ON THE DOUBLE LUSIN CONDITION AND CONVERGENCE THEOREM FOR KURZWEIL-HENSTOCK TYPE INTEGRALS

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Abstract. Equiintegrability in a compact interval E may be defined as a uniform integrability property that involves both the integrand f_n and the corresponding primitive F_n. The pointwise convergence of the integrands f_n to some f and the equiintegrability of the functions f_n together imply that f is also integrable with primitive F and that the primitives F_n converge uniformly to F. In this paper, another uniform integrability property called uniform double Lusin condition introduced in the papers E. Cabral and P. Y. Lee (2001/2002) is revisited. Under the assumption of pointwise convergence of the integrands f_n, the three uniform integrability properties, namely equiintegrability and the two versions of the uniform double Lusin condition, are all equivalent. The first version of the double Lusin condition and its corresponding uniform double Lusin convergence theorem are also extended into the division space.

Keywords: Kurzweil-Henstock integral; g-integral; double Lusin condition; uniform double Lusin condition

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1. Introduction

It is now known that a function f on a closed and bounded interval E in $\mathbb{R}^n$ is Kurzweil-Henstock integrable with primitive F if and only if f and F satisfy the following: for every $\varepsilon > 0$ there exists a gauge δ on E such that

$$(D)\sum |f(x)||I| < \varepsilon \quad \text{and} \quad (D)\sum |F(I)| < \varepsilon$$

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whenever D is a δ-fine partial division of E in Γ_ε, where

$$\Gamma_\varepsilon = \{(x \cdot I)\colon\ I \subset E,\ x \text{ is a vertex of } I \text{ and } |F(I) - f(x)|I|| \geqslant \varepsilon|I|\}.$$

This condition was introduced in [3] and called the *double Lusin condition* in [4]. A sequence $\{f_n\}$ of Kurzweil-Henstock integrable functions with the corresponding primitives $\{F_n\}$ is said to satisfy the *uniform double Lusin condition* or simply UI_1 if given $\varepsilon > 0$ there is a common gauge δ for all f_n such that

$$(D)\sum |f_n(x)||I| < \varepsilon \quad \text{and} \quad (D)\sum |F_n(I)| < \varepsilon$$

whenever D is a δ-fine partial division of E in $\Gamma_{\varepsilon,n}$, where

$$\Gamma_{\varepsilon,n} = \{(x \cdot I)\colon\ I \subset E,\ x \text{ is a vertex of } I \text{ and } |F_n(I) - f_n(x)|I|| \geqslant \varepsilon|I|\}.$$

It was shown in [3] that if the functions f_n satisfy the UI_1 condition and $f_n \to f$ pointwise everywhere then f is Kurzweil-Henstock integrable and

$$\int f = \lim_{n\to\infty} \int f_n.$$

The proof of this convergence theorem makes use of the fact that UI_1 implies equi-integrability of the functions f_n. In this paper this convergence theorem is proved by looking at the behavior of $\Gamma_{\varepsilon,n}$ as n approaches infinity and by comparing it with Γ_ε. Furthermore, supposing the functions f_n are integrable with primitives F_n and $f_n \to f$ pointwise everywhere then the following will be shown to be equivalent:

(i) the functions f_n satisfy the UI_1 condition;

(ii) the functions f_n satisfy the UI_2 condition, that is, for every $\varepsilon > 0$ there exists a gauge δ on E such that for all n

$$(D)\sum |I| < \varepsilon \quad \text{and} \quad (D)\sum |F_n(I)| < \varepsilon$$

whenever D is a δ-fine partial division of E in $\Gamma_{\varepsilon,n}$;

(iii) the functions f_n are equiintegrable, that is, for every $\varepsilon > 0$ there exists a gauge δ on E such that for all n

$$(D)\sum |f_n(x)|I| - F_n(I)| < \varepsilon$$

whenever D is a δ-fine partial division of E.

An axiomatic approach to the Kurzweil-Henstock integral can be found in [5]. This general theory is called the division space. In the division space we have defined an integral called the g-integral, which includes the Kurzweil-Henstock integral, the McShane integral, and the approximate Perron integral. The double Lusin characterization of the g-integral and the corresponding convergence theorem will also be given.

2. The Kurzweil-Henstock integral in $\mathbb{R}^n$

Let E be a compact interval in $\mathbb{R}^n$ described by $E = \prod_{j=1}^{n} [a_j, b_j]$, where $[a_j, b_j]$, $j = 1, 2, \ldots, n$, is a compact interval in $\mathbb{R}$. From this point onwards, E will always refer to an interval while I will denote any subinterval of E in general. It is also useful to denote $E = [a, b]$, where $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. The measure of E is its outer Lebesgue measure $|E|$ given by $|E| = \prod_{j=1}^{m} (b_j - a_j)$. In general, the measure of any interval I is equal to its outer Lebesgue measure $|I|$.

In our discussion, $\mathbb{R}^m$ will be equipped with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

for $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $r > 0$ we set

$$B(x, r) = \{y \in \mathbb{R}^n \colon \ \|x - y\| < r\},$$

where $x - y = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$.

A partial division $D = \{(x, I)\}$ of E is any finite set of point-interval pairs with x a vertex of the corresponding subinterval $I \subset E$ and with the interiors of the subintervals I disjoint. If for some partial division $D = \{(x, I)\}$,

$$\bigcup_{(x,I)\in D} I = E,$$

then D is said to be a division of E. A gauge on E is a function $\delta\colon E \to (0, \infty)$. A partial division $D = \{(x, I)\}$ is said to be δ-fine if for each pair (x, I) in D, $I \subset B(x, \delta(x))$.

A function $f\colon E \to \mathbb{R}$ is said to be Kurzweil-Henstock integrable if there is a real number A such that given $\varepsilon > 0$ there is a gauge δ on E such that for any δ-fine division of E we have

$$\Bigl|(D) \sum f(x)|I| - A\Bigr| < \varepsilon,$$

where $(D) \sum f(x)|I|$ denotes the sum over all the pairs (x, I) in D. The number A is called the integral of f over E and we write

$$\int_E f = A.$$

If f is Kurzweil-Henstock integrable on E then f is Kurzweil-Henstock integrable on any subinterval I of E. Hence, we can define an additive interval function F by

$$F(I) = \int_I f.$$

We call F the primitive of f. Then F is an additive interval function in the sense that for any finite collection $\{I_i\colon\ i = 1, \ldots, n\}$ whose union is a subinterval I of E we have

$$F\left(\bigcup_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} F(I_i).$$

Note that if we have a primitive interval function F, we can define a point function corresponding to F, and conversely. Let $x \in E$, where $x = (x_1, x_2, \ldots, x_m)$. If for some i, $x_i = a_i$, set $F(x) = 0$. In case $x_i \neq a_i$ for all i, set $F(x) = F([a, x])$. Then we have a unique point function. Conversely, given a point F on E, we can define an interval function as follows: let $I = [\alpha, \beta]$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Write $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, where $\gamma_i = \alpha_i$ or β_i and $n(\gamma)$ denotes the number of terms for which $\gamma_i = \alpha_i$. Define

$$F(I) = \sum_{\gamma} (-1)^{n(\gamma)} F(\gamma),$$

where the summation is over all the vertices γ. We have a unique interval function F. Given a primitive point function, we may recover the primitive interval function as described above. Hence, we may identify them with each other.

The following theorems were proved in [3].

Theorem 2.1. *Let f and F be functions defined on E, where the interval function corresponding to F is additive. Then f is Kurzweil-Henstock integrable on E with primitive F if and only if given $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial division D of E we have*

$$(D \cap \Gamma_\varepsilon) \sum |f(x)||I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_\varepsilon) \sum |F(I)| < \varepsilon.$$

Theorem 2.2. *Let f and F be functions defined on E where the interval function corresponding to F is additive. Then f is Kurzweil-Henstock integrable on E with primitive F if and only if given $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial division D of E we have*

$$(D \cap \Gamma_\varepsilon) \sum |I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_\varepsilon) \sum |F(I)| < \varepsilon.$$

3. Convergence theorems involving uniform integrability

Let $\{f_n\}$ be a sequence of Kurzweil-Henstock integrable functions on E with the corresponding primitives F_n and $\Gamma_{\varepsilon,n}$ being as defined above.

Theorem 3.1. *Suppose the functions f_n satisfy the UI_1 condition. If $f_n(x) \to f(x)$ for all $x \in E$, then f is Kurzweil-Henstock integrable on E and*

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Theorem 3.2. *Suppose the functions f_n satisfy the UI_2 condition. If $f_n(x) \to f(x)$ for all $x \in E$, then f is Kurzweil-Henstock integrable on E and*

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

We shall show alternative proofs of Theorem 3.1 and Theorem 3.2 in the succeeding discussions.

Lemma 3.1. *Suppose the functions f_n are Kurzweil-Henstock integrable on E with the corresponding primitives F_n such that $f_n(x) \to f(x)$ for all $x \in E$. If given $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial division D of E there exists a positive integer N_D such that for $n > N_D$*

$$(D \cap \Gamma_{\varepsilon,n}) \sum |f_n(x)||I| < \varepsilon \quad \textit{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon,$$

then the following hold:

(i) *there exists a function F on E such that $F_n(x) \to F(x)$ for all $x \in E$,*

(ii) *the function f is integrable with primitive F and $\int_E f = \lim_{n\to\infty} \int_E f_n$.*

Proof. (i) Let $\varepsilon > 0$ and $x \in E$, where $[a, x]$ is a nondegenerate interval. We will show that $\{F_n(x)\}$ is a Cauchy sequence. Hence it is convergent.

There exists a gauge δ_x such that for every δ_x-fine division D of $[a, x]$ there exists a positive integer N_D for which for $n \geqslant N_D$

$$(D \cap \Gamma_{\varepsilon/k,n}) \sum |f_n(\xi)||I| < \frac{\varepsilon}{k} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon/k,n}) \sum |F_n(I)| < \frac{\varepsilon}{k},$$

where $k = 5 + 2|E|$.

Let D_x be a fixed δ_x-fine division of $[a, x]$. We can find $M_x \geqslant N_{D_x}$ such that for all $n, m \geqslant M_x$

$$|f_n(\xi) - f_m(\xi)| < \frac{\varepsilon}{k|E|}.$$

It follows that for $n, m \geqslant N_x$,

$$\begin{aligned}
|F_n(x) - F_m(x)| &= |F_n([a, x]) - F_m([a, x])| \\
&\leqslant \left|F_n([a, x]) - (D_x) \sum f_m(\xi)|I|\right| \\
&\quad + \left|(D_x) \sum f_n(\xi)|I| - (D_x) \sum f_m(\xi)|I|\right| \\
&\quad + \left|(D_x) \sum f_m(\xi)|I| - F_m([a, x])\right| < \varepsilon.
\end{aligned}$$

Choose $F(x) = \lim_{n\to\infty} F_n(x)$.

We will now show that f is integrable and F is its primitive.

(ii) Given $\varepsilon > 0$ there exists a gauge δ on E such that if D is a δ-fine partial division of E, there is a positive integer N_D such that for $n > N_D$

$$(D \cap \Gamma_{\varepsilon/2,n}) \sum |f(x)||I| < \frac{\varepsilon}{2} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon/2,n}) \sum |F_n(I)| < \frac{\varepsilon}{2}.$$

Since $f_n(x) \to f(x)$ and $F_n(x) \to F(x)$ for all $x \in E$, there exists a positive integer $M_1 > N_D$ such that for $n > M_1$ the inequalities

$$(D) \sum |f_n(x) - f(x)||I| < \frac{\varepsilon}{2} \quad \text{and} \quad (D) \sum |F_n(I) - F(I)| < \frac{\varepsilon}{2}$$

hold. Furthermore, there exists a positive integer $M_2 > M_1$ such that for $n > M_2$,

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2,n}.$$

Then for $n > M_2$,

$$\begin{aligned}
(D \cap \Gamma_\varepsilon) \sum |f(x)||I| &\leqslant (D \cap \Gamma_\varepsilon) \sum |f(x) - f_n(x)||I| + (D \cap \Gamma_\varepsilon) \sum |f_n(x)||I| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
(D \cap \Gamma_\varepsilon) \sum |F(I)| &\leqslant (D \cap \Gamma_\varepsilon) \sum |F(I) - F_n(I)| + (D \cap \Gamma_\varepsilon) \sum |F_n(I)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The proof is complete. □

Correspondingly, we have the following result for UI_2.

Lemma 3.2. *Suppose the functions f_n are Kurzweil-Henstock integrable on E with the corresponding primitives F_n such that $f_n(x) \to f(x)$ for all $x \in E$. If given $\varepsilon > 0$ there exists a gauge δ on E such that for any δ-fine partial division D of E there exists a positive integer N_D such that for $n > N_D$*

$$(D \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon \quad \textit{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon,$$

then f is Kurzweil-Henstock integrable and $\int_E f = \lim\limits_{n\to\infty} \int_E f_n$.

Theorem 3.1 and Theorem 3.2 follow from Lemma 3.1 and Lemma 3.2, respectively. We will now show the equivalence that we described in the introduction.

Theorem 3.3. *Let $\{f_n\}$ be a sequence of integrable functions on E with the corresponding primitives F_n and suppose that $f_n(x)$ converges for all $x \in E$. Then the following statements are equivalent:*
(1) *f_n satisfy the* UI_1 *condition,*
(2) *f_n satisfy the* UI_2 *condition,*
(3) *f_n are equiintegrable.*

Given f_n pointwise convergent everywhere to a function f, each of the statements above implies that f is Kurzweil-Henstock integrable and $\int_E f = \lim\limits_{n\to\infty} \int_E f_n$.

Proof. (1) $\Rightarrow$ (2): Suppose the functions f_n satisfy the UI_1 condition. Given $\varepsilon > 0$ there exists a gauge δ on E independent of n such that for any δ-fine partial division D of E we have

$$(D \cap \Gamma_{\varepsilon^2/2,n}) \sum |f_n(x)||I| < \frac{\varepsilon^2}{2} \quad \text{and} \quad (D \cap \Gamma_{\varepsilon^2/2,n}) \sum |F_n(I)| < \frac{\varepsilon^2}{2}.$$

We may assume $\varepsilon < 1$. Then $\Gamma_{\varepsilon,n} \subset \Gamma_{\varepsilon^2/2,n}$. Hence, for any δ-fine partial division D of E, we have

$$(D \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon \quad \text{and} \quad (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon.$$

(2) $\Rightarrow$ (1): Suppose the functions f_n satisfy the UI_2 condition. Given $\varepsilon > 0$ and a positive integer i, there exists a gauge δ_i on E independent of n such that for any δ_i-fine partial division D_i of E we have

$$(D_i \cap \Gamma_{\varepsilon_i,n}) \sum |I| < \varepsilon_i,$$

where $\varepsilon_i = \varepsilon/(i2^i)$. Since for every x the sequence $\{f_n(x)\}$ is convergent, $\{f_n(x)\}$ is bounded. Let

$$X_i = \{x \in E\colon\ |f_n(x)| < i,\ \forall\, n \in \mathbb{N}\}$$

and put $Y_1 = X_1$ and for $i = 2, 3, \dots$

$$Y_i = X_i \setminus \bigcup_{k=1}^{i-1} X_k.$$

Define $\delta(x) = \delta_i(x)$ when $x \in Y_i$. Let D be a δ-fine partial division of E. Split D into D_i, $i = 1, 2, \dots$, where D_i contains those pairs with tags in Y_i. For any n and i

$$(D_i \cap \Gamma_{\varepsilon_i,n}) \sum |I| < \varepsilon_i.$$

Thus, for any n,

$$(D \cap \Gamma_{\varepsilon,n}) \sum |f_n(x)||I| \leqslant \sum_{i=1}^{\infty} i(D_i \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon.$$

Furthermore, since f_n satisfy the UI_2 condition, we may choose δ appropriately so that

$$(D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| < \varepsilon.$$

Hence f_n satisfy UI_1.

In [3], it was shown that (1) implies (3).

(3) $\Rightarrow$ (1): Suppose the functions f_n are equiintegrable. Then for any positive number ε and positive integer i, there exists a gauge δ_i on E independent of n such that for any δ_i-fine partial division D of E we have

$$(D) \sum |F_n(I) - f_n(x)|I|| < \varepsilon_i^2,$$

where $\varepsilon_i = \varepsilon/(i2^{i+1})$.

Then for any δ_i-fine partial division D of E

$$(D \cap \Gamma_{\varepsilon_i}) \sum |I| \leqslant \frac{1}{\varepsilon_i}(D \cap \Gamma_{\varepsilon_i,n}) \sum |F_n(I) - f_n(x)|I|| < \varepsilon_i.$$

Following the argument we used in the proof that (2) implies (1), there exists a gauge δ on E independent of n such that for any δ-fine partial division D of E we have

$$(D \cap \Gamma_{\varepsilon/2,n}) \sum |f_n(x)||I| < \frac{\varepsilon}{2}.$$

We may assume that $\varepsilon < 1$. Then for any δ-fine partial division D and any n we have $(D \cap \Gamma_{\varepsilon,n}) \sum |I| < \varepsilon$ and

$$\begin{aligned}(D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I)| &\leqslant (D \cap \Gamma_{\varepsilon,n}) \sum |F_n(I) - f_n(x)|I|| \\ &\quad + (D \cap \Gamma_{\varepsilon,n}) \sum |f_n(x)||I| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

The proof is complete. □

To describe the division space we isolate the properties of δ-fine divisions that make integration work. These are properties (i) to (iv) given below. In addition to these four properties, a fifth property called decomposability (v) has to be added in order to prove convergence theorems. A decomposable division space consists of three mathematical objects: a space T, a family $\mathcal{I}$ of intervals I in T, and a collection $\mathcal{A}$ of families of some point-interval pairs (x, I) satisfying certain conditions. A set E is an elementary set of T if it is a finite union of intervals in $\mathcal{I}$. We shall agree that for any elementary set E and any $I \in \mathcal{I}$, $E \setminus I$ is an elementary set or it is empty. A division D of E is the family of a finite number of mutually disjoint intervals I with union E. A subfamily $\mathcal{I}_1$ of $\mathcal{I}$ divides E if a division D of E exists with the intervals of D belonging to $\mathcal{I}_1$. Let $U = \{(x, I)\} \in \mathcal{A}$. Then U divides E if $\{I\colon (x, I) \in U\}$ divides E.

The triple $(T, \mathcal{I}, \mathcal{A})$ is called a decomposable division space if the following conditions are satisfied:

(i) For every elementary set E of T there is $U \in \mathcal{A}$ dividing E.

(ii) If both $U_1, U_2 \in \mathcal{A}$ are dividing E there is $U_3 \in \mathcal{A}$ dividing E with $U_3 \subset U_1 \cap U_2$.

(iii) If $U_0 \in \mathcal{A}$ divides the union of two disjoint elementary sets E_1 and E_2 then a family $U_1 = \{(x, I)\} \subset U_0$ with $I \subset E_1$ belongs to $\mathcal{A}$ and divides E_1.

(iv) Given disjoint elementary sets E_1 and E_2, if $U_1 \in \mathcal{A}$ divides E_1 with $I \subset E_1$ for all $(x, I) \in U_1$, and $U_2 \in \mathcal{A}$ divides E_2 with $I \subset E_2$ for all $(x, I) \in U_2$, then there is $U_3 \in \mathcal{A}$ dividing $E_1 \cup E_2$ with $U_3 \subset U_1 \cup U_2$.

(v) (*Decomposability*) For all elementary sets K, all sequences U_j dividing K, and all sequences $\{E_j\}$ of mutually disjoint subsets of E there is $U_0 \in \mathcal{A}$ dividing K such that

$$U_0[E_j] \subset U_j[E_j], \quad j = 1, 2, \ldots,$$

where

$$U[X] = \{(x, I) \in U\colon x \in X\}.$$

Definition 4.1. Let $X \subset E$ and $g\colon \bigcup\limits_{U \in \mathcal{A}} U \to \mathbb{R}$. We say that g has a bounded Riemann sums on X (BRS(X)) if there exists $U \in \mathcal{A}$ such that

$$(D) \sum |g(x, I)| < M, \quad \text{for some } M > 1$$

whenever D is a partial division of E from $U[X]$. The function $g(x, I)$ is said to have countably bounded Riemann sums on E (CBRS(E)) if

▷ there exists a sequence of set-number pairs $\{(X_i, M_i)\}$ with $\bigcup\limits_{i=1}^{\infty} X_i = E$ and

$$1 < M_1 \leqslant M_2 \leqslant M_3 \ldots$$

▷ and for every i, there exists $U_i \in \mathcal{A}$ dividing E such that

$$(D)\sum |g(x,I)| < M_i$$

whenever D is a partial division of E from $U_i[X_i]$.

Supposing $g(x,I)$ has countably bounded Riemann sums (using X_i, M_i, U_i) we denote

$$V_H(g,X_i,U_i) = \sup_{D\in U_i[X_i]} \sum |g(x,I)| \leqslant M_i.$$

In what follows, let $g\colon \bigcup_{U\in\mathcal{A}} U \to \mathbb{R}$ have countably bounded Riemann sums on E (using X_i, M_i, U_i).

Definition 4.2. A function $h\colon \bigcup_{U\in\mathcal{A}} U \to \mathbb{R}$ is said to be g-integrable with primitive $\mathcal{H}$ if for every positive integer i, for every $\varepsilon > 0$ there exists $U_i^\varepsilon \in \mathcal{A}$ dividing E with $U_i^\varepsilon[X_i] \subset U_i[X_i]$ such that

$$(D)\sum_{x\in X_i} |h(x,I) - \mathcal{H}(I)| < \varepsilon V_H(g;X_i,U_i)$$

for all partial divisions D of E from U_i^ε.

The family U_i^ε in Definition 4.2 is a function of i. That is, the inequality stated in the definition holds when the sum is taken over all $x \in X_i$ using U_i^ε but may not hold when the sum is taken over all $x \in X_j$, $j \neq i$. However, there is a family U^ε which is uniform with respect to i. That is, inequality in Definition 4.2 holds regardless of, where the sum is being taken over. This is what the next theorem says. The advantage of saying Definition 4.2 with U_i^ε instead of U^ε is in the proof of the double Lusin formulation (Theorem 4.3) where U_i^ε is more quickly obtained than U^ε.

Theorem 4.1. *If $h(x,I)$ is g-integrable on E with primitive $\mathcal{H}$ then for every $\varepsilon > 0$ there exists $U^\varepsilon \in \mathcal{A}$ dividing E with $U^\varepsilon[X_i] \subset U_i[X_i]$ for each i, such that for each i,*

$$(D)\sum_{x\in Y_i} |h(x,I) - \mathcal{H}(I)| < \varepsilon V_H(g;X_i,U_i)$$

for all partial divisions D of E from U^ε.

Proof. Let $Y_1 = X_1$. For $i > 1$, let

$$Y_i = X_i \setminus \bigcup_{j=1}^{i-1} X_j.$$

The sets Y_i are pairwise disjoint. In Definition 4.1, we can actually let the sequence $\{X_i\}$ to be nondecreasing and

$$U[X_{i-1}] = U_{i-1}[X_{i-1}]$$

so that for all i,

$$V_H(g; X_i, U_i) \geqslant V_H(g; X_{i-1}, U_{i-1}).$$

By Definition 4.2, for every $i \in \mathbb{N}$, for every $\varepsilon > 0$ there exists $U_i^\varepsilon \in \mathcal{A}$ dividing E with $U_i^\varepsilon[X_i] \subset U_i[X_i]$ such that

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \frac{\varepsilon}{2^i} V_H(g; X_i, U_i)$$

for all partial divisions D in U_i^ε. By the decomposability property, we can choose $U^\varepsilon \in \mathcal{A}$ dividing E such that

$$U^\varepsilon[Y_i] \subset U_i^\varepsilon[Y_i].$$

Then for each $i \in \mathbb{N}$,

$$\begin{aligned}(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| &= \sum_{k=1}^{i} (D) \sum_{x \in Y_k} |h(x, I) - \mathcal{H}(I)| \\ &< \sum_{k=1}^{i} \frac{\varepsilon}{2^k} V_H(g; X_k, U_k) \\ &\leqslant V_H(g; X_i, U_i) \sum_{k=1}^{i} \frac{\varepsilon}{2^k} < \varepsilon V_H(g; X_i, U_i)\end{aligned}$$

for all partial divisions D in U^ε. The proof is complete. □

The next theorem says that Definition 4.2 is just a special case of the generalized Riemann integral defined in ([5], page 165).

Theorem 4.2. *If $h(x, I)$ is g-integrable with primitive $\mathcal{H}$ then $h(x, I)$ is generalized Riemann integrable and the two integrals coincide.*

Proof. Let Y_i be as in the proof of the preceding theorem.

By Definition 4.2, for every positive integer i, for every $\varepsilon > 0$ there exists $U_i^\varepsilon \in \mathcal{A}$ dividing E with $U_i^\varepsilon[X_i] \subset U_i[X_i]$ such that,

$$(D) \sum_{x \in X_i} |h(x, I) - \mathcal{H}(I)| < \frac{\varepsilon}{2^i M_i} V_H(g; X_i, U_i)$$

for all partial divisions D of E from U_i^ε. By the decomposability property, we can choose $U^\varepsilon \in \mathcal{A}$ dividing E such that $U^\varepsilon[Y_i] \subset U_i^\varepsilon[Y_i]$.

Then for all divisions D of E from U^ε, we have

$$(D)\sum |h(x,I)-\mathcal{H}(I)| = \sum_{i=1}^{\infty}(D)\sum_{x\in Y_i}|h(x,I)-\mathcal{H}(I)|$$
$$< \sum_{i=1}^{\infty}\frac{\varepsilon}{2^i M_i}V_H(g;X_i,U_i)\leqslant \varepsilon.$$

This is precisely the definition of the generalized Riemann integral. □

Definition 4.3. We say that a function $h(x,I)$ satisfies the αg-condition on E if for every $x\in E$ there is a minimum number $\alpha(x)\in\mathbb{Z}^+$ such that for any (x,I) in the division space we have

$$|h(x,I)|\leqslant \alpha(x)|g(x,I)|.$$

Given functions $\mathcal{H}$ and $h(x,I)$, and $U\in\mathcal{A}$, we denote

$$\Gamma_\varepsilon = \{(x,I)\in U\colon\ |\mathcal{H}(I)-h(x,I)|\geqslant \varepsilon|g(x,I)|\}.$$

We are now ready to present the double Lusin formulation for the g-integral.

Theorem 4.3. *Let $h\colon \bigcup_{U\in\mathcal{A}} U\to\mathbb{R}$ satisfy the αg-condition and $H\colon \mathcal{I}(E)\to\mathbb{R}$. Then $h(x,I)$ is g-integrable on E with primitive $\mathcal{H}$ if and only if for every i, for every $\varepsilon>0$ there exists $U_i^\varepsilon\in\mathcal{A}$ with $U_i^\varepsilon[X_i]\subset U_i[X_i]$ such that*

$$(D\cap\Gamma_\varepsilon)\sum_{x\in X_i}|h(x,I)|<\varepsilon V_H(g;X_i,U_i)$$

and

$$(D\cap\Gamma_\varepsilon)\sum_{x\in X_i}|H(I)|<\varepsilon V_H(g;X_i,U_i)$$

for all partial divisions D of E from U_i^ε.

P r o o f. (⇒) Suppose $h(x,I)$ is g-integrable with primitive $\mathcal{H}$. Let

$$E_k=\{x\in E\colon\ \alpha(x)=k\}.$$

Given $0<\varepsilon<1$ and a positive integer i, for every k there exists $U_i^{\varepsilon,k}\in\mathcal{A}$ dividing E with $U_i^{\varepsilon,k}[X_i]\subset U_i[X_i]$ such that

$$(D)\sum_{x\in X_i}|h(x,I)-\mathcal{H}(I)|<\frac{\varepsilon^2}{k2^{k+1}}V_H(g;X_i,U_i)$$

for all partial divisions D of E from $U_i^{\varepsilon,k}$.

By decomposability, there exists $U_i^\varepsilon \in \mathcal{A}$ dividing E with

$$U_i^\varepsilon[E_k] \subset U_i^{\varepsilon,k}[E_k].$$

Then for all partial divisions D of E from U_i^ε, we have

$$\begin{aligned}
(D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x,I)| &\leqslant \sum_{k=1}^{\infty} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i \cap E_k} k|g(x,I)| \\
&\leqslant \sum_{k=1}^{\infty} \frac{k}{\varepsilon} (D \cap \Gamma_\varepsilon) \sum_{x \in X_i \cap E_k} |h(x,I) - \mathcal{H}(I)| \\
&< \sum_{k=1}^{\infty} \frac{k}{\varepsilon} \frac{\varepsilon^2}{k 2^{k+1}} V_H(g; X_i, U_i) \\
&< \varepsilon V_H(g; X_i, U_i)
\end{aligned}$$

and

$$\begin{aligned}
(D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I)| &\leqslant (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I) - h(x,I)| + (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x,I)| \\
&< \frac{\varepsilon}{2} V_H(g; X_i, U_i) + \frac{\varepsilon}{2} V_H(g; X_i, U_i) = \varepsilon V_H(g; X_i, U_i).
\end{aligned}$$

($\Leftarrow$) For the converse, given i, for every $\varepsilon > 0$ choose U_i^ε such that

$$(D \cap \Gamma_{\varepsilon/3}) \sum_{x \in X_i} |h(x,I)| < \frac{\varepsilon}{3} V_H(g; X_i, U_i)$$

and

$$(D \cap \Gamma_{\varepsilon/3}) \sum_{x \in X_i} |\mathcal{H}(I)| < \frac{\varepsilon}{3} V_H(g; X_i, U_i)$$

for all partial divisions D of E from U_i^ε. Then for a partial division D of E from U_i^ε we have

$$\begin{aligned}
(D) \sum_{x \in X_i} |h(x,I) - \mathcal{H}(I)| &\leqslant (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |h(x,I)| + (D \cap \Gamma_\varepsilon) \sum_{x \in X_i} |\mathcal{H}(I)| \\
&\quad + (D \setminus \Gamma_\varepsilon) \sum_{x \in X_i} |h(x,I) - \mathcal{H}(I)| \\
&< \frac{\varepsilon}{3} V_H(g; X_i, U_i) + \frac{\varepsilon}{3} V_H(g; X_i, U_i) \\
&\quad + (D \setminus \Gamma_\varepsilon) \sum_{x \in X_i} \frac{\varepsilon}{3} |g(x,I)| \\
&< \varepsilon V_H(g; X_i, U_i).
\end{aligned}$$

The proof is complete. □

We now present a generalization of Theorem 3.1 to the decomposable division space.

Theorem 4.4. *Let $\{h_n(x,I)\}$ be a sequence of g-integrable functions satisfying the αg-condition with the corresponding primitives $\mathcal{H}_n$. If the following conditions are satisfied:*

(1) *$h_n(x,I) \to h(x,I)$ for all (x,I) in the division space, where $h(x,I)\colon \bigcup\limits_{U\in\mathcal{A}} U \to \mathbb{R}$,*

(2) *for every i, for every $\varepsilon > 0$ there exists $U_i^\varepsilon \in \mathcal{A}$ independent of n with $U_i^\varepsilon[X_i] \subset U_i[X_i]$ such that for any partial division D of E in $U_i^\varepsilon[X_i]$ we have*

$$(D\cap\Gamma_{\varepsilon,n})\sum|h_n(x,I)| < \varepsilon V_H(g;X_i,U_i)$$

and

$$(D\cap\Gamma_{\varepsilon,n})\sum|\mathcal{H}_n(I)| < \varepsilon V_H(g;X_i,U_i).$$

Then the following statements hold:

(i) *there exists function $\mathcal{H}$ such that $\mathcal{H}_n(I) \to \mathcal{H}(I)$ for all $I \in \mathcal{I}(E)$,*

(ii) *$h(x,I)$ is g-integrable with*

$$\int_E h(x,I) = \lim_{n\to\infty}\int_E h_n(x,I).$$

P r o o f. Let $I_0 \in \mathcal{I}$, $U \in \mathcal{A}$ dividing I_0 and D a division of I_0. By the triangle inequality

$$\begin{aligned}(D)\sum|\mathcal{H}_m(I)-\mathcal{H}_n(I)| \leqslant {} & (D)\sum|\mathcal{H}_m(I)-h_m(x,I)| \\ & + (D)\sum|h_m(x,I)-h_n(x,I)| \\ & + (D)\sum|h_n(x,I)-\mathcal{H}_n(I)|.\end{aligned}$$

In view of condition (2), the fact that $h_n(x,I)$ converges for all (x,I), and the inequality above it follows that $\{\mathcal{H}_n(I)\}$ is Cauchy and hence convergent. We let $\mathcal{H}(I)$ be its limit.

Consider X_i. Let $X_i = X$ and $\varepsilon > 0$. In view of condition (2), there exists $U^{\varepsilon/2} \in \mathcal{A}$ independent of n with $U^{\varepsilon/2}[X] \subset U[X]$ such that for any partial division D of E in $U^{\varepsilon/2}[X]$ we have

$$(D\cap\Gamma_{\varepsilon/2,n})\sum|h_n(x,I)| < \frac{\varepsilon}{2}V_H(g;X,U)$$

and

$$(D\cap\Gamma_{\varepsilon/2,n})\sum|\mathcal{H}_n(I)| < \frac{\varepsilon}{2}V_H(g;X,U).$$

Let D be any partial division of E in $V^\varepsilon[X]$. By condition (3) there exists a positive integer N_1 such that for any $n > N_1$ we have

$$|\mathcal{H}(I) - \mathcal{H}_n(I)| < \frac{\varepsilon}{2}|g(x,I)| \quad \forall\,(x,I) \in D.$$

From condition (1), given the same there exists a positive integer $N_2 > N_1$ such that for any $n > N_2$ we have

$$|h(x,I) - h_n(x,I)| < \frac{\varepsilon}{2}|g(x,I)| \quad \forall\,(x,I) \in D.$$

Finally, in view of conditions (1) and (3) there exists a positive integer $N_D > N_2$ such that for $n > N_D$ we have

$$D \cap \Gamma_\varepsilon \subset D \cap \Gamma_{\varepsilon/2,n}.$$

Then for $n > N_D$,

$$\begin{aligned}(D\cap\Gamma_\varepsilon)\sum|h(x,I)| &\leqslant (D\cap\Gamma_\varepsilon)\sum|h(x,I)-h_n(x,I)| + (D\cap\Gamma_\varepsilon)\sum|h_n(x,I)| \\ &< \frac{\varepsilon}{2}(D\cap\Gamma_\varepsilon)\sum|g(x,I)| + \frac{\varepsilon}{2}V_H(g;X,U) \\ &\leqslant \varepsilon V_H(g;X,U)\end{aligned}$$

and

$$\begin{aligned}(D\cap\Gamma_\varepsilon)\sum|\mathcal{H}(I)| &\leqslant (D\cap\Gamma_\varepsilon)\sum|\mathcal{H}(I)-\mathcal{H}_n(I)| + (D\cap\Gamma_\varepsilon)\sum|\mathcal{H}_n(I)| \\ &< \frac{\varepsilon}{2}(D\cap\Gamma_\varepsilon)\sum|g(x,I)| + \frac{\varepsilon}{2}V_H(g;X,U) \\ &\leqslant \varepsilon V_H(g;X,U).\end{aligned}$$

Therefore $h(x,I)$ is g-integrable and $\mathcal{H}(I)$ is its primitive. □

The double Lusin condition that is used to characterize the Kurzweil-Henstock integral uses point-interval pairs (x,I) such that x is contained in I. It can be shown that the McShane integral has a similar double Lusin characterization with tags x floating around I (see [2]). The approximate Perron (AP) integral was given in [5]. A double Lusin characterization of AP integral can also be given. The Henstock-Stieltjes integral was given in [1] with a function g of bounded variation (BV). In our characterization of the g-integral it is possible for g to be BVG^*, where VBG^* is defined as follows: Given a subset X of E, an interval function g on E is said to be $BV^*(X)$ if there exists a nonnegative number M such that for any X-tagged partial division D of E we have $(D)\sum|g(I)| \leqslant M$ and g is BVG^* on E if $E = \bigcup_{i=1}^{\infty} X_i$ such that for each i, g is $BV^*(X_i)$. It remains to be explored what happens when g is BVG^*.

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