Y}^1_{q,\varrho}(0,0,\gamma)=\mathfrak{Y}_{q,\varrho}(\gamma)$ and $\mathfrak{K}^1_{q,\varrho}(0,0,\gamma)=\mathfrak{K}_{q,\varrho}(\gamma)$ if

$$\operatorname{Re}\Big(\frac{zD_q(\mathfrak{D}_{q,\varrho}f(z))}{\mathfrak{D}_{q,\varrho}f(z)}\Big)<\gamma,\quad 0<q<1,\ \gamma>1,\ 0\leqslant\varrho<2,$$

or, equivalently

$$\Big|\Big(\frac{zD_q(\mathfrak{D}_{q,\varrho}f(z))}{\mathfrak{D}_{q,\varrho}f(z)}-1\Big)\Big(\frac{zD_q(\mathfrak{D}_{q,\varrho}f(z))}{\mathfrak{D}_{q,\varrho}f(z)}-(2\gamma-1)\Big)^{-1}\Big|<1$$

and

$$\operatorname{Re}\Big(\frac{D_q(zD_q(\mathfrak{D}_{q,\varrho}f(z)))}{D_q(\mathfrak{D}_{q,\varrho}f(z))}\Big)<\gamma,\quad 0<q<1,\ \gamma>1,\ 0\leqslant\varrho<2,$$

or, equivalently

$$\Big|\Big(\frac{D_q(zD_q(\mathfrak{D}_{q,\varrho}f(z)))}{D_q(\mathfrak{D}_{q,\varrho}f(z))}-1\Big)\Big(\frac{D_q(zD_q(\mathfrak{D}_{q,\varrho}f(z)))}{D_q(\mathfrak{D}_{q,\varrho}f(z))}-(2\gamma-1)\Big)^{-1}\Big|<1,$$

respectively;

(ii) $\mathfrak{Y}^m_{q,0}(l,\lambda,\gamma)=\mathfrak{Y}^m_q(l,\lambda,\gamma)$ and $\mathfrak{K}^m_{q,0}(l,\lambda,\gamma)=\mathfrak{K}^m_q(l,\lambda,\gamma)$ if

$$\operatorname{Re}\Big(\frac{zD_q(\mathfrak{D}^m_q(\lambda,l)f(z))}{\mathfrak{D}^m_q(\lambda,l)f(z)}\Big)<\gamma,\quad 0<q<1,\ \gamma>1,$$

or, equivalently

$$\Big|\Big(\frac{zD_q(\mathfrak{D}^m_q(\lambda,l)f(z))}{\mathfrak{D}^m_q(\lambda,l)f(z)}-1\Big)\Big(\frac{zD_q(\mathfrak{D}^m_q(\lambda,l)f(z))}{\mathfrak{D}^m_q(\lambda,l)f(z)}-(2\gamma-1)\Big)^{-1}\Big|<1$$

and

$$\operatorname{Re}\Big(\frac{D_q(zD_q(\mathfrak{D}_q^m(\lambda,l)f(z)))}{D_q(\mathfrak{D}_q^m(\lambda,l)f(z))}\Big)<\gamma,\quad 0<q<1,\ \gamma>1,$$

or, equivalently

$$\Big|\Big(\frac{D_q(zD_q(\mathfrak{D}_q^m(\lambda,l)f(z)))}{D_q(\mathfrak{D}_q^m(\lambda,l)f(z))}-1\Big)\Big(\frac{D_q(zD_q(\mathfrak{D}_q^m(\lambda,l)f(z)))}{D_q(\mathfrak{D}_q^m(\lambda,l)f(z))}-(2\gamma-1)\Big)^{-1}\Big|<1,$$

respectively, where $\lambda\geqslant 0$, $l>-1$ and $m\in\mathbb{N}_0$;

(iii) $\mathfrak{Y}_{q,0}^m(0,\lambda,\gamma)=E_q(m,\lambda,\gamma)$ and $\mathfrak{K}_{q,0}^m(0,\lambda,\gamma)=G_q(m,\lambda,\gamma)$ (see Aouf et al. [4]);

(iv) $\lim\limits_{q\to 1^-}\mathfrak{Y}_{q,\varrho}^1(0,0,\gamma)=\mathfrak{Y}_\varrho(\gamma)$ and $\lim\limits_{q\to 1^-}\mathfrak{K}_{q,\varrho}^1(0,0,\gamma)=\mathfrak{K}_\varrho(\gamma)$ if

$$\operatorname{Re}\Big(\frac{z(\mathfrak{D}_\varrho f(z))'}{\mathfrak{D}_\varrho f(z)}\Big)<\gamma,\quad \gamma>1,\ 0\leqslant\varrho<2,$$

or, equivalently

$$\Big|\Big(\frac{z(\mathfrak{D}_\varrho f(z))'}{\mathfrak{D}_\varrho f(z)}-1\Big)\Big(\frac{z(\mathfrak{D}_\varrho f(z))'}{\mathfrak{D}_\varrho f(z)}-(2\gamma-1)\Big)^{-1}\Big|<1$$

and

$$\operatorname{Re}\Big(1+\frac{z(\mathfrak{D}_\varrho f(z))''}{(\mathfrak{D}_\varrho f(z))'}\Big)<\gamma,\quad \gamma>1,\ 0\leqslant\varrho<2,$$

or, equivalently

$$\Big|\Big(\frac{z(\mathfrak{D}_\varrho f(z))''}{(\mathfrak{D}_\varrho f(z))'}\Big)\Big(\frac{z(\mathfrak{D}_\varrho f(z))''}{(\mathfrak{D}_\varrho f(z))'}-2(\gamma-1)\Big)^{-1}\Big|<1,$$

respectively;

(v) $\lim\limits_{q\to 1^-}\mathfrak{Y}_{q,0}^m(l,\lambda,\gamma)=\mathfrak{Y}^m(l,\lambda,\gamma)$ and $\lim\limits_{q\to 1^-}\mathfrak{K}_{q,0}^m(l,\lambda,\gamma)=\mathfrak{K}^m(l,\lambda,\gamma)$ if

$$\operatorname{Re}\Big(\frac{z(\mathfrak{D}^m(\lambda,l)f(z))'}{\mathfrak{D}^m(\lambda,l)f(z)}\Big)<\gamma,\quad \gamma>1,\ \lambda\geqslant 0,\ l>-1,\ m\in\mathbb{N}_0,$$

or, equivalently

$$\Big|\Big(\frac{z(\mathfrak{D}^m(\lambda,l)f(z))'}{\mathfrak{D}^m(\lambda,l)f(z)}-1\Big)\Big(\frac{z(\mathfrak{D}^m(\lambda,l)f(z))'}{\mathfrak{D}^m(\lambda,l)f(z)}-(2\gamma-1)\Big)^{-1}\Big|<1$$

and

$$\operatorname{Re}\Big(1+\frac{z(\mathfrak{D}^m(\lambda,l)f(z))''}{(\mathfrak{D}^m(\lambda,l)f(z))'}\Big)<\gamma,\quad \gamma>1,\ \lambda\geqslant 0,\ l>-1,\ m\in\mathbb{N}_0,$$

or, equivalently

$$\Big|\Big(\frac{z(\mathfrak{D}^m(\lambda,l)f(z))''}{(\mathfrak{D}^m(\lambda,l)f(z))'}\Big)\Big(\frac{z(\mathfrak{D}^m(\lambda,l)f(z))''}{(\mathfrak{D}^m(\lambda,l)f(z))'}-2(\gamma-1)\Big)^{-1}\Big|<1,$$

respectively;

(vi) $\lim_{q\to 1^-} \mathfrak{Y}^0_{q,\varrho}(l,\lambda,\gamma) = M(\gamma)$ and $\lim_{q\to 1^-} \mathfrak{K}^0_{q,\varrho}(l,\lambda,\gamma) = N(\gamma)$, $\gamma > 1$ (see Nishiwaki and Owa [16]);

(vii) $\lim_{q\to 1^-} \mathfrak{Y}^0_{q,\varrho}(l,\lambda,\gamma) = M(\gamma)$ and $\lim_{q\to 1^-} \mathfrak{K}^0_{q,\varrho}(l,\lambda,\gamma) = N(\gamma)$, $1 < \gamma \leqslant \frac{4}{3}$ (see Uralegaddi et al. [26]).

Definition 1.9 (Subordination Principle, see [8], Chapter 6, page 190). For two functions f and g, analytic in $\mathbb{D}$, we say that the function f is subordinate to g in $\mathbb{D}$, and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in \mathbb{D},$$

if there exists a Schwarz function $\varphi(z)$ analytic in $\mathbb{D}$ with

$$\varphi(0) = 0 \quad \text{and} \quad |\varphi(z)| < 1, \quad z \in \mathbb{D},$$

such that

$$f(z) = g(\varphi(z)), \quad z \in \mathbb{D}.$$

In particular, if the function $g(z)$ is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Definition 1.10 (Subordinating Factor Sequence). An infinite sequence $\{c_\kappa\}_{\kappa=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in $\mathbb{D}$, we have the subordination given by

$$\sum_{\kappa=1}^{\infty} a_\kappa c_\kappa z^\kappa \prec f(z), \quad z \in \mathbb{D};\ a_1 = 1. \tag{1.11}$$

A finite sequence $\{c_\kappa\}_{\kappa=1}^{n}$ is said to be a subordinating factor sequence if (1.1) implies (1.11) whenever $c_{n+1} = c_{n+2} = \ldots = 0$. The class of such infinite sequences will be denoted by $\mathcal{F}$, and the class of sequences of length n by $\mathcal{F}_n$.

Lemma 1.1 ([27], page 690, Theorem 2). *The sequence $\{c_\kappa\}_{\kappa=1}^{\infty}$ of complex numbers is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left(1 + 2\sum_{\kappa=1}^{\infty} c_\kappa z^\kappa\right) > 0, \quad z \in \mathbb{D}.$$

Following the technique of Owa and Nishiwaki (see [17]), we can obtain the following lemmas:

Lemma 1.2. *If $f(z)$ satisfies the coefficient inequality*

$$\sum_{\kappa=2}^{\infty}(([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(\kappa)|a_\kappa|\leqslant 2(\gamma-1), \tag{1.12}$$

$$\lambda\geqslant 0,\ l>-1,\ \gamma>1,\ m\in\mathbb{N}_0,\ 0<q<1,\ \varrho<2,$$

then $f(z)\in\mathfrak{Y}_{q,\varrho}^{m}(l,\lambda,\gamma)$.

Lemma 1.3. *If $f(z)$ satisfies the coefficient inequality*

$$\sum_{\kappa=2}^{\infty}[\kappa]_q(([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(\kappa)|a_\kappa|\leqslant 2(\gamma-1), \tag{1.13}$$

$$\lambda\geqslant 0,\ l>-1,\ \gamma>1,\ m\in\mathbb{N}_0,\ 0<q<1,\ \varrho<2,$$

then $f(z)\in\mathfrak{K}_{q,\varrho}^{m}(l,\lambda,\gamma)$.

In view of Lemma 1.2 and Lemma 1.3, we define the subclasses $\mathfrak{Y}_{q,\varrho}^{*m}(l,\lambda,\gamma)\subset\mathfrak{Y}_{q,\varrho}^{m}(l,\lambda,\gamma)$ and $\mathfrak{K}_{q,\varrho}^{*m}(l,\lambda,\gamma)\subset\mathfrak{K}_{q,\varrho}^{m}(l,\lambda,\gamma)$ which consist of functions $f\in\mathcal{A}$ whose coefficients satisfy the inequality (1.12) and (1.13), respectively.

Here we investigate some subordination results for the functions in the classes $\mathfrak{Y}_{q,\varrho}^{*m}(l,\lambda,\gamma)$ and $\mathfrak{K}_{q,\varrho}^{*m}(l,\lambda,\gamma)$ employing the technique used earlier by Attiya (see [5]); Srivastava and Attiya (see [25]). Also, we derive integral means results for these classes.

2. Main results

Theorem 2.1. *Let $f(z)\in\mathfrak{Y}_{q,\varrho}^{*m}(l,\lambda,\gamma)$ and let $\mathcal{K}$ be the familiar class of functions belong to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))}(f\star g)(z)\prec g(z), \tag{2.1}$$

$$\lambda\geqslant 0,\ l>-1,\ \gamma>1,\ m\in\mathbb{N}_0,\ 0<q<1,\ 0\leqslant\varrho<2$$

for every function $g\in\mathcal{K}$. Further,

$$\operatorname{Re}(f(z))>-\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)},\quad z\in\mathbb{D}. \tag{2.2}$$

The constant factor

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))}$$

is the best estimate.

P r o o f. Let $f(z) \in \mathfrak{Y}^{*m}_{q,\varrho}(l,\lambda,\gamma)$, and suppose that

$$g(z) = z + \sum_{\kappa=2}^{\infty} c_\kappa z^\kappa \in \mathcal{K}.$$

Then

$$\begin{aligned}
&\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}(f\star g)(z)\\
&\quad= \frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}\Big(z+\sum_{\kappa=2}^{\infty} a_\kappa c_\kappa z^\kappa\Big)
\end{aligned}$$

By Definition 1.10, the subordination result holds true if

$$\Big\{\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}a_\kappa\Big\}_{\kappa=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 1.1, this is equivalent to the next inequality:

(2.3)
$$\operatorname{Re}\Big(1+\sum_{\kappa=1}^{\infty}\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}a_\kappa z^\kappa\Big) > 0, \quad z\in\mathbb{D}.$$

Since $\Phi(\kappa) = (([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)$ is an increasing function of κ ($\kappa > 2$), we have, for $|z| = r < 1$

$$\begin{aligned}
(2.4)\qquad &\operatorname{Re}\Big(1+\sum_{\kappa=1}^{\infty}\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}a_\kappa z^\kappa\Big)\\
&= \operatorname{Re}\Big(1+\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}z\\
&\quad+\sum_{\kappa=2}^{\infty}\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}a_\kappa z^\kappa\Big)\\
&\geqslant 1-\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&\quad-\sum_{\kappa=2}^{\infty}\frac{(([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}|a_\kappa|r^\kappa\\
&> 1-\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&\quad-\frac{2(\gamma-1)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&> 0
\end{aligned}$$

where we use the assertion (1.12) of Lemma 1.2. This evidently proves the inequality (2.3) and hence also the subordination result (2.1) asserted by Theorem 2.1. The inequality (2.2) follows from (2.1) by taking

$$g(z)=\frac{z}{1-z}=z+\sum_{\kappa=2}^{\infty}z^{\kappa}\in\mathcal{K}. \tag{2.5}$$

To prove the sharpness of the constant

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))},$$

we consider the function $\mathcal{H}\in\mathfrak{Y}_{q,\varrho}^{*m}(l,\lambda,\gamma)$ defined by

$$\mathcal{H}(z)=z-\frac{2(\gamma-1)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}z^2.$$

Thus, from (2.1), we have

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))}\mathcal{H}(z)\prec\frac{z}{1-z},\quad z\in\mathbb{D}.$$

It is easily verified that

$$\min_{|z|=r\leqslant 1}\Big\{\mathrm{Re}\Big(\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))}\mathcal{H}(z)\Big)\Big\}=-\frac{1}{2}.$$

This shows that the constant

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)+2(\gamma-1))}$$

cannot be replaced by any larger one. The proof of Theorem 2.1 is completed. □

Putting $l=\lambda=0$ and $m=1$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let $f(z)\in\mathfrak{Y}_{q,\varrho}^{*}(\gamma)$ and let $\mathcal{K}$ be the familiar class of functions belonging to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)+2(\gamma-1))}(f\star g)(z)\prec g(z),$$

$\gamma>1$, $0<q<1$, $0\leqslant\varrho<2$ for every function $g\in\mathcal{K}$. Further,

$$\mathrm{Re}(f(z))>-\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)+2(\gamma-1)}{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)},\quad z\in\mathbb{D}.$$

The constant factor

$$\frac{(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)}{2((([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)+2(\gamma-1))}$$

is the best estimate.

Putting $\varrho = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let $f(z) \in \mathfrak{Y}_q^{*m}(l, \lambda, \gamma)$ and let $\mathcal{K}$ be the familiar class of functions belonging to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2)}{2((([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2) + 2(\gamma - 1))}(f \star g)(z) \prec g(z),$$

$\lambda \geqslant 0$, $l > -1$, $\gamma > 1$, $m \in \mathbb{N}_0$, $0 < q < 1$ for every function $g \in \mathcal{K}$. Further,

$$\operatorname{Re}(f(z)) > -\frac{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2) + 2(\gamma - 1)}{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2)}, \quad z \in \mathbb{D}.$$

The constant factor

$$\frac{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2)}{2((([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_q^{m,\lambda,l}(2) + 2(\gamma - 1))}$$

is the best estimate.

Remark 2.1.

- ▷ Putting $q \to 1^-$ and $m = 0$ in Theorem 2.1, we obtain the result which was obtained by Srivastava and Attiya in [25], page 3, Theorem 1, with $\lambda = 1$;
- ▷ Putting $q \to 1^-$, $m = 0$ and $1 < \gamma \leqslant \frac{3}{2}$ in Theorem 2.1, we obtain the result which was obtained by Srivastava and Attiya, see [25], page 5, Corollary 2.

Theorem 2.2. *Let $f(z) \in \mathfrak{K}_{q,\varrho}^{*m}(l, \lambda, \gamma)$ and let $\mathcal{K}$ be the familiar class of functions belonging to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))}(f \star g)(z) \prec g(z), \tag{2.6}$$
$$\lambda \geqslant 0,\ l > -1,\ \gamma > 1,\ m \in \mathbb{N}_0,\ 0 < q < 1,\ 0 \leqslant \varrho < 2$$

for every function $g \in \mathcal{K}$. Further,

$$\operatorname{Re}(f(z)) > -\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1)}{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}, \quad z \in \mathbb{D}. \tag{2.7}$$

The constant factor

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))}$$

is the best estimate.

Proof. Let $f(z) \in \mathfrak{K}^{*m}_{q,\varrho}(l,\lambda,\gamma)$, and suppose that

$$g(z) = z + \sum_{\kappa=2}^{\infty} c_\kappa z^\kappa \in \mathcal{K}.$$

Then

$$\begin{aligned}&\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}(f\star g)(z)\\&\quad=\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}\Big(z+\sum_{\kappa=2}^{\infty}a_\kappa c_\kappa z^\kappa\Big).\end{aligned}$$

By Definition 1.10, the subordination result holds true if

$$\Big\{\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}a_\kappa\Big\}_{\kappa=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 1.1, this is equivalent to the next inequality:

(2.8)

$$\operatorname{Re}\Big(1+\sum_{\kappa=1}^{\infty}\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1))}a_\kappa z^\kappa\Big) > 0,\ z\in\mathbb{D}.$$

Since $\Phi(\kappa) = [\kappa]_q(([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)$ is an increasing function of κ ($\kappa > 2$), we have, for $|z| = r < 1$

$$\begin{aligned}(2.9)\quad &\operatorname{Re}\Big(1+\sum_{\kappa=1}^{\infty}\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}a_\kappa z^\kappa\Big)\\
&=\operatorname{Re}\Big(1+\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}z\\
&\quad+\sum_{\kappa=2}^{\infty}\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}a_\kappa z^\kappa\Big)\\
&\geqslant 1-\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&\quad-\sum_{\kappa=2}^{\infty}\frac{[\kappa]_q(([\kappa]_q-1)+|[\kappa]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}|a_\kappa|r^\kappa\\
&> 1-\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&\quad-\frac{2(\gamma-1)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)+2(\gamma-1)}r\\
&> 0\end{aligned}$$

where we use the assertion (1.13) of Lemma 1.3. This evidently proves the inequality (2.8) and hence also the subordination result (2.6) asserted by Theorem 2.2. The inequality (2.7) follows from (2.6) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{\kappa=2}^{\infty} z^{\kappa} \in \mathcal{K}. \tag{2.10}$$

To prove the sharpness of the constant

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))},$$

we consider the function $\mathcal{G} \in \mathfrak{K}_q^{*m}(l, \lambda, \gamma)$ defined by

$$\mathcal{G}(z) = z - \frac{2(\gamma - 1)}{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)} z^2.$$

Thus, from (2.6), we have

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))}\mathcal{G}(z) \prec \frac{z}{1-z}, \quad z \in \mathbb{D}.$$

It is easily verified that

$$\min\left\{\mathrm{Re}\Big(\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))}\mathcal{G}(z)\Big)\right\} = -\frac{1}{2}.$$

This shows that the constant

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}^{m,\lambda,l}(2) + 2(\gamma - 1))}$$

cannot be replaced by any larger one. The proof of Theorem 2.2 is completed. □

Putting $l = \lambda = 0$ and $m = 1$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.3. *Let $f(z) \in \mathfrak{K}_{q,\varrho}^{*}(\gamma)$ and let $\mathcal{K}$ be the familiar class of functions belong to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}(2)}{2([2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}(2) + 2(\gamma - 1))}(f \star g)(z) \prec g(z),$$

$\gamma > 1$, $0 < q < 1$, $0 \leqslant \varrho < 2$ *for every function* $g \in \mathcal{K}$. *Further,*

$$\mathrm{Re}(f(z)) > -\frac{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}(2) + 2(\gamma - 1)}{[2]_q(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta_{q,\varrho}(2)}, \quad z \in \mathbb{D}.$$

The constant factor

$$\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_{q,\varrho}(2)+2(\gamma-1))}$$

is the best estimate.

Putting $\varrho=0$ in Theorem 2.2, we obtain the following corollary:

Corollary 2.4. *Let $f(z)\in\mathfrak{K}_q^{*m}(l,\lambda,\gamma)$ and let $\mathcal{K}$ be the familiar class of functions belong to $\mathcal{A}$ which are univalent and convex in $\mathbb{D}$. Then*

$$\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)+2(\gamma-1))}(f\star g)(z)\prec g(z),$$
$$\lambda\geqslant 0,\ l>-1,\ \gamma>1,\ m\in\mathbb{N}_0,\ 0<q<1$$

for every function $g\in\mathcal{K}$. Further,

$$\operatorname{Re}(f(z))>-\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)+2(\gamma-1)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)},\quad z\in\mathbb{D}.$$

The constant factor

$$\frac{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)}{2([2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta_q^{m,\lambda,l}(2)+2(\gamma-1))}$$

is the best estimate.

Remark 2.2.

- ▷ Putting $q\to 1^-$ and $m=0$ in Theorem 2.2, we obtain the result which was obtained by Srivastava and Attiya in [25], page 5, Theorem 2, with $\lambda=1$;
- ▷ Putting $q\to 1^-$, $m=0$ and $1<\gamma\leqslant\frac{3}{2}$ in Theorem 2.2, we obtain the result which was obtained by Srivastava and Attiya, see [25], page 6, Corollary 4.

3. Integral means inequalities

Lemma 3.1 ([15], Theorem 2, page 484)**.** *If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with $g(z)\prec f(z)$, then*

$$\int_0^{2\pi}|g(re^{i\theta})|^\sigma\,d\theta\leqslant\int_0^{2\pi}|f(re^{i\theta})|^\sigma\,d\theta,\quad\sigma>0,\ 0<r<1. \tag{3.1}$$

Silverman in [21] found that the function $f_2(z)=z-z^2/2$ is often extremal over the family $\mathcal{T}$ denoting the subset of $\mathcal{A}$ comprising of functions

$$f(z)=z-\sum_{\kappa=2}^{\infty}|a_\kappa|z^\kappa,\quad z\in\mathbb{D}$$

and applied this function to resolve his integral means inequality, conjectured in [22] and settled in [23], that

$$\int_0^{2\pi} |f(re^{i\theta})|^\sigma \, d\theta \leqslant \int_0^{2\pi} |f_2(re^{i\theta})|^\sigma \, d\theta, \quad \sigma > 0,\ 0 < r < 1,\ f \in \mathcal{T}.$$

Applying Lemma 1.2 and Lemma 3.1, we prove the following result:

Theorem 3.1. *Let* $f(z) \in \mathfrak{Y}^{*m}_{q,\varrho}(l,\lambda,\gamma) \cap \mathcal{T}$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{2(\gamma-1)}{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)} z^2,$$

then we have

$$\int_0^{2\pi} |f(z)|^\sigma \, d\theta \leqslant \int_0^{2\pi} |f_2(z)|^\sigma \, d\theta, \quad \sigma > 0, \quad z = re^{i\theta}, \quad 0 < r < 1. \tag{3.2}$$

P r o o f. For $f(z) = z - \sum\limits_{\kappa=2}^{\infty} |a_\kappa| z^\kappa$, the inequality (3.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{\kappa=2}^{\infty} |a_\kappa| z^{\kappa-1} \right|^\sigma d\theta \leqslant \int_0^{2\pi} \left| 1 - \frac{2(\gamma-1)}{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)} z \right|^\sigma d\theta.$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{\kappa=2}^{\infty} |a_\kappa| z^{\kappa-1} \prec 1 - \frac{2(\gamma-1)}{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)} z.$$

Setting

$$1 - \sum_{\kappa=2}^{\infty} |a_\kappa| z^{\kappa-1} = 1 - \frac{2(\gamma-1)}{(([\kappa]_q - 1) + |[\kappa]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)} \varphi(z),$$

and using (1.12), we obtain that $\varphi(z)$ is analytic in $\mathbb{D}$, $\varphi(0) = 0$ and

$$\begin{aligned} |\varphi(z)| &= \left| \sum_{\kappa=2}^{\infty} \frac{(([2]_q - 1) + |[2]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}{2(\gamma-1)} |a_\kappa| z^{\kappa-1} \right| \\ &\leqslant |z| \sum_{\kappa=2}^{\infty} \frac{(([\kappa]_q - 1) + |[\kappa]_q + 1 - 2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(\kappa)}{2(\gamma-1)} |a_\kappa| \leqslant |z|. \end{aligned}$$

The proof of Theorem 3.1 is completed. □

Our proof of Theorem 3.2 below is much akin to that of Theorem 3.1. Here we make use of Lemma 1.3.

Theorem 3.2. *Let $f(z) \in \mathfrak{K}^{*m}_{q,\varrho}(l,\lambda,\gamma) \cap \mathcal{T}$ and $f_2(z)$ be defined by*

$$f_2(z) = z - \frac{2(\gamma-1)}{[2]_q(([2]_q-1)+|[2]_q+1-2\gamma|)\Theta^{m,\lambda,l}_{q,\varrho}(2)}z^2;$$

then we have

$$\int_0^{2\pi} |f(z)|^\sigma \, \mathrm{d}\theta \leqslant \int_0^{2\pi} |f_2(z)|^\sigma \, \mathrm{d}\theta, \quad \sigma > 0, \quad z = r\mathrm{e}^{\mathrm{i}\theta}, \quad 0 < r < 1.$$

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