SOME MONOUNARY ALGEBRAS WITH EKP

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Abstract. An algebra $\mathcal{A}$ is said to have the endomorphism kernel property (EKP) if every congruence on $\mathcal{A}$ is the kernel of some endomorphism of $\mathcal{A}$. Three classes of monounary algebras are dealt with. For these classes, all monounary algebras with EKP are described.

Keywords: monounary algebra; endomorphism; congruence; kernel

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1. Introduction

The notions of homomorphism and congruence in universal algebra are of cardinal importance. The well-known fundamental homomorphism theorem says that there is a correspondence between congruences on an algebra $\mathcal{A}$ and kernels of homomorphisms of the same algebra.

We deal with algebras possessing the endomorphism kernel property (EKP). An algebra is defined to have EKP if every congruence on $\mathcal{A}$ is the kernel of an endomorphism of $\mathcal{A}$. This notion was introduced in [2] for distributive lattices. EKP was studied in finite distributive lattices and de Morgan algebras (see [2]), Stone algebras and modular p-algebras (see [8]–[10]). Further, the strong EKP (i.e. each congruence of $\mathcal{A}$ is a kernel of a strong endomorphism of $\mathcal{A}$) was investigated in [3], [4], [6], [7], and [11]–[13].

We focus on EKP in monounary algebras. The importance of theory of unary and monounary algebras is pointed out for example in the monographs [24], [15], [19], [25]. The advantage of monounary algebras is their relatively easy visualization as they can be represented as planar directed graphs. Several authors concentrate

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on endomorphisms of monounary algebras, see e.g. [1], [5], [16], [17], [18], [21], [26], [27] and of injective monounary algebras (see [20], [22], [23]).

The main result of the paper is a characterization of a monounary algebra $\mathcal{A}$ with EKP if

(i) $\mathcal{A}$ consists of finitely many connected components (Theorem 3.1),
(ii) $\mathcal{A}$ is injective (Theorem 4.1),
(iii) each cyclic element of $\mathcal{A}$ has only finitely many ancestors (Theorem 5.1 and 5.2).

2. Preliminaries

The set of all positive integers is denoted by $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of a set A is denoted by $\|A\|$. If ψ is a mapping from a set A into a set B, then $\ker(\psi)$ denotes the kernel of ψ.

We deal with monounary algebras. The fundamental operation will be mostly denoted by f. The identity operation is denoted by id.

Let $\mathcal{A} = (A, f)$ be a monounary algebra. We denote by $\mathbf{S}(\mathcal{A})$ the class of all algebras that are isomorphic to a subalgebra of $\mathcal{A}$. The algebra $\mathcal{A}$ is *connected* if for every $a, b \in A$ there exist $m, n \in \mathbb{N}$ such that $f^n(a) = f^m(b)$. We say that a set B is a *component* of the algebra $\mathcal{A}$ if B has the following properties:

(1) $B \subseteq A$,
(2) $f(B) \subseteq B$,
(3) (B, f) is connected,
(4) if $a \in A$ is such that $f(a) \in B$, then $a \in B$.

If $\|A\| = 1$, then the algebra $\mathcal{A}$ is called *trivial.* A component B of an algebra $\mathcal{A}$ is called trivial if the algebra (B, f) is trivial.

We say that a set C is a *cycle* of the algebra $\mathcal{A}$ if C has the following properties:

(1) C is a finite subset of A,
(2) $f(C) = C$,
(3) (C, f) is connected.

If C is a cycle, then $\|C\|$ is called the length of the cycle C. Algebra $\mathcal{A}$ is called a cycle if A is a cycle of $\mathcal{A}$.

A subset B of A is termed as a *chain of* $\mathcal{A}$ if for every $a, b \in B$ there is $n \in \mathbb{N}_0$ such that either $f^n(a) = b$ or $f^n(b) = a$. If A is a chain of $\mathcal{A}$, then we will say that $\mathcal{A}$ is a *basic algebra.* Basic monounary algebras were introduced in [14].

A connected monounary algebra with a one-element cycle is called a *root monounary algebra* or simply a *root*, cf. [15]. By a c-root we mean the root with the cycle $\{c\}$.

Let $b \in A$. We denote

$$\begin{aligned} f^{-1}(b) &= \{a \in A\colon\ f(a) = b\}, \\ \downarrow b &= \{a \in A\colon\ f^k(a) = b \text{ for some } k \in \mathbb{N}_0\}, \\ \uparrow b &= \{f^k(b), k \in \mathbb{N}\}, \\ C_{\mathcal{A}} &= \{a \in A\colon\ a \text{ is an element of some cycle of } \mathcal{A}\}, \\ C^*_{\mathcal{A}} &= \{a \in A\colon\ f(a) = a\}. \end{aligned}$$

Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$. Let κ be a cardinal number. The following condition is denoted by (γ).

(γ) If $\mathcal{A}$ contains κ cycles of length k, then $\mathcal{A}$ contains $\kappa \cdot \aleph_0$ cycles of length l.

Let $\mathcal{B} = (B, f)$ and $A \cap B = \emptyset$. The algebra $(A \cup B, f)$ will be denoted by $\mathcal{A} + \mathcal{B}$. Let $a \in A$, $b \in B$. Let $\mathcal{A}$ be an a-root and $\mathcal{B}$ be a b-root. Then

$$\mathcal{A} \oplus \mathcal{B} = ((A \cup B) \setminus \{b\}, g),$$

where

$$g(x) = \begin{cases} f(x) & \text{if } x \in (A \cup B) \setminus f^{-1}(b), \\ a & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (A, f)$. Now we present three lemmas without proofs; they are easy to show directly from definitions.

Lemma 2.1. *Let $\mathcal{B} = (B, f)$ be a subalgebra of $\mathcal{A}$ and $\theta \in \mathrm{Con}(\mathcal{B})$. If $\theta' = \theta \cup \{(a, a), a \in A \setminus B\}$, then $\theta' \in \mathrm{Con}(\mathcal{A})$.*

Lemma 2.2. *Let $\mathcal{B} = (B, f) \in \mathbf{S}(\mathcal{A})$. Then the following statements are valid:*

(1) *If $k \in \mathbb{N}$, κ is a cardinal number and $\mathcal{B}$ contains κ cycles of length k, then $\mathcal{A}$ contains κ cycles of length k.*

(2) *If $c \in C^*_{\mathcal{B}}$, then there exists $c' \in C^*_{\mathcal{A}}$ such that*

$$\|f^{-1}(c)\| \leqslant \|f^{-1}(c')\|.$$

(3) *If f is injective on the set $A \setminus C_{\mathcal{A}}$, then f is injective on the set $B \setminus C_{\mathcal{B}}$.*

(4) *Let $\mathcal{D} = (D, \mathrm{id})$ and $D \cap B = \emptyset$. If $\|D\| + \|C^*_{\mathcal{B}}\| \leqslant \|C^*_{\mathcal{A}}\|$, then $\mathcal{B} + \mathcal{D} \in \mathbf{S}(\mathcal{A})$.*

(5) *Let $\mathcal{B}$ be connected and its operation be not injective.*
If $\mathcal{A} = \mathcal{D}_1 + \mathcal{D}_2$ and the operation of $\mathcal{D}_1$ is injective, then $\mathcal{B}$ is a subalgebra of $\mathcal{D}_2$.

Lemma 2.3. *The following statements are valid:*

(1) *If $\mathcal{A}$ is connected, then any homomorphic image of $\mathcal{A}$ is connected.*

(2) *Let $k \in \mathbb{N}$. If $\mathcal{A}$ is a cycle of length k and $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$, then $\mathcal{B}$ is a cycle of length l, where l divides k.*

(3) *Let κ be the number of components of $\mathcal{A}$. Then the algebra (B, id) such that $\|B\| \leqslant \kappa$ is a homomorphic image of $\mathcal{A}$.*

(4) *Let $k \in \mathbb{N}$. If $\mathcal{A}$ is connected without a cycle, then there are at least two homomorphic images of $\mathcal{A}$ such that they are basic algebras with a cycle of length k.*

(5) *If $\mathcal{B}$ is a homomorphic image of $\mathcal{A}$ such that $C^*_{\mathcal{B}} \neq \emptyset$, then $\mathcal{B}$ is a homomorphic image of $\mathcal{A} + \mathcal{D}$ for every algebra $\mathcal{D}$.*

Lemma 2.4. *Let $\{A_i \colon i \in I\}$ be the set of all components of $\mathcal{A}$ without a cycle. Let $a_i \in A_i$ be such that $f^{-1}(a_i) \neq \emptyset$ for every $i \in I$.*

Then there exist $c \notin A$ and a c-root $\mathcal{D} = (D, g)$ and a homomorphism φ from $\mathcal{A}$ onto $\mathcal{D}$ such that

(1) *φ is injective on the set $A \setminus \Bigl(C_{\mathcal{A}} \cup \bigcup_{i \in I} \uparrow a_i\Bigr)$,*

(2) *$\|g^{-1}(c)\| = \sum_{x \in C_{\mathcal{A}}} (\|f^{-1}(x)\| - 1) + \sum_{i \in I} \|f^{-1}(a_i)\| + 1$.*

Proof. Put $D = \{c\} \cup \Bigl[A \setminus \Bigl(C_{\mathcal{A}} \cup \bigcup_{i \in I} \uparrow a_i\Bigr)\Bigr]$. For $z \in D$ put

$$g(z) = \begin{cases} c & \text{if } z = c \text{ or } f(z) \in C_{\mathcal{A}} \text{ or } f(z) = a_i \text{ for some } i \in I, \\ f(z) & \text{otherwise.} \end{cases}$$

Then the algebra (D, g) is the c-root such that

$$\|g^{-1}(c)\| = \sum_{x \in C_{\mathcal{A}}} (\|f^{-1}(x)\| - 1) + \sum_{i \in I} \|f^{-1}(a_i)\| + 1.$$

It is a homomorphic image of $\mathcal{A}$ since

$$\varphi(x) = \begin{cases} x & \text{if } x \in D, \\ c & \text{otherwise,} \end{cases}$$

is a homomorphism from $\mathcal{A}$ onto (D, g). □

3. Several properties of algebras with EKP

We say that an algebra $\mathcal{A}$ has an *endomorphism kernel property* if every congruence relation on $\mathcal{A}$ is a kernel of some endomorphism of $\mathcal{A}$, i.e.

$$\mathrm{Con}(\mathcal{A}) = \{\ker(\varphi)\colon\ \varphi \text{ is an endomorphism of } \mathcal{A}\}.$$

Shortly, we will write that $\mathcal{A}$ *has EKP*.

The next lemma is a very useful tool for manipulation with EKP in monounary algebras. It will be often used in this paper.

Lemma 3.1. *The algebra $\mathcal{A}$ has EKP if and only if $\mathcal{B} \in \mathbf{S}(\mathcal{A})$ for every homomorphic image $\mathcal{B}$ of $\mathcal{A}$.*

P r o o f. It follows immediately from Lemma 2.1 of [13]. □

E x a m p l e 3.1.

(1) An n-element cycle, $n > 1$, has not EKP.

(2) Let $A \neq \emptyset$. Algebras (A, id), (A, const), where const is a constant operation, have EKP.

(3) Let κ be an infinite cardinal. Let $\mathcal{A} = (A, f)$ be such that
   (a) $f^2(x) = f(x)$ for every $x \in A$,
   (b) $\mathcal{A}$ consists of at most κ components,
   (c) every component has the cardinality κ.
   Then $\mathcal{A}$ has EKP.

Lemma 3.2. *Let $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, \mathrm{id})$ and $A \cap B = \emptyset$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ has EKP,*

(ii) *$\mathcal{A} + \mathcal{B}$ has EKP.*

Lemma 3.3. *Let $\mathcal{A} = (A, f)$ have EKP. If B is a component of $\mathcal{A}$, then the algebra $(A \setminus B, f)$ has EKP.*

P r o o f. Denote $D = A \setminus B$ and $\mathcal{D} = (D, f)$. Let $\theta \in \mathrm{Con}\mathcal{D}$. Consider $\theta' = \theta \cup \{(b, b), b \in B\}$. Then $\theta' \in \mathrm{Con}\mathcal{A}$ according to Lemma 2.1. The assumption $\mathcal{A}$ has EKP implies that there exists an endomorphism φ of $\mathcal{A}$ such that $\ker(\varphi) = \theta'$. We have that $\ker(\varphi|D) = \theta$ and φ is injective on B. If $\varphi(D) \subseteq D$, then $\varphi|D$ is an endomorphism of $\mathcal{D}$.

Suppose that $\varphi(D) \cap B \neq \emptyset$. Denote $E = \varphi^{-1}(B)$. Then E consists of some components of $\mathcal{A}$. Let $\mathcal{B}' = (B', f)$ be a component of $\mathcal{A}$ such that $\varphi(B) \subseteq B'$.

Assume that there exists $a \in D$ such that $\varphi(a) \in B'$. Take $b \in B$. Then there exist $m, n \in \mathbb{N}$ such that $f^n(\varphi(a)) = f^m(\varphi(b))$ since $\mathcal{B}'$ is connected. Therefore $\varphi(f^n(a)) = \varphi(f^m(b))$. We have

$$f^n(a) \notin B, \; f^m(b) \in B \text{ and } (f^n(a), f^m(b)) \in \ker(\varphi),$$

a contradiction. Hence $\varphi(D) \cap B' = \emptyset$.

Take $\mathcal{B}'' = (\varphi(B), f)$. We have $\mathcal{B}'' \cong \mathcal{B}$ because φ is injective on B. Let us define ε, the mapping from A into A, such that

$$\varepsilon(x) = \begin{cases} x & \text{if } x \in B, \\ \varphi^2(x) & \text{if } x \in E, \\ \varphi(x) & \text{otherwise.} \end{cases}$$

We obtain that ε is an endomorphism of $\mathcal{A}$, $\ker(\varepsilon) = \theta'$ and $\varepsilon|D$ is an endomorphism of $\mathcal{D}$, $\ker(\varepsilon|D) = \theta$. □

Lemma 3.4. *Let $\mathcal{A} = (A, f)$ have EKP. Then the algebra $\mathcal{A}$ satisfies condition (γ).*

Proof. Let $k, l \in \mathbb{N}$ be such that l divides k, $l < k$. Let κ be a cardinal number.

Suppose that $\mathcal{A}$ contains κ cycles of length k. Then these κ cycles can be homomorphically mapped onto κ cycles of length l. That means $\mathcal{A}$ contains κ cycles of length k and κ cycles of length l. These 2κ cycles can be mapped by a homomorphism onto 2κ cycles of length l. Therefore $\mathcal{A}$ contains κ cycles of length k and 2κ cycles of length l, etc. It yields that $\mathcal{A}$ contains $\aleph_0 \cdot \kappa$ cycles of length l. □

Let $\{A_i \colon i \in I\}$ be the component partition of $\mathcal{A}$.

Lemma 3.5. *Let $\mathcal{A} = (A, f)$ have EKP. Then $\|C_{\mathcal{A}}\| = \|C^*_{\mathcal{A}}\| = \|I\|$.*

Proof. Every connected monounary algebra contains at most one cycle. This yields $\|C^*_{\mathcal{A}}\| \leqslant \|I\|$.

The algebra (I, id) is a homomorphic image of $\mathcal{A}$ and $C^*_{\mathcal{A}} \subseteq C_{\mathcal{A}}$. It implies that $\|C_{\mathcal{A}}\| \geqslant \|C^*_{\mathcal{A}}\| \geqslant \|I\|$ in view of Lemma 3.1.

Let I be finite. Then every component of $\mathcal{A}$ contains a 1-element cycle according to $(I, \mathrm{id}) \in \mathbf{S}(\mathcal{A})$. Thus $\|C_{\mathcal{A}}\| = \|C^*_{\mathcal{A}}\|$.

Let I be infinite. Then

$$\|C_{\mathcal{A}}\| \leqslant \|I\| \sum_{k=1}^{\infty} k = \|I\| \cdot \aleph_0 = \|I\|.$$

□

Corollary 3.1. *Let $\mathcal{A}$ be a basic algebra. The algebra $\mathcal{A}$ has EKP if and only if $\mathcal{A}$ has a 1-element cycle.*

P r o o f. If $\mathcal{A}$ contains a 1-element cycle, then it has EKP by Lemma 3.1. If $\mathcal{A}$ has EKP, then the previous assertion gives $C^*_{\mathcal{A}} \neq \emptyset$. $\square$

Theorem 3.1. *Let I be a finite set and $\{A_i\colon\ i \in I\}$ be the component partition of the algebra $\mathcal{A}$. The algebra $\mathcal{A}$ has EKP if and only if*

(1) *the algebra (A_i, f) has EKP for every $i \in I$,*
(2) *if $J \subseteq I$, $J = \{j_1, \ldots, j_m\}$, then there exists $j \in J$ such that*

$$\mathcal{A}_{j_1} \oplus \ldots \oplus \mathcal{A}_{j_m} \in \mathbf{S}(\mathcal{A}_j).$$

P r o o f. Suppose that $\mathcal{A}$ has EKP. In view of Lemma 3.2 we suppose that $\mathcal{A}$ has no trivial component. The first assertion follows from Lemma 3.3.

Denote $\mathcal{D} = (D, f) = \mathcal{A}_{j_1} \oplus \ldots \oplus \mathcal{A}_{j_m}$. Consider the algebra $\mathcal{A}'$ that has a component partition $\{D\} \cup \{A_i\colon\ i \in I \setminus J\}$. Then $\mathcal{A}'$ is a homomorphic image of $\mathcal{A}$. In view of Lemma 3.1 we have $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. Therefore there exists $k \in I$ such that $\mathcal{D} \in \mathbf{S}(\mathcal{A}_k)$. If $k \in J$, then condition (2) is satisfied. Assume that $k \in I \setminus J$. Then there exists $k_1 \in I \setminus \{k\}$ such that $\mathcal{A}_k \in \mathbf{S}(\mathcal{A}_{k_1})$ according to $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. We have $\mathcal{D} \in \mathbf{S}(\mathcal{A}_k) \subseteq \mathbf{S}(\mathcal{A}_{k_1})$. Therefore if $k_1 \in J$, then condition (2) is satisfied. If $k_1 \in I \setminus J$, then we will continue to take $k_2 \in I \setminus \{k, k_1\}$ such that $\mathcal{A}_k \in \mathbf{S}(\mathcal{A}_{k_1})$ according to $\mathcal{A}' \in \mathbf{S}(\mathcal{A})$. After at most $\|I\| - m$ steps we obtain an element from J.

Suppose that (1), (2) are valid. Let $\mathcal{B} = (B, f)$ be a homomorphic image of $\mathcal{A}$, the mapping φ be a corresponding homomorphism and $\{B_k, k \in K\}$ be a component partition of $\mathcal{B}$. Define $\psi\colon\ I \to K$ such that if $\varphi(A_i) \subseteq B_k$, then $\psi(i) = k$. Take $k \in K$. Denote $L = \psi^{-1}(k)$. If $\|L\| = 1$, then $\varphi(A_{\psi^{-1}(k)}) = B_k$ and $\mathcal{B}_k \in \mathbf{S}(\mathcal{A}_{\psi^{-1}(k)})$ according to (1). Let $\|L\| > 1$. Then $\bigcup\limits_{i \in L} \varphi(A_i) = B_k$. Take $j \in L$ such that $\sum\limits_{i \in L} \mathcal{A}_i \in \mathbf{S}(\mathcal{A}_j)$ according to (2). We obtain $\varphi(A_i) \in \mathbf{S}(A_i) \subseteq \mathbf{S}(\mathcal{A}_j)$ for every $i \in L$. Therefore $B_k = \bigcup\limits_{i \in L} \varphi(A_i) \in \mathbf{S}(\mathcal{A}_j)$. $\square$

Corollary 3.2. *Let $\mathcal{A}$ consist of finitely many components. If $\mathcal{A}$ has EKP, then there exists at most one $c \in C^*_{\mathcal{A}}$ such that*

$$1 < \|f^{-1}(c)\| < \aleph_0.$$

Corollary 3.3. *Let $\mathcal{A}$ be such that*

(1) *$\mathcal{A}$ consists of finitely many components,*
(2) *$f^2(a) = f(a)$ for every $a \in A$,*
(3) *at most one component of $\mathcal{A}$ is finite.*

Then $\mathcal{A}$ has EKP.

4. Injective algebras

A monounary algebra is called *injective* if its fundamental operation is injective. Finite components of injective algebras are cycles and infinite components contain no cycle. If $C_{\mathcal{A}} \neq \emptyset$, then the algebra $(C_{\mathcal{A}}, f)$ is injective.

In this section, we describe all injective monounary algebras with EKP. Then a method how to obtain a new algebra with EKP from an injective one will be derived. Namely, we will see that an algebra with EKP that has finitely many components can be added.

Denote by $\mathcal{I}$ the class of all injective monounary algebras.

Theorem 4.1. *Let $\mathcal{A} \in \mathcal{I}$. Then the following statements are equivalent:*

(i) *$\mathcal{A}$ has EKP.*
(ii) *Every component of $\mathcal{A}$ is a cycle and $\mathcal{A}$ satisfies (γ).*

Proof. Let (i) be valid. Assume that $\mathcal{A}$ contains a component without a cycle. Then this component can be mapped by a homomorphism onto a connected monounary algebra with a cycle and a non-injective operation according to Lemma 2.3 (4). That means $\mathcal{A}$ has not EKP by Lemma 3.1. Condition (γ) is valid by Lemma 3.4.

Suppose that (ii) is valid. Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$. Then every component of $\mathcal{B}$ is a cycle. For every $i \in \mathbb{N}$ we denote by ν_i the number of cycles of length i of $\mathcal{A}$ and by μ_i the number of cycles of length i of $\mathcal{B}$.

Assume that $i \in \mathbb{N}$. We need to prove that $\mu_i \leqslant \nu_i$. Then (i) is satisfied according to Lemma 3.1. In view of Lemma 2.3 (2) we have

$$\mu_i \leqslant \sum_{j \in \mathbb{N}} \nu_{i \cdot j}.$$

Suppose that there exists $j \neq 1$ such that $\nu_{i \cdot j} \neq 0$. Then condition (γ) implies that $\nu_i \geqslant \aleph_0$ and $\nu_{i \cdot k} \leqslant \nu_i$ for every $k \in \mathbb{N}$. Therefore

$$\sum_{j \in \mathbb{N}} \nu_{i \cdot j} = \nu_i.$$

□

Corollary 4.1. *If $\mathcal{A}$ has EKP, then the algebra $(C_{\mathcal{A}}, f)$ has EKP.*

Proof. We have $(C_{\mathcal{A}}, f) \in \mathcal{I}$. The assertion follows from Lemma 3.4, Lemma 3.5 and the previous theorem. □

Theorem 4.2. *Let algebras $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, f)$ be such that*

(a) $\mathcal{A} \in \mathcal{I}$,

(b) $A \cap B = \emptyset$,

(c) *$\mathcal{B}$ consists of finitely many components,*

(d) *$(D, f) \notin \mathcal{I}$ for every component D of $\mathcal{B}$.*

Then the following statements are equivalent:

(1) *The algebra $\mathcal{A} + \mathcal{B}$ has EKP.*

(2) *Algebras $\mathcal{A}$, $\mathcal{B}$ have EKP.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ have EKP. Thus, $\mathcal{A}$ consists of cycles according to Theorem 4.1 and every component of $\mathcal{B}$ has a cycle according to Lemma 3.5. Assume that ε is a homomorphism from $\mathcal{A}+\mathcal{B}$ onto $\mathcal{D}' = (D', f)$. Then $\varepsilon(B)$ is a union of some components of $\mathcal{D}'$ according to (a). Therefore $(\varepsilon(B), f) \in \mathbf{S}(\mathcal{B})$ since $\mathcal{B}$ has EKP. Further, the set $\varepsilon(A) \setminus \varepsilon(B)$ is a union of some components of $\mathcal{D}'$, too, and it determines a subalgebra of the algebra $(\varepsilon(A), f)$. Thus, the algebra $(\varepsilon(A) \setminus \varepsilon(B), f) \in \mathbf{S}(\mathcal{A})$ since $\mathcal{A}$ has EKP. We conclude that $\mathcal{D}' \in \mathbf{S}(\mathcal{A} + \mathcal{B})$.

Suppose that $\mathcal{A}+\mathcal{B}$ has EKP. Then $\mathcal{A}$ has EKP according to (c) and Lemma 3.3. Let n be the number of components of $\mathcal{B}$. Then $n \in \mathbb{N}$ according to (c) and the number of nontrivial roots of $\mathcal{A}+\mathcal{B}$ is at most n according to (a). Suppose that B' is a component of $\mathcal{B}$ that does not determine a root. In view of (d) take $b \in B'$ such that $\|f^{-1}(b)\| > 1$. Let $d \notin A \cup B$. For $x \in A \cup B$ we define

$$\zeta(x) = \begin{cases} d & \text{if } x \in \uparrow b, \\ x & \text{otherwise.} \end{cases}$$

Then the algebra $(\zeta(A \cup B), f)$, where $f(d) = d$, is a homomorphic image of $\mathcal{A}+\mathcal{B}$ that contains $n+1$ nontrivial roots. That means that this algebra is not a subalgebra of $\mathcal{A}+\mathcal{B}$, a contradiction. We obtain that every component of $\mathcal{B}$ is a nontrivial root. Let $C_{\mathcal{B}} = \{c_1, \ldots, c_n\}$.

Assume that φ is a homomorphism from $\mathcal{B}$ onto $\mathcal{D} = (D, f)$, $A \cap D = \emptyset$. Then every component of $\mathcal{D}$ is a root and $\mathcal{D}$ consists of at most n components. If every component of $\mathcal{D}$ is trivial, then $\mathcal{D} \in \mathbf{S}(\mathcal{B})$. Suppose that $\mathcal{D}$ contains nontrivial components. Let $d_1, \ldots, d_m$ be all cyclic points of $\mathcal{D}$ such that $\|f^{-1}(d_i)\| > 1$, $i = 1, \ldots, m$. Then $m \leqslant n$. Put

$$\psi(x) = \begin{cases} x & \text{if } x \in A, \\ \varphi(x) & \text{if } x \in B. \end{cases}$$

Then ψ is a homomorphism from $\mathcal{A}+\mathcal{B}$ onto $\mathcal{A}+\mathcal{D}$. Thus $\mathcal{A}+\mathcal{D} \in \mathbf{S}(\mathcal{A}+\mathcal{B})$. Assume that ξ is an embedding of $\mathcal{A}+\mathcal{D}$ into $\mathcal{A}+\mathcal{B}$. Then $\xi(d_i) \in B$ for every $i = 1, \ldots, m$ according to (a). Therefore $\mathcal{D} \in \mathbf{S}(\mathcal{B})$ according to $\left\| D \setminus \left(\bigcup\limits_{i=1}^{m} \downarrow d_i \right) \right\| + m \leqslant n$ and Lemma 2.2 (6). □

The sum of two algebras with EKP need not have EKP:

Example 4.1. Let $A = \{a, a'\}$ and $f(a) = f(a') = a$. The algebra $\mathcal{A} = (A, f)$ has EKP. Let $B \cap A = \emptyset$ and $\mathcal{B} = (B, f)$ be isomorphic to $\mathcal{A}$. The algebra $\mathcal{A} + \mathcal{B}$ does not have EKP.

5. Class $\mathcal{F}$

Denote by $\mathcal{F}$ the class of all monounary algebras $\mathcal{A} = (A, f)$ such that the set $f^{-1}(a)$ is finite for every $a \in C_{\mathcal{A}}$. Thus $\mathcal{I} \subset \mathcal{F}$. In this section, we describe all algebras with EKP from the class $\mathcal{F}$. We will see that the non injective ones are exactly the algebras of the form $\mathcal{B} + \mathcal{D}$, where $\mathcal{B} \in \mathcal{I}$ has EKP and $\mathcal{D}$ is connected with EKP.

The next statement follows from the definition of $\mathcal{F}$ immediately.

Lemma 5.1. *Let $\mathcal{A} = (A, f) \in \mathcal{F}$. Then*

(1) *if $A \neq \bigcup\limits_{b \in C_{\mathcal{A}}} \downarrow b$, then $\mathcal{A}$ contains a component without a cycle,*

(2) *if $C^*_{\mathcal{A}} \neq \emptyset$ and $D = \bigcup\limits_{b \in C^*_{\mathcal{A}}} \downarrow b$, then one of the following cases occurs:*

   (a) *$D = A$;*

   (b) *$\mathcal{A} = (D, f) + \mathcal{B}$, where $C^*_{\mathcal{B}} = \emptyset$.*

We denote by $\mathcal{F}^\circ$ the class of all monounary algebras $\mathcal{A} \in \mathcal{F}$ such that every component of $\mathcal{A}$ has a 1-element cycle. Thus, $\mathcal{A} \in \mathcal{F}^\circ$ if and only if $C^*_{\mathcal{A}} \neq \emptyset$ and equality (2)(a) from Lemma 5.1 is valid.

We denote by $\mathcal{F}^*$ the class of all monounary algebras $\mathcal{A}$ such that $\mathcal{A} = (A, f) \in \mathcal{F}^\circ$ with $\mathcal{A} = \mathcal{B} \oplus \mathcal{D}$, where $\mathcal{B}$ is basic and $\mathcal{D}$ has a constant operation. The next assertion is obvious.

Lemma 5.2. *Let $\mathcal{A} = (A, f) \in \mathcal{F}^*$. Then*

(1) *$\mathcal{A}$ is connected,*

(2) *if $c \in C^*_{\mathcal{A}}$, then $f^{-1}(c) = A$ or $A \setminus f^{-1}(c)$ is a chain of $\mathcal{A}$.*

The next lemma says that connected monounary algebras with EKP from $\mathcal{F}$ are precisely the algebras of $\mathcal{F}^*$.

Theorem 5.1. *Let $\mathcal{A} \in \mathcal{F}$ be connected. Then the following conditions are equivalent*

(i) *$\mathcal{A}$ has EKP,*
(ii) *$\mathcal{A} \in \mathcal{F}^*$.*

P r o o f. Let $\mathcal{A}$ be nontrivial. The implication (ii) $\Rightarrow$ (i) follows from Lemma 3.1.

Suppose that (i) is valid and (ii) fails to hold. Then there is $c \in A$ such that $f(c) = c$ according to Lemma 3.5.

Let $d \in A \setminus \{c\}$ be such that $f^{-1}(d)$ has at least two elements. Then there exists $a \in A \setminus \{c\}$ such that

(1) $\|f^{-1}(a)\| > 1$,
(2) if $n \in \mathbb{N} \setminus \{1\}$ is such that $f^n(a) = c$ and $f^{n-1}(a) \neq c$, then for every $a_0 \in A \setminus \{c\}$ such that $f^{n-1}(a_0) = c$, the equality $\|f^{-1}(a_0)\| = 1$ is valid.

Take $B = (A \setminus \uparrow a) \cup \{c\}$ and for $x \in B$ put

$$g(x) = \begin{cases} c & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise.} \end{cases}$$

The algebra (B, g) is a homomorphic image of $\mathcal{A}$. The relationship $\|g^{-1}(c)\| > \|f^{-1}(c)\|$ is satisfied. Therefore $(B, g) \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction. We obtain that the set $f^{-1}(d)$ possesses at most one element. This is equivalent to injectivity of f on the set $A \setminus \{c\}$.

Assume that $A \setminus f^{-1}(c) \neq \emptyset$ and it is not a chain of $\mathcal{A}$. Then there are $a, b \in f^{-1}(c) \setminus \{c\}$ such that $a \neq b$, $f^{-1}(a) \neq \emptyset$ and $f^{-1}(b) \neq \emptyset$. Take $B' = A \setminus \{a\}$ and put for $x \in B'$

$$h(x) = \begin{cases} b & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise.} \end{cases}$$

The algebra (B', h) is a homomorphic image of the algebra $\mathcal{A}$ that is not isomorphic to a subalgebra of $\mathcal{A}$ according to Lemma 2.2 (3), a contradiction. $\square$

Lemma 5.3. *Let $\mathcal{A} \in \mathcal{F}^\circ$ be not connected. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ has EKP,*
(ii) *$\mathcal{A} = \mathcal{B} + \mathcal{D}$, where $\mathcal{B} = (B, \mathrm{id})$ and $\mathcal{D} \in \mathcal{F}^*$.*

P r o o f. Every algebra satisfying (ii) has EKP according to Lemmas 5.1 and 3.2. Let $\mathcal{A}$ have EKP. Assume that $\mathcal{A}$ contains more than one nontrivial component. Remark that $C_{\mathcal{A}} = C^*_{\mathcal{A}}$ since $\mathcal{A} \in \mathcal{F}^\circ$. Let $c \notin A$. Put $B = (A \cup \{c\}) \setminus C_{\mathcal{A}}$. For $x \in A$ define

$$\varphi(x) = \begin{cases} c & \text{if } x \in C_{\mathcal{A}}, \\ x & \text{otherwise} \end{cases}$$

and for $y \in \varphi(A)$ define

$$h(y) = \begin{cases} c & \text{if } y = c \text{ or } f(y) \in C_{\mathcal{A}}, \\ f(y) & \text{otherwise.} \end{cases}$$

Then $(\varphi(A), h)$ is a homomorphic image of $\mathcal{A}$. This algebra is a c-root. In view of Lemma 2.2 (2) we obtain that $\varphi(A) = B$ and $(B, h) \notin \mathbf{S}(\mathcal{A})$, a contradiction. Thus, $\mathcal{A}$ has exactly one nontrivial component $\mathcal{D} = (D, f)$.

Let $\mathcal{D}$ do not have EKP. That means $\mathcal{D}$ can be mapped by a homomorphism onto an algebra $\mathcal{D}'$ such that $\mathcal{D}' \notin \mathbf{S}(\mathcal{D})$ according to Lemma 3.1. The algebra $\mathcal{D}'$ is a connected element of $\mathcal{F}^\circ$. Therefore the operation of $\mathcal{D}'$ is not injective. Further, $\mathcal{D}'$ is a homomorphic image of $\mathcal{A}$ too and it is not isomorphic to a subalgebra of $\mathcal{A}$ according to Lemma 2.2 (5), a contradiction. We obtain $\mathcal{D} \in \mathcal{F}^*$ by Theorem 5.1. Therefore (ii) is valid. □

Theorem 5.2. *Let $\mathcal{A} \in \mathcal{F}$ be not connected. Then the following statements are equivalent:*

(i) *The algebra $\mathcal{A}$ has EKP.*

(ii) *Denote $D_1 = \bigcup_{b \in C^*_{\mathcal{A}}} \downarrow b$. Then*
  - (a) *every component of $\mathcal{A}$ has a cycle,*
  - (b) *$(C_{\mathcal{A}}, f)$ has EKP,*
  - (c) *$C^*_{\mathcal{A}} \neq \emptyset$ and the algebra (D_1, f) has EKP,*
  - (d) *if $\mathcal{A} \notin \mathcal{F}^\circ$, then $C_{\mathcal{A}} \setminus C^*_{\mathcal{A}} = A \setminus D_1$.*

(iii) *$\mathcal{A} = \mathcal{B} + \mathcal{D}$, where*
  - (a) *$\mathcal{B} \in \mathcal{I}$,*
  - (b) *$\mathcal{B}$ has EKP,*
  - (c) *$\mathcal{D} \in \mathcal{F}^*$.*

P r o o f. If $\mathcal{A} \in \mathcal{I}$, then properties (i), (ii) and (iii) are equivalent according to Theorem 4.1. We suppose that $\mathcal{A} \notin \mathcal{I}$.

Let (i) be fulfilled. Then (ii)(b) is true according to Corollary 4.1. Therefore $C^*_{\mathcal{A}} \neq \emptyset$ in view of Theorem 4.1. We have $C_{\mathcal{A}} \setminus C^*_{\mathcal{A}} \subseteq A \setminus D_1$. Let a_i, $i \in I$, $c \notin A$

and $\mathcal{D}$ be as in Lemma 2.4. If (ii)(a) is not valid, then $I \neq \emptyset$ and $\sum_{i\in I} \|f^{-1}(a_i)\| > 0$. Take $b \in C_{\mathcal{A}}$. We obtain

$$\|g^{-1}(c)\| = \|f^{-1}(b)\| + \sum_{x\in C_{\mathcal{A}}\setminus\{b\}} (\|f^{-1}(x)\| - 1) + \sum_{i\in I} \|f^{-1}(a_i)\| > \|f^{-1}(b)\|.$$

That means that $\mathcal{D} \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction. If (ii)(d) is not valid, then there is $a \in C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}$ such that $\|f^{-1}(a)\| > 1$. Take $b_1 \in C^*_{\mathcal{A}}$. We obtain

$$\begin{aligned}\|g^{-1}(c)\| &= \|f^{-1}(b_1)\| + \sum_{x\in C_{\mathcal{A}}\setminus\{b_1\}} (\|f^{-1}(x)\| - 1) + \sum_{i\in I} \|f^{-1}(a_i)\| \\ &\geqslant \|f^{-1}(b_1)\| + (\|f^{-1}(a)\| - 1) + \sum_{x\in C_{\mathcal{A}}\setminus\{a,b_1\}} (\|f^{-1}(x)\| - 1) > \|f^{-1}(b_1)\|.\end{aligned}$$

That means that $\mathcal{D} \notin \mathbf{S}(\mathcal{A})$ according to Lemma 2.2 (2), a contradiction.

Let $\mathcal{D}_0$ be a homomorphic image of (D_1, f). Then $\mathcal{D}_0$ is a homomorphic image of $\mathcal{A}$ by Lemma 2.3 (5). Therefore $\mathcal{D}_0 \in \mathbf{S}(\mathcal{A})$. That means $\mathcal{D}_0 \in \mathbf{S}(D_1, f)$ in view of Lemma 5.1 (2). Hence (ii)(c) holds.

Now let (ii) be valid. Property (c) and Lemma 5.3 imply that there exist $B_1, D \subset D_1$ such that $(D_1, f) = (B_1, \mathrm{id}) + (D, f)$ and $(D, f) \in \mathcal{F}^*$. Denote $\mathcal{D} = (D, f)$; thus (iii)(c) is valid. If $\mathcal{A} \in \mathcal{F}^\circ$, then we denote $\mathcal{B} = (B_1, \mathrm{id})$, else we denote $\mathcal{B} = (B_1, \mathrm{id}) + (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}, f)$. We obtain

$$B_1 \cup (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}) \cup D = (C_{\mathcal{A}} \setminus C^*_{\mathcal{A}}) \cup D_1 = A$$

according to (d). The algebra $\mathcal{B}$ consists of all components of the algebra $(C_{\mathcal{A}}, f)$ except one 1-element cycle. Thus, $\mathcal{B}$ has EKP according to (ii)(b) and Theorem 4.1. We have shown that (ii) implies (iii).

If (iii) is satisfied, then $\mathcal{A}$ has EKP according to Lemma 5.2 and Theorem 4.2. □

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