GENERALIZATION OF THE WEAK AMENABILITY ON VARIOUS BANACH ALGEBRAS

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Abstract. The generalized notion of weak amenability, namely (φ,ψ)-weak amenability, where φ,ψ are continuous homomorphisms on a Banach algebra $\mathcal{A}$, was introduced by Bodaghi, Eshaghi Gordji and Medghalchi (2009). In this paper, the (φ,ψ)-weak amenability on the measure algebra $M(G)$, the group algebra $L^1(G)$ and the Segal algebra $S^1(G)$, where G is a locally compact group, are studied. As a typical example, the (φ,ψ)-weak amenability of a special semigroup algebra is shown as well.

Keywords: Banach algebra; (φ,ψ)-derivation; group algebra; locally compact group; measure algebra; Segal algebra; weak amenability

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. Let the products of $a \in \mathcal{A}$ and $x \in X$ be denoted by $a \cdot x$ and $x \cdot a$. A derivation $D\colon \mathcal{A} \to X$ is a linear map which satisfies $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in \mathcal{A}$. The derivation δ is said to be *inner* if there exists $x \in X$ such that $\delta(a) = \delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. The linear space of bounded derivations from $\mathcal{A}$ into X is denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations is denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X. A Banach algebra $\mathcal{A}$ is amenable if every continuous derivation from $\mathcal{A}$ into every dual Banach $\mathcal{A}$-module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-module X. This definition was introduced by Johnson in [12]. One of the important resuls that Johnson obtained was that the group algebra $L^1(G)$ is amenable if and only if the

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locally compact group G is amenable. Also, a Banach algebra $\mathcal{A}$ is called *weakly amenable* if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. Weak amenability of the group algebra $L^1(G)$ was proved by Johnson in [13] for the first time. After that, Despić and Ghahramani in [8] gave a different and shorter proof for it; see also [1] and [10]. Dales and Pandey studied the weak amenability of special case of Segal algebras in [7], and then the weak amenability of Segal algebras and the Lebesgue-Fourier algebra of a locally compact group were proved by Ghaharamani and Lau in [9]. For more details about Segal algebras refer to [16].

Let X be an $\mathcal{A}$-bimodule and let φ,ψ be continuous homomorphisms of $\mathcal{A}$ into itself. A bounded linear mapping $d\colon \mathcal{A}\to X$ is called a (φ,ψ)*-derivation* if

$$d(ab)=d(a)\cdot\varphi(b)+\psi(a)\cdot d(b),\quad a,b\in\mathcal{A}.$$

A bounded linear mapping $d\colon \mathcal{A}\to X$ is called a (φ,ψ)*-inner derivation* if there exists $x\in X$ such that

$$d(a)=x\cdot\varphi(a)-\psi(a)\cdot x,\quad a\in\mathcal{A}.$$

Let $\mathcal{A}$ be Banach algebra and let φ and ψ be as in the above. We consider the following module actions on $\mathcal{A}$:

$$a\cdot x:=\varphi(a)x,\quad x\cdot a:=x\psi(a),\quad a,x\in\mathcal{A}.$$

We denote the above $\mathcal{A}$-bimodule by $\mathcal{A}_{(\varphi,\psi)}$. Then $\mathcal{A}$ is called (φ,ψ)-weakly amenable if $H^1(\mathcal{A},(\mathcal{A}_{(\varphi,\psi)})^*)=\{0\}$. These concepts are introduced and investigated in [4] and [15] (for the generalization of n-weak amenability refer to [5]). It is also proved in [4], Example 4.2 that for any locally compact group algebra G, $L^1(G)$ is $(\varphi,0)$ and $(0,\psi)$-weakly amenable. For the module versions of these notions refer to [3] and [2].

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Similarly to [4], we denote by $\mathrm{Hom}(\mathcal{A},\mathcal{B})$ the space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$ and denote $\mathrm{Hom}(\mathcal{A},\mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$. Suppose that $\mathcal{A}$ is a Banach algebra, X is a Banach $\mathcal{A}$-module and $\varphi,\psi\in\mathrm{Hom}(\mathcal{A})$. A derivation $D\colon \mathcal{A}\to X$ is called approximately (φ,ψ)-inner if there exists a net (x_α) in X such that for all $a\in\mathcal{A}$, $D(a)=\lim_\alpha(x_\alpha\cdot\varphi(a)-\psi(a)\cdot x_\alpha)$ in norm. A Banach algebra $\mathcal{A}$ is approximately (φ,ψ)-weakly amenable if every derivation $D\colon \mathcal{A}\to(\mathcal{A}_{(\varphi,\psi)})^*$ is approximately (φ,ψ)-inner.

In this work, we prove that if the group algebra $L^1(G)$ is a two-sided G-module, then it is (φ,ψ)-amenable. We also show that (φ,ψ)-weak amenability of the measure algebra $M(G)$ necessitates G being discrete and amenable locally compact group. Finally, we investigate (approximate) (φ,ψ)-weak amenability of Segal algebras.

2. Results for group algebras

In this section, by providing some new results on (φ,ψ)-weak amenability of Banach algebras, we study the (φ,ψ)-weak amenability of group algebra $L^1(G)$, where $\varphi,\psi\in\operatorname{Hom}(L^1(G))$ and G is a locally compact group. From now on, the notation $\langle f,x\rangle$ means the value of the mapping f at x.

Proposition 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule and $\varphi,\psi\in\operatorname{Hom}(\mathcal{A})$. Suppose that $\Phi\colon\,\mathcal{A}\to\mathcal{B}$ is an $\mathcal{A}$-bimodule morphism with dense range, and $d\colon\,\mathcal{A}\to(\mathcal{B}_{(\varphi,\psi)})^*$ is a nonzero (φ,ψ)-derivation. Then $D:=\Phi^*\circ d$ is a nonzero (φ,ψ)-derivation.*

Proof. For every $x,y,z\in\mathcal{A}$ we have

$$\begin{aligned}\langle D(xy),z\rangle &= \langle \Phi^*\circ d(xy),z\rangle = \langle \Phi^*(d(x)\cdot\varphi(y)+\psi(x)\cdot d(y)),z\rangle\\ &= \langle d(x)\cdot\varphi(y)+\psi(x)\cdot d(y),\Phi(z)\rangle\\ &= \langle d(x),\varphi(y)\cdot\Phi(z)\rangle+\langle d(y),\Phi(z)\cdot\psi(x)\rangle\\ &= \langle d(x),\Phi(\varphi(y)z)\rangle+\langle d(y),\Phi(z\psi(x))\rangle\\ &= \langle \Phi^*(d(x)),\varphi(y)z\rangle+\langle \Phi^*(d(y)),z\psi(x)\rangle\\ &= \langle D(x)\cdot\varphi(y)+\psi(x)\cdot D(y),z\rangle.\end{aligned}$$

Therefore D is a (φ,ψ)-derivation. If $D=0$, then for every $x,y\in\mathcal{A}$ we have $\langle D(x),y\rangle=0$. Thus, $\langle\Phi^*\circ d(x),y\rangle=\langle d(x),\Phi(y)\rangle=0$. This means that $d(\mathcal{A})=0$, and so $d=0$. $\square$

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}$, and $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ such that $\mathcal{A}=\mathcal{B}\oplus\mathcal{I}$. If $\mathcal{A}$ is (φ,ψ)-weakly amenable, where $\varphi,\psi\in\operatorname{Hom}(\mathcal{A})$ and $\varphi(\mathcal{B}),\psi(\mathcal{B})\subseteq\mathcal{B}$, then $\mathcal{B}$ is (φ,ψ)-weakly amenable.*

Proof. Let $\pi\colon\,\mathcal{A}\to\mathcal{B}$ be the natural projection from $\mathcal{A}$ onto $\mathcal{B}$. For every $a,b\in\mathcal{A}$ there are $x,y\in\mathcal{B}$ such that $a=x+I$ and $b=y+I$. Then $\pi(ab)=xy=x\pi(b)=\pi(a)y$. Suppose that $d\colon\,\mathcal{B}\to(\mathcal{B}_{(\varphi,\psi)})^*$ is an arbitrary (φ,ψ)-derivation. From Proposition 2.1, $D=\pi^*\circ d\colon\,\mathcal{A}\to(\mathcal{A}_{(\varphi,\psi)})^*$ is a (φ,ψ)-derivation. Since $\mathcal{A}$ is (φ,ψ)-weakly amenable, then there exists an element $\xi\in(\mathcal{A}_{(\varphi,\psi)})^*$ such that

$$D(x)=\xi\cdot\varphi(x)-\psi(x)\cdot\xi,\quad x\in\mathcal{A}.$$

Set $\eta=\xi|_{\mathcal{B}}$. Then

$$\begin{aligned}\langle d(x),y\rangle &= \langle d(x),\pi(y)\rangle=\langle\pi^*\circ d(x),y\rangle=\langle D(x),y\rangle\\ &= \langle\xi\cdot\varphi(x)-\psi(x)\cdot\xi,y\rangle=\langle\xi,\varphi(x)y\rangle-\langle\xi,y\psi(x)\rangle\\ &= \langle\eta,\varphi(x)y\rangle-\langle\eta,y\psi(x)\rangle=\langle\eta\cdot\varphi(x)-\psi(x)\cdot\eta,y\rangle\end{aligned}$$

for all $x, y \in \mathcal{B}$ (note that $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$). Hence, d is a (φ,ψ)-inner derivation. $\square$

Let G be a non-discrete locally compact group. Then $M(G) = M_{\rm d}(G) \oplus M_{\rm c}(G)$, where $M_{\rm d}(G)$ and $M_{\rm c}(G)$ are discrete measures and continuous measures on G, respectively (see Theorem 19.20 of [11]).

Corollary 2.3. *Let $M(G)$ be (φ,ψ)-weakly amenable, where $\varphi,\psi \in \operatorname{Hom}(M(G))$. If $\varphi(M_{\rm d}(G)) \subseteq M_{\rm d}(G)$ and $\psi(M_{\rm d}(G)) \subseteq M_{\rm d}(G)$, then $M_{\rm d}(G)$ is (φ,ψ)-weakly amenable.*

Remark 2.4. In [6], Theorem 1.2, Dales, Ghahramani and Helemskii showed that $M(G)$ is weakly amenable if and only if G is discrete if and only if there is no nonzero continuous point derivation at a character of $M(G)$. Similarly, suppose that $\varphi \in \operatorname{Hom}(M(G))$ which is onto. If $M(G)$ is (φ,φ)-weakly amenable and ψ is a nonzero character on $M(G)$, then by Theorem 2.10 of [4], there are no nonzero point derivations at $\psi \circ \varphi$. This means that G is discrete and $M(G)$ is weakly amenable. Converse of this assertion is true when φ and ψ are identity homomorphisms. This shows that the class of (φ,ψ)-weakly amenable Banach algebras is wider than that of weakly amenable Banach algebras.

Let X be a Banach space, a net $(m_\alpha) \subset X^*$ is said to converge *weak*$^\sim$ to $m \in X^*$ if $m_\alpha \overset{w^*}{\to} m$ and $\|m_\alpha\| \to \|m\|$. This notion was introduced by Lau and Loy in [14]. For a particular case, if $\mu \in M(G)$, suppose that $\nu \in L^\infty(G)^*$ is a norm preserving extension of μ. Then there exists a net $(f_\gamma) \subset L^1(G)$ with $\|f_\gamma\| \leqslant \|\mu\|$, and $f_\gamma \overset{w^*}{\to} \nu$. Passing to a suitable subnet, we can write $\|f_\gamma\| \to \|\mu\|$, so we have $f_\gamma \to \mu$ in weak$^\sim$. Let $\mathcal{A}$ be a Banach algebra. We say that an operator φ satisfies the weak$^\sim$ condition if for every net $(m_\alpha) \subset \mathcal{A}^*$ which converges weak$^\sim$ to $m \in \mathcal{A}^*$, $m_\alpha \overset{w^*}{\to} \varphi(m)$ holds.

Theorem 2.5. *Let G be a locally compact group and X be a $M(G)$-bimodule with the module actions $\mu \cdot x = \widetilde{\varphi}(\mu) \cdot x$ and $x \cdot \mu = x \cdot \widetilde{\psi}(\mu)$, where $\varphi,\psi \in \operatorname{Hom}(L^1(G))$ and $\widetilde{\varphi},\widetilde{\psi}$ are extensions of φ and ψ, respectively. Then every (φ,ψ)-derivation $D\colon L^1(G) \to X^*$ extends to a unique $(\widetilde{\varphi},\widetilde{\psi})$-derivation from $M(G)$ into X^*.*

Proof. Let $\mu \in M(G)$. Take a net $(f_\gamma) \subset L^1(G)$ such that $f_\gamma \to \mu$ weak$^\sim$. Then the net (Df_γ) converges in w^*-topology. Define $\overline{D}\mu = w^*\text{-}\lim_\gamma Df_\gamma$. Thus, $\overline{D}$ is a bounded linear operator which extends D. For every $x \in X$, $g_1, g_2 \in L^1(G)$ and $\mu \in M(G)$ we have

$$
\begin{aligned}
(2.1)\qquad \langle \overline{D}(\mu), g_1 \cdot x \cdot g_2\rangle &= w^*\text{-}\lim_\gamma \langle D(f_\gamma), g_1 \cdot x \cdot g_2\rangle = w^*\text{-}\lim_\gamma \langle D(f_\gamma)\cdot g_1, x \cdot g_2\rangle \\
&= w^*\text{-}\lim_\gamma \langle D(f_\gamma)\cdot \varphi(g_1), x \cdot g_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= w^*\text{-}\lim_\gamma \langle D(f_\gamma \cdot g_1) - \psi(f_\gamma)\cdot D(g_1), x\cdot g_2\rangle \\
&= w^*\text{-}\lim_\gamma \langle D(f_\gamma \cdot g_1), x\cdot g_2\rangle - w^*\text{-}\lim_\gamma \langle \psi(f_\gamma)\cdot D(g_1), x\cdot g_2\rangle \\
&= \langle \overline{D}(\mu\cdot g_1), x\cdot g_2\rangle - \langle \widetilde{\psi}(\mu)\cdot D(g_1), x\cdot g_2\rangle \\
&= \langle \overline{D}(\mu\cdot g_1), x\cdot g_2\rangle - \langle D(g_1), x\cdot g_2\cdot \widetilde{\psi}(\mu)\rangle.
\end{aligned}
$$

Now, we show that $\overline{D}$ is a $(\widetilde{\varphi}, \widetilde{\psi})$-derivation. As above, assume that $f_\gamma \to \mu$ weak$^\sim$. By (2.1) we get

$$
\begin{aligned}
\langle \overline{D}(\eta\mu), g\cdot x\rangle &= \lim_\gamma \langle \overline{D}(\eta\cdot f_\gamma), g\cdot x\rangle \\
&= \lim_\gamma \langle \overline{D}(\eta), f_\gamma g\cdot x\rangle + \lim_\gamma \langle D(f_\gamma), g\cdot x\cdot \widetilde{\psi}(\eta)\rangle \\
&= \lim_\gamma \langle \overline{D}(\eta)\cdot \varphi(f_\gamma), g\cdot x\rangle + \lim_\gamma \langle D(f_\gamma), g\cdot x\cdot \widetilde{\psi}(\eta)\rangle \\
&= \lim_\gamma \langle \overline{D}(\eta)\cdot \varphi(f_\gamma), g\cdot x\rangle + \lim_\gamma \langle \widetilde{\psi}(\eta)\cdot D(f_\gamma), g\cdot x\rangle \\
&= \langle \overline{D}(\eta)\cdot \widetilde{\varphi}(\mu) + \widetilde{\psi}(\eta)\cdot \overline{D}(\mu), g\cdot x\rangle
\end{aligned}
$$

for every $\eta \in M(G)$, $g \in L^1(G)$ and $x \in X$ (note that $\|f_\gamma * g - \mu * g\| \to 0$ and $L^1(G)\cdot X = X$). For the uniqueness of $\overline{D}$, let D' be another $(\widetilde{\varphi}, \widetilde{\psi})$-derivation which extends D to $M(G)$. Let f_γ and μ be as above, then $D'(\mu) = \lim_\gamma D(f_\gamma) = \overline{D}(\mu)$. $\square$

Let G be a locally compact group. A Banach space X is called a *left Banach G-module* if the following statements hold:

(1) There is $k \geqslant 0$ such that $\|g\cdot x\| \leqslant k\|x\|$ for every $g \in G, x \in X$;
(2) For $x \in X$, the map $G \to X\colon\ g \mapsto g\cdot x$ is continuous.

Similarly for the right Banach G-modules and two-sided Banach G-modules, where in the latter case we require the map $G\times G \to X\colon\ (g_1, g_2) \mapsto g_1\cdot x\cdot g_2$ to be continuous. If X^* is the dual of X, then X^* is a two-sided G-module with the actions defined as follows:

$$
\langle f\cdot\theta, x\rangle = \langle f, \theta\cdot x\rangle \quad \text{and} \quad \langle \theta\cdot f, x\rangle = \langle f, x\cdot\theta\rangle
$$

for every $\theta \in G$, $x \in X$ and $f \in X^*$. Here we use the technique of the proof in [8] to show that $L^1(G)$ is (φ, ψ)-weakly amenable. In fact, we generalize the result of [4], Example 4.2 which asserts that for any locally compact group G, $L^1(G)$ is (φ, ψ)-weakly amenable in which either φ or ψ is zero homomorphism.

Theorem 2.6. *Let $L^1(G)$ be a two-sided Banach G-module and let $\varphi, \psi \in \mathrm{Hom}(L^1(G))$. Then $L^1(G)$ is (φ, ψ)-weak amenable.*

Proof. Let $D\colon L^1(G) \to L^\infty(G)$ be a (φ, ψ)-derivation. By Theorem 2.5, it suffices to show that the extension of D to $M(G)$ is inner. For $t \in G$, by δ_t we mean the point mass at t. Then

$$\begin{aligned}
(2.2) \qquad \psi(\delta_{t^{-1}}) \cdot D(\delta_t) &= \psi(\delta_{t^{-1}}) \cdot D(\delta_{tx^{-1}} * \delta_x) \\
&= \psi(\delta_{t^{-1}})\psi(\delta_{tx^{-1}}) \cdot D(\delta_x) + \psi(\delta_{t^{-1}}) \cdot D(\delta_{tx}) \cdot \varphi(\delta_x) \\
&= \psi(\delta_{x^{-1}}) \cdot D(\delta_x) + \psi(\delta_{x^{-1}}) \cdot \psi(\delta_{(tx^{-1})^{-1}}) \cdot D(\delta_{tx}) \cdot \varphi(\delta_x).
\end{aligned}$$

For every $\lambda \in L^\infty(G)$ let $\mathrm{Re}(\lambda)$ denote the real part of λ and let

$$S = \{\mathrm{Re}(\psi(\delta_{t^{-1}}) \cdot D(\delta_t))\colon\ t \in G\}.$$

Take $\xi = \sup(S)$. Since $L^1(G)$ is two-sided G-module, then we obtain

$$\begin{aligned}
(2.3) \qquad \sup(\psi(\delta_{x^{-1}}) \cdot S \cdot \varphi(\delta_x)) &= \psi(\delta_{x^{-1}}) \cdot \sup(S) \cdot \varphi(\delta_x) \quad \text{and} \\
\sup(\lambda + S) &= \lambda + \sup(S), \quad x \in G,\ \lambda \in L^\infty(G).
\end{aligned}$$

Now, by (2.2) and (2.3) we have

$$\xi = \psi(\delta_{x^{-1}}) \cdot \mathrm{Re}(D(\delta_x)) + \psi(\delta_{x^{-1}}) \cdot \xi \cdot \varphi(\delta_x),$$

then

$$\mathrm{Re}(D(\delta_x)) = \psi(\delta_x) \cdot \xi - \xi \cdot \varphi(\delta_x)$$

for every $x \in G$. Similarly, for imaginary part there exists an element $\zeta \in L^\infty(G)$ such that

$$\mathrm{Im}(D(\delta_x)) = \psi(\delta_x) \cdot \zeta - \zeta \cdot \varphi(\delta_x)$$

for every $x \in G$. Therefore by taking $\varsigma = \xi + \mathrm{i}\zeta$, we find

$$D(\delta_x) = \psi(\delta_x) \cdot \varsigma - \varsigma \cdot \varphi(\delta_x)$$

for every $x \in G$. Let $\mu \in M(G)$, then there exists a net (μ_α) with each μ_α a linear combination of point masses such that $\mu_\alpha \to \mu$ in strong topology. Note that point masses are extreme points of $M(G)$. On the other hand, we can take this net to be the one from the proof of Theorem 2.5. Then

$$\overline{D}(\mu) = \widetilde{\psi}(\mu) \cdot \varsigma - \varsigma \cdot \widetilde{\varphi}(\mu)$$

for every $\mu \in M(G)$. This means that $L^1(G)$ is (φ, ψ)-weak amenable. $\square$

E x a m p l e 2.7. Suppose that S is a discrete infinite semigroup and s_0 is a fixed element in S. Define an algebra product in $l^1(S)$ via $st := s(s_0)t,\ s,t \in l^1(S)$. This Banach algebra has been introduced by Yong Zang in [17]. For every $\varphi,\psi \in \mathrm{Hom}(l^1(S))$ we show that $l^1(S)$ is (φ,ψ)-weakly amenable. This Banach algebra has a left identity e_0 defined by

$$e_0(s) = \begin{cases} 1 & \text{if } s = s_0, \\ 0 & \text{if } s \neq s_0. \end{cases}$$

The $l^1(S)$-bimodule actions on the dual module $l^1(S)^* = l^\infty(S)$ are in fact defined as follows:

$$f \cdot s = s(s_0)f, \quad s \cdot f = f(s)e_0^*, \quad s \in l^1(S),\ f \in l^\infty(S),$$

where e_0^* is the element of $l^\infty(S)$ satisfying $e_0^*(s_0) = 1$ and $s_0^*(s) = 0$ for $s \neq s_0$. Let $\varphi\colon\ l^\infty(S) \to l^\infty(S)$ be a nonzero homomorphism. Then

$$s(s_0)\varphi(t) = \varphi(s(s_0)t) = \varphi(st) = \varphi(s)\varphi(t) = \varphi(s)(s_0)\varphi(t).$$

Hence, $\varphi(t)(\varphi(s)(s_0) - s(s_0))$ for all $s,t \in l^1(S)$. Since φ is nonzero,

$$\varphi(s)(s_0) = s(s_0), \quad s \in l^1(S). \tag{2.4}$$

Now, suppose that $\varphi,\psi \in \mathrm{Hom}(l^1(S))$ and $D\colon\ l^1(S) \to l^\infty(S)$ is a bounded (φ,ψ)-derivation. For each $s,t \in l^1(S)$ we have

$$\begin{aligned} s(s_0)D(t) &= D(s(s_0)t) = D(st) \\ &= D(s)\cdot\varphi(t) + \psi(s)\cdot D(t) = \varphi(t)(s_0)D(s) + \psi(s)\cdot D(t). \end{aligned}$$

Replacing t by s in the last equalities and using (2.4), we get $\psi(s)\cdot D(s) = 0$ for all $s \in l^1(S)$. The last equality implies that $\psi(s)\cdot D(t) = -\psi(t)\cdot D(s)$ for all $s,t \in l^1(S)$. Thus

$$\begin{aligned} D(s) &= D(s_0 s) = D(s_0)\cdot\varphi(s) + \psi(s_0)\cdot D(s) \\ &= D(s_0)\cdot\varphi(s) - \psi(s)\cdot D(s_0) \end{aligned}$$

for all $s \in l^1(S)$. Therefore $l^1(S)$ is (φ,ψ)-weakly amenable. Note that $l^1(S)$ cannot be amenable because this Banach algebra does not have a bounded right approximate identity (see [17]).

3. Results for Segal algebras

Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Then $(\mathfrak{B}, \|\cdot\|')$ is an abstract Segal algebra with respect to $(\mathcal{A}, \|\cdot\|)$ if

(1) $\mathfrak{B}$ is a dense left ideal in $\mathcal{A}$ and $\mathfrak{B}$ is a Banach algebra with respect to $\|\cdot\|'$;
(2) There exists $M > 0$ such that $\|b\| \leqslant M\|b\|'$ for each $b \in \mathfrak{B}$;
(3) There exists $C > 0$ such that $\|ab\|' \leqslant C\|a\|'\|b\|'$ for each $a, b \in \mathfrak{B}$.

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions:

(i) $S^1(G)$ is dense in $L^1(G)$;
(ii) If $f \in S^1(G)$, then $L_x f \in S^1(G)$, where $L_x f(a) = f(xa)$, i.e. $S^1(G)$ is left translation invariant;
(iii) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ and $\|L_x f\|_S = \|f\|_S$ for all $f \in S^1(G)$ and $x \in G$;
(iv) Map $x \mapsto L_x f$ from G into $S^1(G)$ is continuous.

In the upcoming result we study approximate (φ, ψ)-weak amenability of Segal algebras.

Theorem 3.1. *Let G be an amenable group and let $S^1(G)$ be a symmetric Segal algebra with approximate identity $(e_\alpha)_\alpha$ contained in the center of $L^1(G)$. Given $\varphi, \psi \in \operatorname{Hom}(S^1(G))$ such that extensions of φ, ψ to $L^1(G)$ are continuous, and ψ is onto, every bounded (φ, ψ)-derivation from $S^1(G)$ into $S^1(G)^*$ is approximately (φ, ψ)-inner.*

P r o o f. Let $(e_\alpha)_\alpha$ be an approximate identity of $S^1(G)$ contained in the center of $L^1(G)$, and let $\widetilde{\varphi}$ and $\widetilde{\psi}$ be the extensions of φ and ψ to $L^1(G)$, respectively. For each α define $D_\alpha\colon L^1(G) \to S^1(G)^*$ by $D_\alpha(f) = D(e_\alpha * f) - D(e_\alpha) \cdot \widetilde{\varphi}(f)$ for every $f \in L^1(G)$. At first, we prove that D_α is a bounded (φ, ψ)-derivation. The boundedness of D_α comes from that $L^1(G)$ acts continuously on $S^1(G)$ on the right, and so $f \mapsto D(e_\alpha * f)$ is continuous from $L^1(G)$ into $S^1(G)^*$. Similarly $L^1(G)$ acts continuously on $S^1(G)$ on the left, and so $f \mapsto D(e_\alpha) \cdot \widetilde{\varphi}(f)$ is continuous, which implies that D_α is continuous. Let $f_1, f_2 \in L^1(G)$. Then

$$\begin{aligned} D_\alpha(f_1 * f_2) &= D(e_\alpha * f_1 * f_2) - D(e_\alpha) \cdot \widetilde{\varphi}(f_1 * f_2) \\ &= \text{norm-}\lim_\beta (D(e_\alpha * f_1 * e_\beta * f_2) - D(e_\alpha) \cdot \widetilde{\varphi}(f_1 * f_2)) \\ &= \text{norm-}\lim_\beta (D(e_\alpha * f_1) \cdot \varphi(e_\beta * f_2) + \psi(e_\alpha * f_1) \cdot D(e_\beta * f_2) \\ &\quad - D(e_\alpha) \cdot \widetilde{\varphi}(f_1 * f_2)) \end{aligned}$$

$$
\begin{aligned}
&= D(e_\alpha * f_1)\cdot\widetilde{\varphi}(f_2) - D(e_\alpha)\cdot\widetilde{\varphi}(f_1 * f_2)\\
&\quad + w^*\text{-}\lim_\beta \psi(f_1 * e_\alpha)\cdot D(e_\beta * f_2)\\
&= (D(e_\alpha * f_1) - D(e_\alpha)\cdot\widetilde{\varphi}(f_1))\cdot\widetilde{\varphi}(f_2)\\
&\quad + w^*\text{-}\lim_\beta (\widetilde{\psi}(f_1)\psi(e_\alpha)\cdot D(e_\beta * f_2))\\
&= (D(e_\alpha * f_1) - D(e_\alpha)\cdot\widetilde{\varphi}(f_1))\cdot\widetilde{\varphi}(f_2)\\
&\quad + w^*\text{-}\lim_\beta (\widetilde{\psi}(f_1)\cdot D(e_\alpha * e_\beta * f_2) - \widetilde{\psi}(f_1)\cdot D(e_\alpha)\cdot\varphi(e_\beta * f_2))\\
&= (D(e_\alpha * f_1) - D(e_\alpha)\cdot\widetilde{\varphi}(f_1))\cdot\widetilde{\varphi}(f_2) + \widetilde{\psi}(f_1)\cdot D(e_\alpha * f_2)\\
&\quad - \widetilde{\psi}(f_1)\cdot D(e_\alpha)\cdot\widetilde{\varphi}(f_2)\\
&= D_\alpha(f_1)\cdot\widetilde{\varphi}(f_2) + \widetilde{\psi}(f_1)\cdot D(f_2).
\end{aligned}
$$

Therefore D_α is a (φ,ψ)-derivation. By Johnson's theorem there exists ξ_α in $S^1(G)^*$ such that

$$D_\alpha(f) = \xi_\alpha\cdot\widetilde{\varphi}(f) - \widetilde{\psi}(f)\cdot\xi_\alpha = D(e_\alpha * f) - D(e_\alpha)\cdot\widetilde{\varphi}(f)$$

for all $f \in L^1(G)$. So we get

$$D(e_\alpha)\cdot\widetilde{\varphi}(f) = D(e_\alpha * f) - \psi(e_\alpha)\cdot D(f) \xrightarrow{w^*} 0$$

for every $f \in S^1(G)$. Then

$$D(f) = w^*\text{-}\lim_\alpha \xi_\alpha\cdot\widetilde{\varphi}(f) - \widetilde{\psi}(f)\cdot\xi_\alpha, \quad f \in S^1(G).$$

Take $f = e_\alpha$. Since (e_α) is in the center of $L^1(G)$, then $D(e_\alpha) = 0$. Therefore for every $f \in S^1(G)$ we have

$$D(f) = \text{norm-}\lim_\alpha \xi_\alpha\cdot\widetilde{\varphi}(f) - \widetilde{\psi}(f)\cdot\xi_\alpha,$$

and so the proof is complete. □

In the upcoming result, we show that the concept of generalized weak amenability on Banach algebras can be induced on Segal algebras.

Theorem 3.2. *Let $\mathcal{A}$ be a commutative Banach algebra and let $\mathfrak{B}$ be an abstract Segal algebra of $\mathcal{A}$ with approximate identity (e_α). Suppose that $\varphi \in \mathrm{Hom}(\mathfrak{B})$ has a continuous extension $\widetilde{\varphi}$ to $\mathcal{A}$. If $\mathcal{A}$ is $(\widetilde{\varphi},\widetilde{\varphi})$-weakly amenable, then $\mathfrak{B}$ is (φ,φ)-weakly amenable.*

P r o o f. Let $D\colon \mathfrak{B} \to \mathfrak{B}^*$ be a continuous (φ,φ)-derivation. Similarly as in the proof of Theorem 3.1, define $D_\alpha\colon \mathcal{A} \to \mathfrak{B}^*$ by $D_\alpha(f) = D(e_\alpha * f) - D(e_\alpha)\cdot\widetilde{\varphi}(f)$ for every $f \in A$. According to the proof of Theorem 3.1, D_α is a continuous (φ,φ)-derivation. Since $\mathcal{A}$ is $(\widetilde{\varphi},\widetilde{\varphi})$-weakly amenable and commutative, $\mathfrak{B}^*$ is a symmetric $\mathcal{A}$-bimodule. This means that $D_\alpha = 0$, and so $D = 0$. $\square$

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